Analytic Expansions of Two- and Three-Particle Excited-State Energies

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in collaboration with

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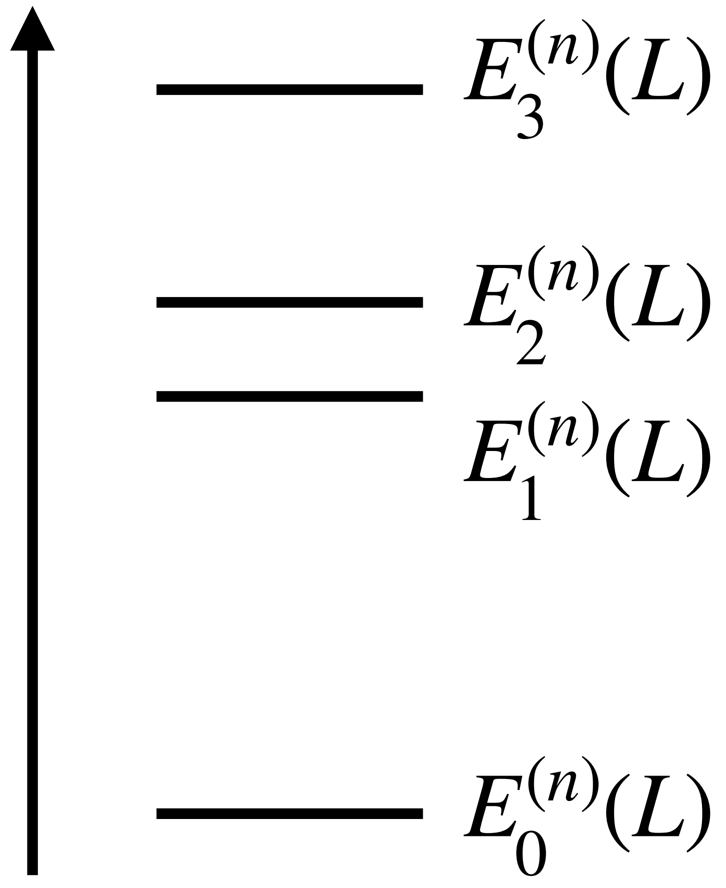
Publications to Appear Shortly

Motivation

Idea: Energy levels of particles in a box map to infinite-volume scattering amplitude, useful for extracting low-energy QCD scattering parameters from lattice simulations



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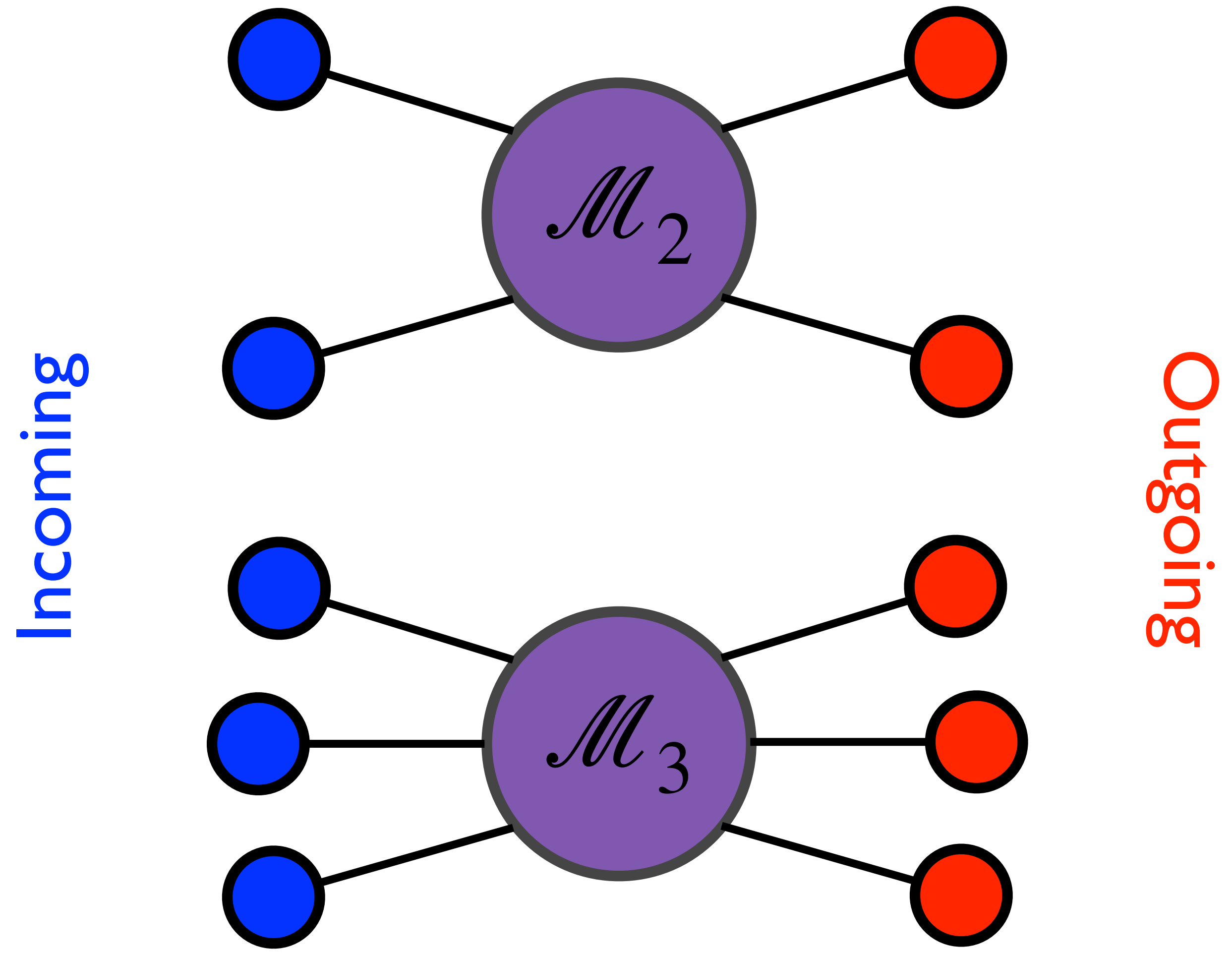
$$\det[1 + F_3\mathcal{K}_{df,3}] = 0$$



Previous work by Huang, Yang, Beane, Detmold, Savage, Hansen, Sharpe, Pang, Wu, Hammer, Meißner, Rusetsky, Romero-López, Schlage, Urbach, ...

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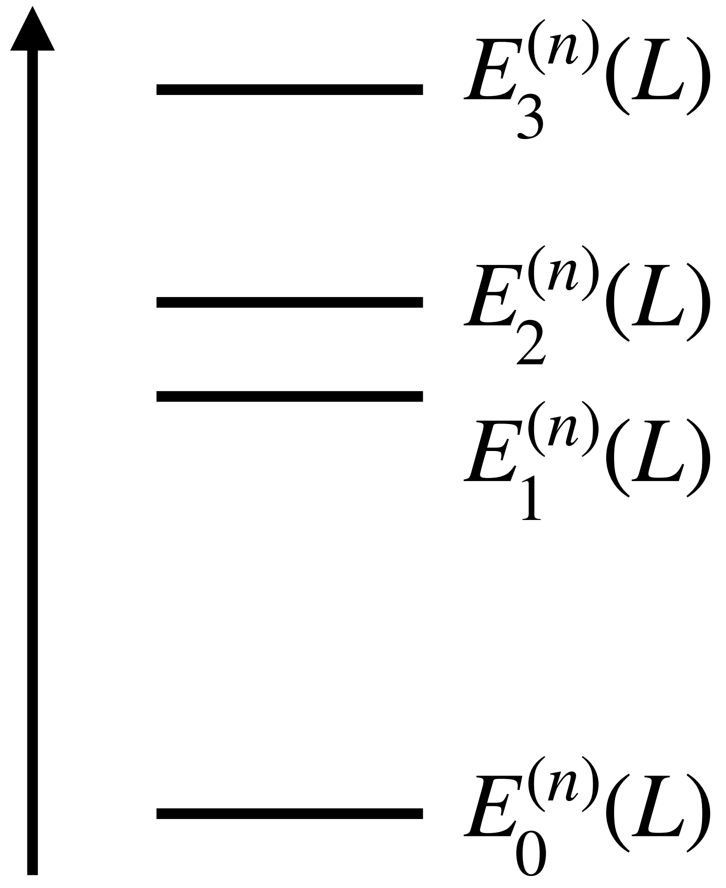

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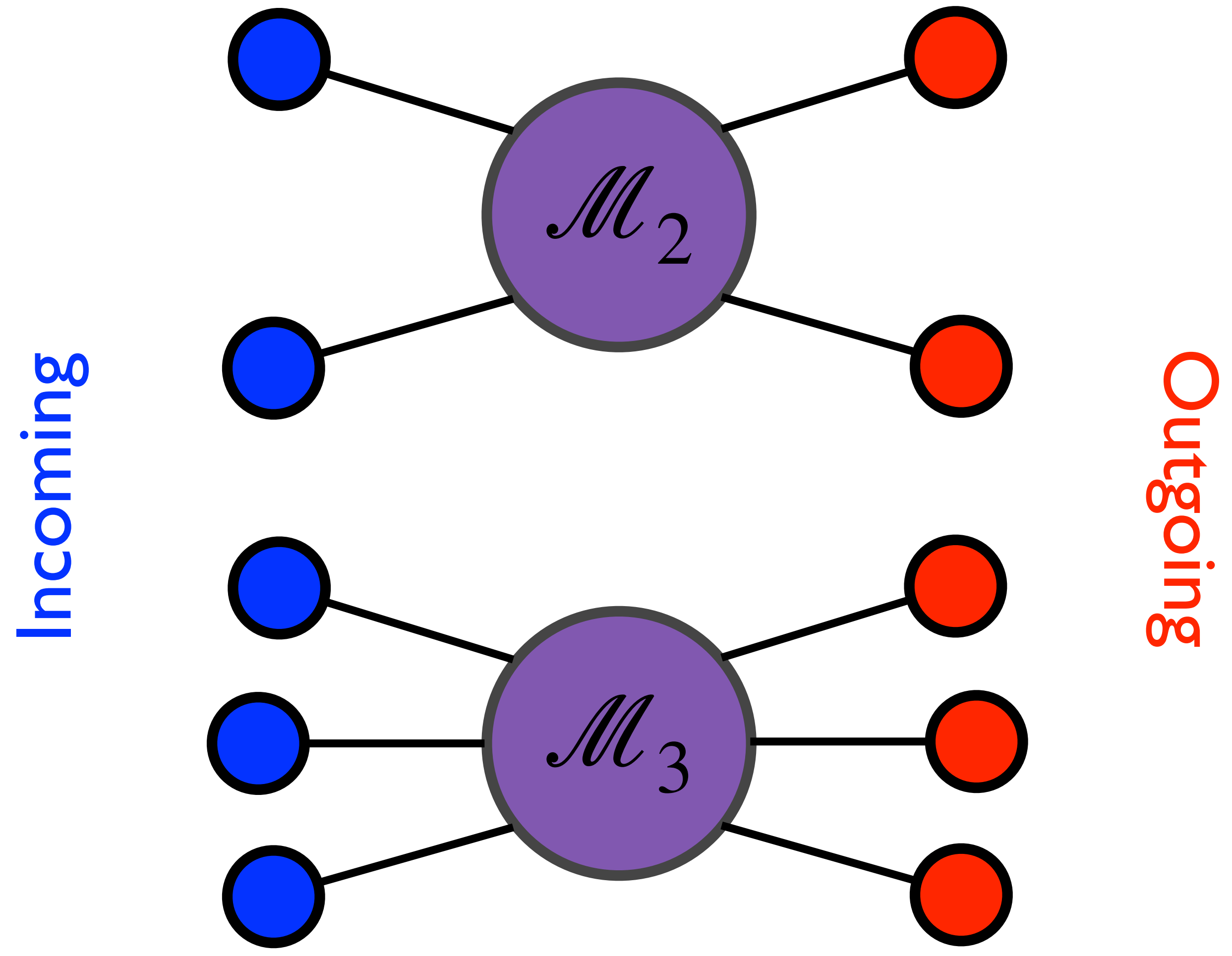

Goal: Derive analytic results for two- and three-particle systems, in a power counting scheme

Motivation for This Work

- Test three-particle quantization condition
- Build intuition for N-particles systems
- Guided root finding
- Test convergence when including higher partial waves

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Utilizing Two-Particle Quantization

(details in manuscript to appear)

Idea: Truncation of F and $\mathscr{M}$ are necessary to make practical use of quantization condition

- Expansion in a given power counting scheme allows for analytic expressions

Truncation to s-wave only:

$$\underbrace{p^\star \cot \delta_0(p^\star)}_{\text{inf. vol.}} = \underbrace{f(q, \boldsymbol{d}, L)}_{\text{fin. vol.}}$$

$\boldsymbol{d}$: total momentum
$p^\star$: relative momentum, in CoM

Power Counting: Treat theory as weakly interacting, expand around non-interacting energy

$$q_{\mathfrak{n}}(L)^2 = q_{\mathfrak{n}}^{(0)}(L)^2 + \sum_{k=1}^{\infty} \epsilon^k \, \Delta_{q[\mathfrak{n}]}^{(k)}(L)$$

q: relative momentum, in CoM; dimensionless
$\mathfrak{n}$: collective index for a given energy level

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Idea: Systematically solve for $\Delta_{q[\mathfrak{n}]}^{(k)}(L)$ to find total energy of the system

- Infinite-volume contribution is simple to expand
- Known geometrical function f must be treated with care due to poles

Expanding Geometric Function

Complication: Geometric function contains a sum over the set of all three vectors

$$f = \frac{1}{\gamma(q_{\mathfrak{n}}, \boldsymbol{d}, L)} \sum_{v \in \mathbb{Z}^3} \mathscr{F}(q_{\mathfrak{n}}, \boldsymbol{d}, L)$$

q: relative momentum, in CoM; dimensionless
$\mathfrak{n}$: collective index for a given energy level

- The summand $\mathscr{F}$ has poles when expanding around the non-interacting energy
  - Defines a set of vectors $S_{\mathfrak{n}}$ for each energy level

$$S_{\mathfrak{n}} = \left\{ v \in \mathbb{Z}^3 \,\middle|\, E_{\mathfrak{n}}^{(0)} - \omega_v - \omega_{\boldsymbol{d}-v} = 0 \right\}$$

ω_v: single particle energy

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Expansion: Break infinite sum into sum over $\boldsymbol{\nu} \in S_{\mathfrak{n}}$ and $\boldsymbol{\nu} \notin S_{\mathfrak{n}}$

$$f = \frac{1}{\gamma(q_{\mathfrak{n}}, \boldsymbol{d}, L)} \left(\sum_{\nu \in S_{\mathfrak{n}}} T(q_{\mathfrak{n}}, \boldsymbol{d}, L) + \sum_{\nu \notin S_{\mathfrak{n}}} B(q_{\mathfrak{n}}, \boldsymbol{d}, L) \right)$$

Starts at $\mathcal{O}(1/\epsilon)$

Starts at $\mathcal{O}(1)$

Two-Particle Result, NLO

NLO Result: General result for any s-wave dominated non-degenerate state*

$$E_{\mathfrak{n}}(L) = E_{\mathfrak{n}}^{(0)}(L) + \epsilon\, g_{\mathfrak{n}} \frac{E_{\mathfrak{n}}^{(0)}(L)}{4\omega_{\nu_{\mathfrak{n}}}\omega_{d-\nu_{\mathfrak{n}}}} \frac{8\pi a_0}{\gamma_{\mathfrak{n}}^{(0)} L^3} + \mathcal{O}(\epsilon^2)$$

$g_{\mathfrak{n}}$: size of set $S_{\mathfrak{n}}$
ω_ν: single particle energy

Holds for states in trivial irrep in any moving frame, assuming $a^2 r \sim \mathcal{O}(\epsilon^2)$

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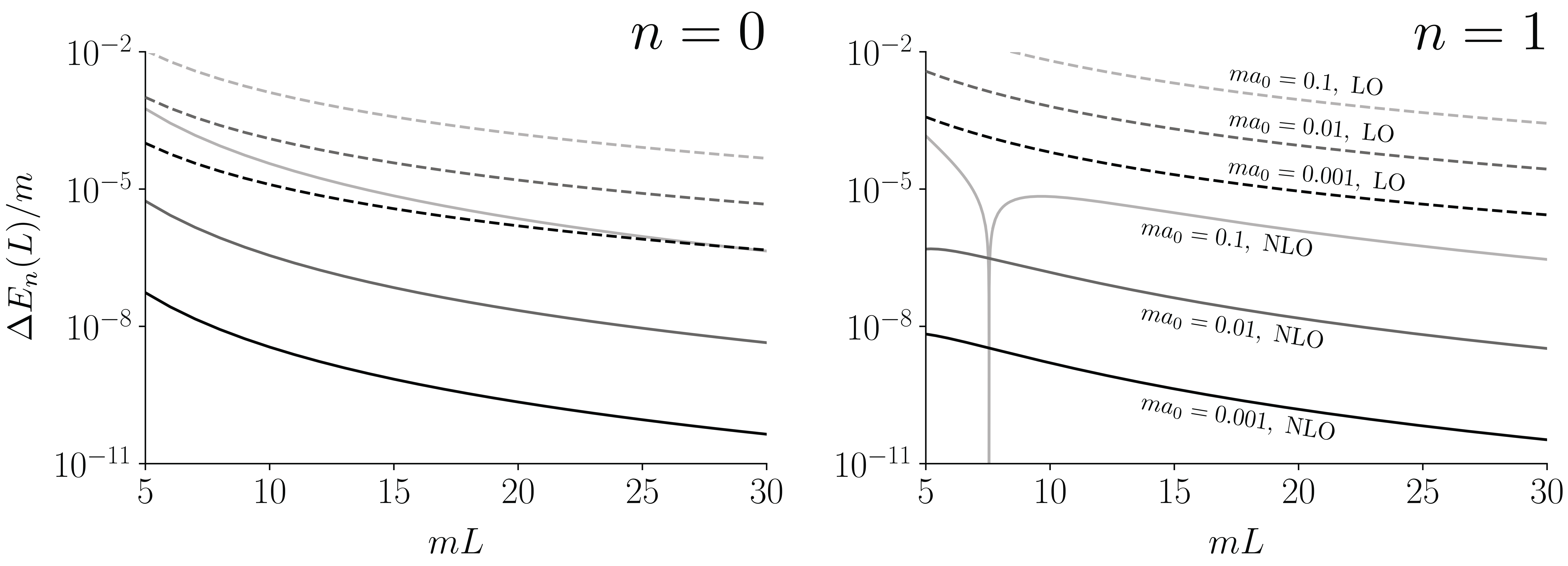
Sanity Check: Does the analytic result converge to the numerical solution?



CoM

- Check expansion validity by subtracting LO and NLO terms from exact numerical solution
- At large mL, error reduction scales with a_0^2, as expected
- Similar test for moving frames have been done

Two-Particle Result, Higher Order

Idea: Higher order corrections can be found by further expanding $F, \mathscr{M}$ and systematically solving for $\Delta^{(k)}_{q[\mathfrak{n}]}$

Complication: NNLO depends on infinite sum over $v \notin S_{\mathfrak{n}}$

- Summand for each energy level is different
- In moving frames the summand depends on mL

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Example: NNLO result in the CoM frame*

$$E_{\mathfrak{n}}(L) = E_{\mathfrak{n}}^{(0)}(L) + g_{\mathfrak{n}} \frac{E_{\mathfrak{n}}^{(0)}(L)}{4\omega_{\nu_{\mathfrak{n}}}\omega_{d-\nu_{\mathfrak{n}}}} \frac{8\pi a_0}{L^3} + \epsilon^2 g_{\mathfrak{n}} \frac{8a_0^2}{E_{\mathfrak{n}}^{(0)}(L) L^4} \left(B_{\mathfrak{n},0} - \frac{4\pi^2 g_{\mathfrak{n}}}{E_{\mathfrak{n}}^{(0)}(L)^2 L^2} \right) + \mathcal{O}(\epsilon^3)$$

$$B_{\mathfrak{n},0} = \lim_{s \to -1} \sum_{v \notin S_{\mathfrak{n}}} \left[q_{\mathfrak{n}}^{(0)2} - v^2 \right]^s \qquad q_{\mathfrak{n}}^{(0)} \in \{0, \mathbb{Z}^+\}$$

* have general NNLO result, just much less compact

Three-Particle Expansion

Idea: Apply same systematic approach to three-particle quantization condition

$$\det[1 + F_3 \mathcal{K}_{df,3}] = 0$$

Complication: Both F_3 and $\mathcal{K}_{df,3}$ have a significantly more complicated structure

$$F_3 \equiv \frac{1}{L^3}\frac{1}{2\omega}\left(\frac{F}{3} - F\frac{1}{\mathcal{K}_2^{-1} + F + G}F\right)$$

F, G: similar to two-particle F matrix

$\mathcal{K}_2$: two-particle K matrix

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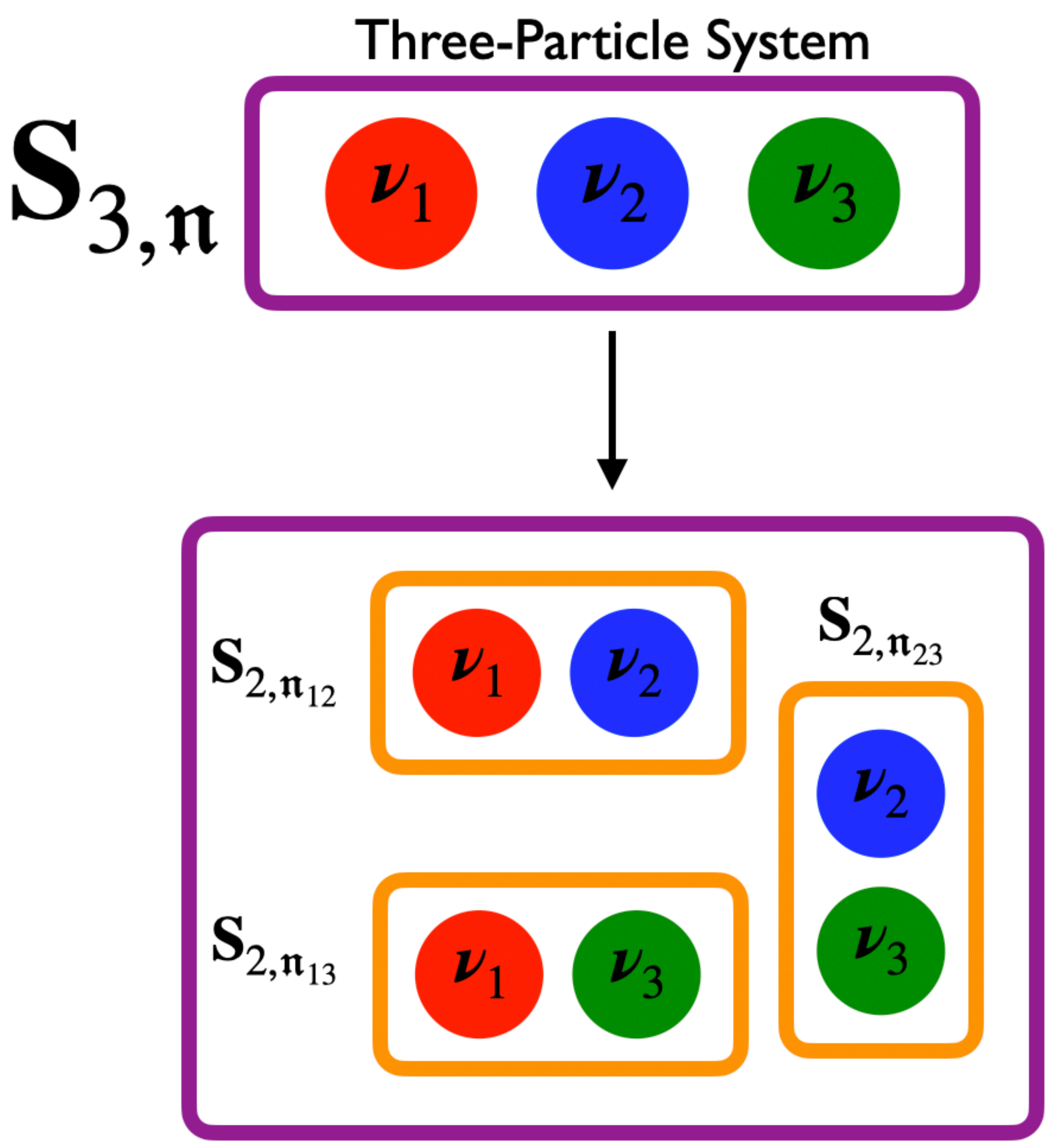
- Similar to two-particle case, F and G have poles that can be used to define $S_{3,\mathfrak{n}}$

$$S_{3,\mathfrak{n}} = \left\{ v_1 \in \mathbb{Z}^3 \,\middle|\, E^{(0)}_{3,\mathfrak{n}} - \left(\omega_{v_1} + \omega_{v_2} + \omega_{d-v_1-v_2}\right) = 0 \quad \forall\ v_2 \in \mathbb{Z}^3 \right\}$$

Way Forward: NLO energy is given by solving for the pole in F_3

Three-Particle NLO Energy

Idea: Set $S_{3,\mathfrak{n}}$ decomposes into at most three two-particle sets $S_{2,\mathfrak{n}_i}$



Collection of Two-Particle Subsystems

(non-degenerate states)

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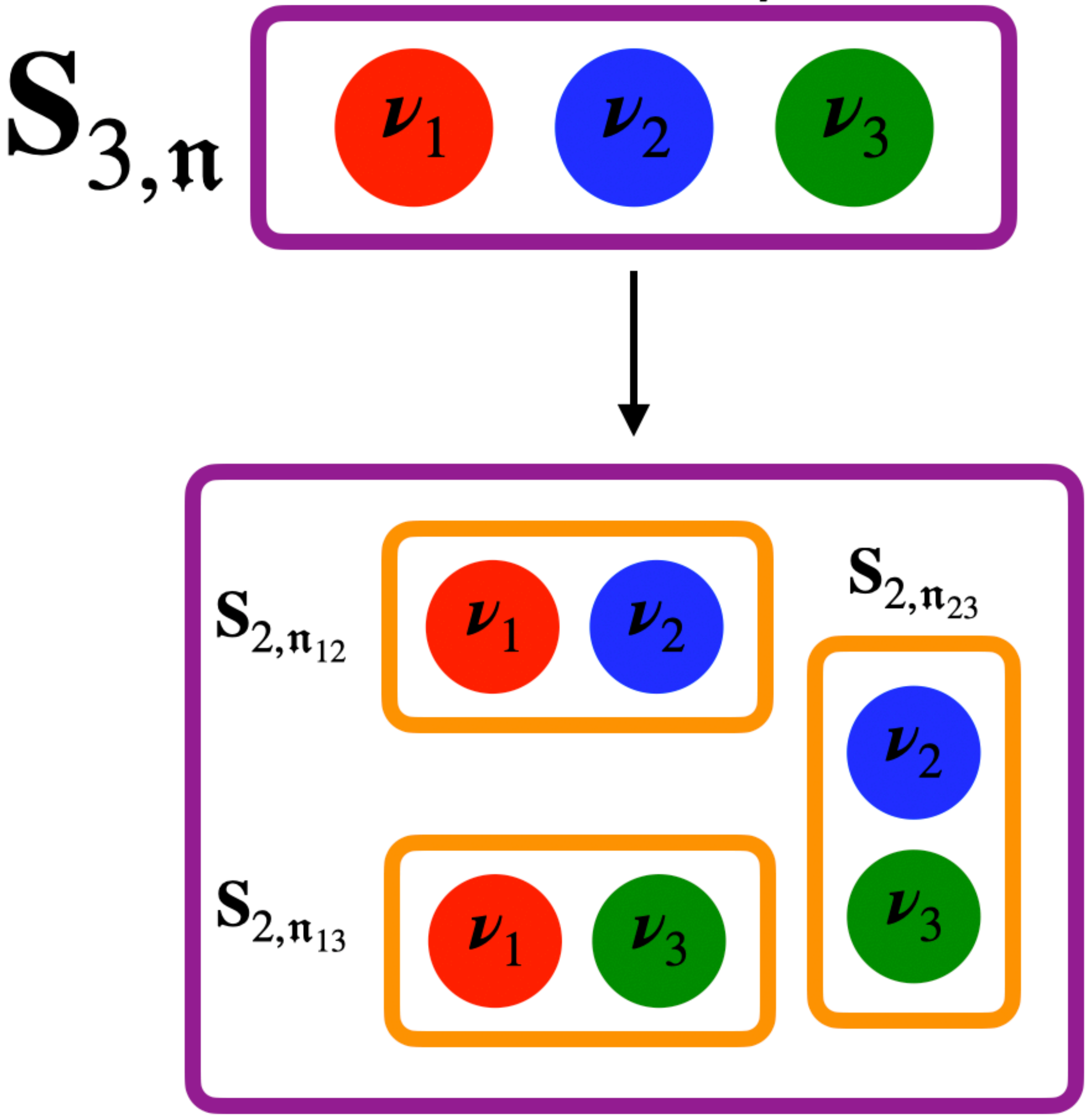
Three-Particle System



Collection of Two-Particle Subsystems

(non-degenerate states)

Idea: Texture of $\mathscr{K}_2, F$ and G allow for the derivation of NLO energy

- $\mathscr{K}_2$ and F are always diagonal
- G has only five classes of textures

$$\widetilde{G} \sim \left(\begin{array}{c|c|c} \lozenge[n_a] & \blacklozenge[g_{\mathfrak{n}a}/2] & \blacklozenge[g_{\mathfrak{n}a}/2] \\ \hline \blacklozenge[g_{\mathfrak{n}b}/2] & \lozenge[n_b] & \blacklozenge[g_{\mathfrak{n}b}/2] \\ \hline \blacklozenge[g_{\mathfrak{n}c}/2] & \blacklozenge[g_{\mathfrak{n}c}/2] & \lozenge[n_c] \end{array}\right)$$

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$$\widetilde{G} \sim \blacklozenge[g_{2,\mathfrak{n};a}] \qquad \widetilde{G} \sim \blacklozenge[1]$$

$\lozenge[n]$: square matrix of all zeros, with dimensions $n \times n$

$\blacklozenge[n]$: rectangular matrix of zeros and ones, with n non-zero entries per row

Three-Particle NLO Result

NLO Result: General result for any s-wave dominated non-degenerate state

$$E_{3,\mathfrak{n}} = E^{(0)}_{3,\mathfrak{n}} + \epsilon \sum_{i=1}^{3} \Delta^{(1)}_{E[2,\mathfrak{n}_i]} + \mathcal{O}(\epsilon^2) \qquad \Delta^{(1)}_{E[2,\mathfrak{n}]} \equiv g_{\mathfrak{n}} \frac{E^{(0)}_{\mathfrak{n}}(L)}{4\omega_{\nu_{\mathfrak{n}}}\omega_{d-\nu_{\mathfrak{n}}}} \frac{8\pi a_0}{\gamma^{(0)}_{\mathfrak{n}} L^3}$$

, ...

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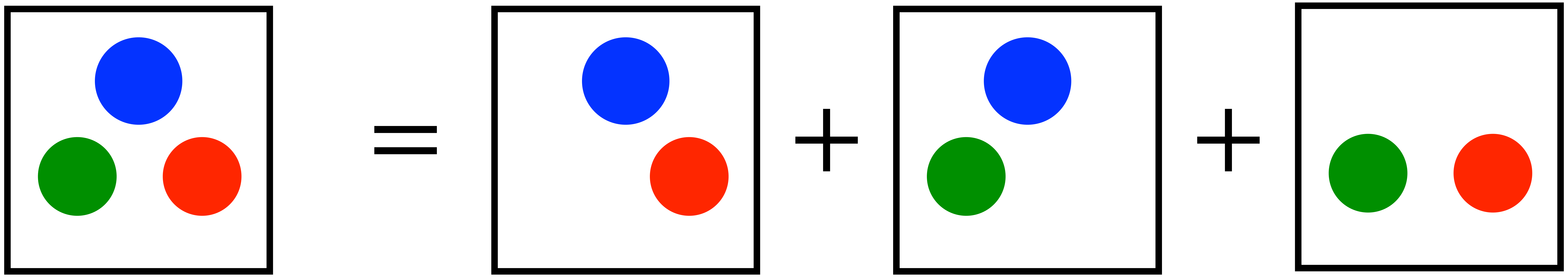
Interpretation: LO energy shift is the sum of all possible two-particle LO energy shifts



$$\Delta E_{LO}(p_1, p_2, p_3) = \Delta E_{LO}(p_1, p_2) + \Delta E_{LO}(p_1, p_3) + \Delta E_{LO}(p_2, p_3)$$

This result has been derived for all non-degenerate s-wave dominated energy levels

Previous work by Huang, Yang, Beane, Detmold, Savage, Hansen, Sharpe, Pang, Wu, Hammer, Meißner, Rusetsky, Romero-López, Schlage, Urbach, ...

Conclusion

- Developed systematic method for expanding two- and three-particle energies for any non-degenerate state in any frame
  - Can also handle degenerate states, though more complicated

$$E_{2,\mathfrak{n}}(L) = E^{(0)}_{2,\mathfrak{n}}(L) + \epsilon\, g_{\mathfrak{n}} \frac{E^{(0)}_{2,\mathfrak{n}}(L)}{4\omega_{\nu_\mathfrak{n}}\omega_{d-\nu_\mathfrak{n}}} \frac{8\pi a_0}{\gamma^{(0)}_{\mathfrak{n}} L^3} + \mathcal{O}(\epsilon^2) \qquad E_{3,\mathfrak{n}} = E^{(0)}_{3,\mathfrak{n}} + \sum_{i=1}^{3} g_{\mathfrak{n}_i} \frac{E^{(0)}_{2,\mathfrak{n}_i}(L)}{4\omega_{\nu_{\mathfrak{n}_i}}\omega_{d_i-\nu_{\mathfrak{n}_i}}} \frac{8\pi a_0}{\gamma^{(0)}_{\mathfrak{n}_i} L^3} + \mathcal{O}(\epsilon^2)$$

- Useful for
  - Guided root finding
  - Testing three-particle quantization condition
  - Testing convergence when including higher partial waves
  - Build intuition for N particles