Approaching the master-field
Hadronic observables in large volumes

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master-fields: motivation

with state-of-the-art supercomputers and algorithms, we can generate gauge field configurations with

[Lüscher Lattice 2017, Francis, Fritzsch, Lüscher, Rago 2020; previous talk by P. Fritzsch]

- 192^4 lattice points • up to $\approx 18\,\mathrm{fm}$ length • $m_\pi L = 25$
- stochastic locality ⇒ expectation values from volume average
- also errors estimated using volume average

standard MC errors

for an observable $O(x)$ localized to a region around x,
the estimator of its field-theoretical expectation value $\langle O(x)\rangle$ is, traditionally, the Monte Carlo average

$$\bar{O}(x) = \frac{1}{n}\sum_{i=1}^{n} O\Big|_{U_i}, \qquad \sigma_{\bar{O}}(x) = \frac{1}{n^{1/2}}\sigma_O(x)$$

where the field-theoretical distribution variance is $\sigma_O^2(x) = \langle [O(x) - \langle O(x)\rangle]^2\rangle$

master-field errors

in the master-field approach, the Monte Carlo average is replaced with the translation average [Lüscher Lattice 2017]

$$\langle\!\langle O(x) \rangle\!\rangle = \frac{1}{V} \sum_z O(x+z), \qquad \langle O(x) \rangle = \langle\!\langle O(x) \rangle\!\rangle + \mathcal{O}\left(V^{-1/2}\right)$$

whose distribution has variance

$$\begin{aligned}
\sigma^2_{\langle\!\langle O \rangle\!\rangle}(x) = \left\langle [\langle\!\langle O(x) \rangle\!\rangle - \langle O(x) \rangle]^2 \right\rangle &= \frac{1}{V} \sum_y \langle O(y) O(0) \rangle_c \\
&= \frac{1}{V} \left[\sum_{|y| \le R} \langle O(y) O(0) \rangle_c + \mathcal{O}\left(\mathrm{e}^{-mR}\right) \right] \\
&= \frac{1}{V} \left[\sum_{|y| \le R} \langle\!\langle O(y) O(0) \rangle\!\rangle_c + \mathcal{O}\left(\mathrm{e}^{-mR}\right) + \mathcal{O}\left(V^{-1/2}\right) \right]
\end{aligned}$$

and for n master-fields

$$\sigma^2_{\langle\!\langle \bar{O} \rangle\!\rangle}(x) = \frac{1}{n} \sigma^2_{\langle\!\langle O \rangle\!\rangle}(x) = \frac{1}{V} \left[\sum_{|y| \le R} \left\langle\!\!\left\langle \bar{O}(y) \bar{O}(0) \right\rangle\!\!\right\rangle_c + \mathcal{O}\left(\mathrm{e}^{-mR}\right) + \mathcal{O}\left(V^{-1/2}\right) \right]$$

hadronic observables

hadron propagator, *e.g.* Wick-connected meson contraction for source $y=0$ to sink x

$$C_{\Gamma\Gamma'}(x,0) = [\bar{u}\Gamma d](x)[\bar{d}\Gamma' u](0) = \mathrm{tr}\left\{\Gamma\gamma_5 D^{-1}(x,0)\gamma_5\Gamma' D^{-1}(x,0)\right\}$$

- $\|D^{-1}(x,0)\| \propto \mathrm{e}^{-m_\pi|x|/2}$: non-*ultralocal* but still localized $\sim m_\pi^{-1}$
- the master-field error is given by the four-point function

$$\left\langle [\langle\!\langle C(x,0)\rangle\!\rangle - \langle C(x,0)\rangle]^2 \right\rangle = \frac{1}{V}\left[\sum_{|y|\leq R} \langle\!\langle C(x+y,y)C(x,0)\rangle\!\rangle_c + \mathcal{O}\left(\mathrm{e}^{-mR}\right) + \mathcal{O}\left(V^{-1/2}\right)\right]$$

- large footprint • y can be sampled $\Rightarrow$ no all-to-all needed

everything works also replacing $C(x,0)$ with time-momentum correlators

$$\tilde{C}(x_0,\vec{p}) = \sum_{\vec{x}} \mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}} C(x,0)$$

but these have a large footprint in space $\Rightarrow$ extract hadronic observables from position-space correlators?
e.g. • basic spectroscopy: m_π, m_N, • decay constants: f_π, • vector correlator physics: a_μ^{HVP}
alternative: inexact momentum projection, localized in space

momentum-projected Euclidean-time correlators

or time-momentum correlators

$$\tilde{C}_{PP}(t,\vec{p}) \to \frac{\left|c_P(\vec{p})\right|^2}{2E_\pi(\vec{p})} e^{-E_\pi(\vec{p})t},$$
$$\tilde{C}_{NN}(t,\vec{p}) \to \frac{\left|c_N(\vec{p})\right|^2}{2E_N(\vec{p})} (-i\not{p} + m_N) e^{-E_N(\vec{p})t},$$

standard approach to lattice spectroscopy

- discrete spectrum in a L^3 box $\Rightarrow \tilde{C}(t,\vec{p})$ is a sum of exponentials, also at $a \neq 0$
- leading finite-T effects (back-propagating states) are easy to take into account

we considered point sources (that also work for position-space) and

- no smearing
- $3d$-fermion smearing with $\kappa_{3d} = 0.180, 0.190, 0.200$ [Papinutto, Scardino, Schaefer 2018]
- gradient-flow smearing ($4d$ smearing, modifies the transfer matrix but ok if $\sqrt{8t_{\mathrm{flow}}} \ll t$) [Lüscher 2013]

position-space correlators

$$C_{PP}(x) \to \frac{|c_P|^2}{4\pi^2}\frac{m_\pi}{|x|}K_1(m_\pi|x|),$$

$$C_{NN}(x) \to \frac{|c_N|^2}{4\pi^2}\frac{m_N^2}{|x|}\left[K_1(m_N|x|) + \frac{\not{x}}{|x|}K_2(m_N|x|)\right].$$

- two structures for the nucleon correlator

$$\mathrm{tr}\, C_{NN}(x) \to \frac{|c_N|^2}{4\pi^2}\frac{m_N^2}{|x|}K_1(m_N|x|), \qquad \mathrm{tr}\, \not{x} C_{NN}(x) \to \frac{|c_N|^2}{4\pi^2}m_N^2 K_2(m_N|x|)$$

- exact description only for $a \to 0$, at $a \neq 0$, each direction has different cut-off effects
- we work with the radial correlator, averaged over $4d$ spheres $\mathring{C}(r) = (1/\mathrm{r}_4(r^2)) \sum_{|x|=r} C(x)$
  $\Rightarrow$ cut-off effects are partially averaged out
- different rôle of finite-L, T effects, more complex treatment in general
  $\Rightarrow$ eventually, not an issue in the large-volume regime $L, T \gg 1/\Lambda, 1/m_\pi$

systematics effects in position-space correlators (pion)

on a $T = 96a$, $L = 64a = 6.02\,\mathrm{fm}$ box ($m_\pi L \approx 8.9$) $\Rightarrow$ too small for master-field, traditional MC error here

- each $r^2 \in \mathbb{Z}$ up to $r = \min(T, L)/2 \Rightarrow 32^2$ points
- correction for finite-L, T effects works great!
  summing over images, with $\mathbb{L} = \mathrm{diag}(T, L, L, L)$

$$\mathring{C}_{L,T}(r) = \sum_{n\in\mathbb{Z}^4} \frac{1}{\mathrm{r}_4(r^2)} \sum_{|x|=r} C(x + \mathbb{L}\cdot n)$$

- "one-state" fit with two parameters a_0, m_π

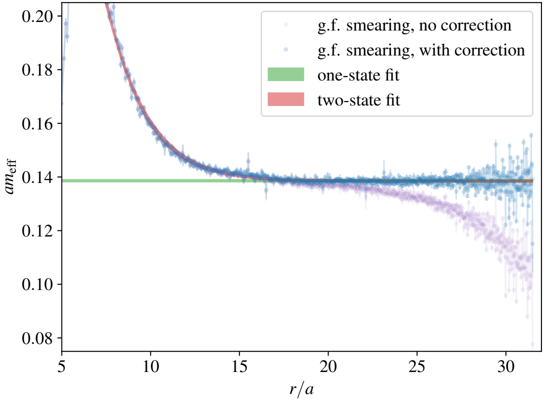
$$\mathring{C}(r) = a_0 \frac{m_\pi}{r} K_1(m_\pi r) + 6a_0 \int \frac{\mathrm{d}\Omega_3}{2\pi^2} \frac{m_\pi K_1(m_\pi |x + L\cdot\hat{z}|)}{|x + L\cdot\hat{z}|}$$

- "two-state" fit with excited state well described by

$$\mathring{C}(r) \rightarrow \mathring{C}(r) + a_1 \frac{m_1}{r} K_1(m_1 r)$$

- varying r_{min}, fixed $r_{\mathrm{max}} = L/2$

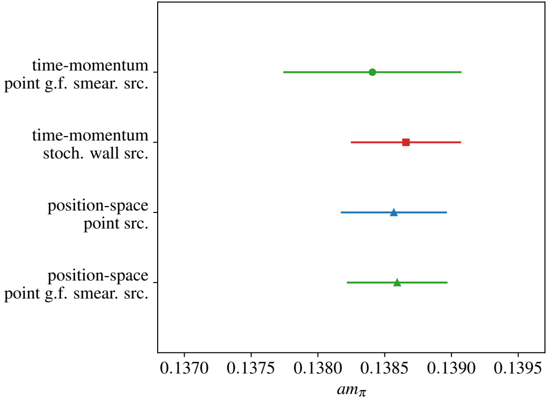
effective mass plot



results (pion)

on a $T = 96a$, $L = 64a = 6.02\,\mathrm{fm}$ box ($m_\pi L \approx 8.9$) $\Rightarrow$ too small for master-field, traditional MC error here

- with the same number of point sources, the radial correlator determination has half the error of the time-momentum correlator
- smearing does not improve the m_π determination $\Rightarrow$ this is expected
- zero-momentum time correlator with stochastic wall sources results in a similar precision (at same number of inversion)



$$m_\pi = 0.1386(4)/a \approx 290.5(8)\,\mathrm{MeV}$$

systematics effects in position-space correlators (nucleon)

on a $T = 96a$, $L = 64a = 6.02\,\mathrm{fm}$ box ($m_\pi L \approx 8.9$) $\Rightarrow$ too small for master-field, traditional MC error here

- each $r^2 \in \mathbb{Z}$ up to $r = \min(T, L)/2 \Rightarrow 32^2$ points
- we observe a upward trend in m_{eff} at large r
  finite-L, T effects? partial correction, in progress
  $\Rightarrow$ the problem goes away on true master-fields
- "one-state" fit with two parameters a_0, m_N

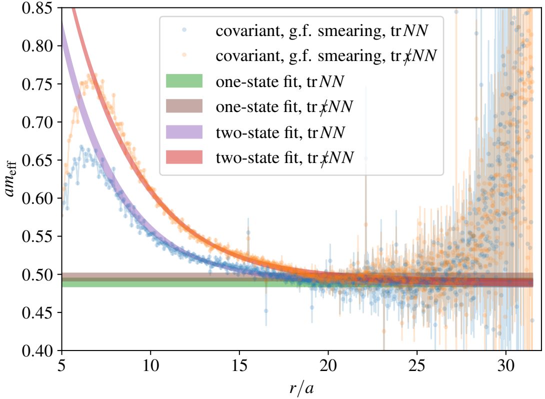
$$\mathring{C}(r) = a_0 \frac{m_N^2}{r} K_1(m_N r) \quad \text{or} \quad a_0 m_N^2 K_2(m_N r)$$

- "two-state" fit with excited state well described by

$$\mathring{C}(r) \to \mathring{C}(r) \cdot \left[1 + a_1 \frac{m_\pi}{r} K_1(m_\pi r)\right]$$

- varying r_{min}, fixed $r_{\mathrm{max}} = 24a$ to avoid the large r

effective mass plot

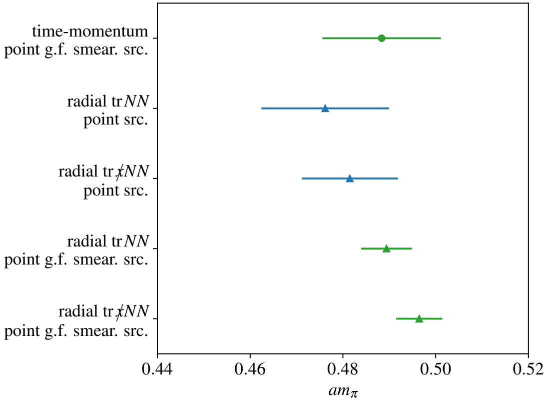


results (nucleon)

on a $T = 96a$, $L = 64a \approx 6.02\,\mathrm{fm}$ box ($m_\pi L \approx 8.9$) $\Rightarrow$ too small for master-field, traditional MC error here

- smearing works! gradient-flow smearing gives the best results
- with the same number of point sources, the radial correlator determination has less than half the error of the time-momentum correlator
- systematic shift between $\mathrm{tr}\, N\bar{N}$ and $\mathrm{tr}\, \not{x} N\bar{N}$

  different cut-off effects?

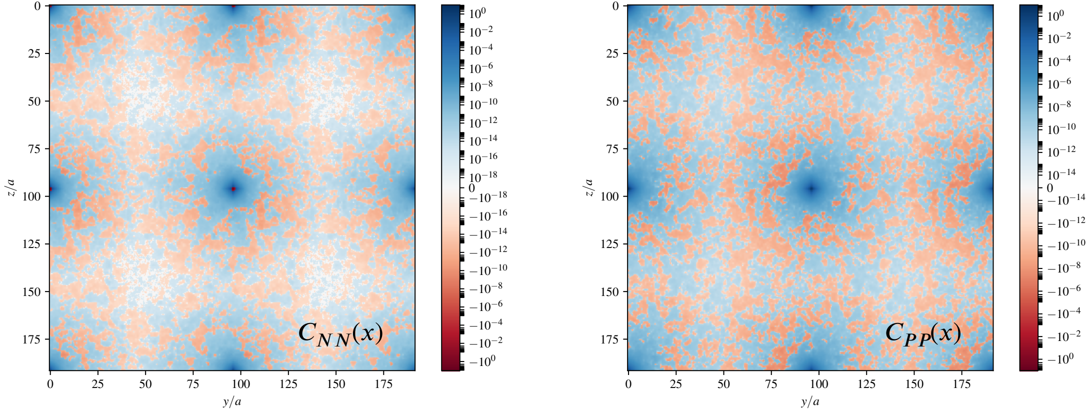


$$m_N = 0.489(5)/a \approx 1025(10)\,\mathrm{MeV}$$

master-field errors: hadronic observables

$L = T = 96a \approx 9.02\,\mathrm{fm}$ and $L = T = 192a \approx 18.05\,\mathrm{fm}$ boxes ⇒ test master-field error estimation

- evaluate the four-point function ⇒ estimate the master-field error
- grid of point sources, work in progress!



conclusions

- position-space correlators can be used to extract hadron masses
- short-distance and cut-off effects are under control
- large volumes are needed to have good control of finite-T, L systematics $\Rightarrow$ master-fields are ideal

outlook

- go beyond the sphere-averaged radial correlator, understanding better cut-off effects on the $a \neq 0$ correlator
- real master-fields are now available [previous talk by P. Fritzsch]
- study more complex observables, *e.g.* • decay constants,
  - hadronic vacuum polarization ($g_\mu - 2$, $\Delta\alpha_{\text{had}}$) [Meyer 2017; MC, Gérardin, Ottnad, Meyer Lattice 2018]
- denser spectrum of excited states $\Rightarrow$ scattering amplitudes from spectral reconstruction

[Hansen, Meyer, Robaina 2017; Bulava, Hansen 2019; J. Bulava talk Wed 27 14:30 EDT; Bruno, Hansen 2021; M. Bruno talk Wed 27 6:45 EDT]

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thanks

for your attention!

questions?