Higher partial wave contamination in the finite-volume formulae for 1-to-2 transitions

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Work completed with Max T Hansen and manuscript should appear shortly

Lellouch and Lüscher, 2000, hep-lat/0003023

The LL factor

- In the early 2000s LL worked out how to get K->ππ from the finite volume:

$$\underbrace{|\langle E_{\pi\pi},\pi\pi,\mathrm{out}|\mathcal{H}(0)|K\rangle|^2}_{\text{Infinite volume}}\Big|_{E^{\mathrm{cm}}_{\pi\pi}=E^{\mathrm{cm}}_{n}(L)} = 8\pi\left[\overbrace{q\frac{\partial\phi}{\partial q}}^{\text{Volume Effects}}+\overbrace{k\frac{\partial\delta_0}{\partial k}}^{\text{Dynamics}}\right]\left(\frac{m_k}{k_\pi}\right)^3\underbrace{|\langle E_n,L|\mathcal{H}(0)|K,L\rangle|^2}_{\text{Finite volume}}$$

$$k = q\frac{2\pi}{L}$$

- However
  - Assumes $\delta_l = 0$ for l>0
  - Only works for the first 8 energy levels

1+J->2 formula for l>0

Conversion Factor

$$\left|\langle E_{\pi\pi}, \pi\pi, \mathrm{out}|\mathcal{H}(0)|K\rangle\right|^2\Big|_{E^{\mathrm{cm}}_{\pi\pi}=E^{\mathrm{cm}}_n(L)} = 2m_K L^6\, \mathcal{C}(E_n(L), L)\, \left|\langle E_n, \boldsymbol{P}, L, A_1|\mathcal{H}(0)|K, \boldsymbol{P}, L\rangle\right|^2$$

Infinite volume

Finite volume

- Evaluated at one of the finite volume energy levels
- Tune to on-shell kaon energy
- We will assume **P**=0, periodic boundary conditions and K->ππ

We will focus on the Conversion Factor

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Conversion Factor

We will factor this out

$$\mathcal{C}(E_n(L), \boldsymbol{P}, L) \equiv \frac{\boxed{\cos^2 \delta_0(E_{\text{cm}})}}{\text{adj}\left[M(E, \boldsymbol{P}, L)\right]_{00}} \frac{\partial \det\left[M(E, \boldsymbol{P}, L)\right]}{\partial E}\bigg|_{E=E_n(L)}$$

2 particle energy levels given by Det(M)=0

Where

Dynamics

Volume Effects

$$M(E, \boldsymbol{P}, L) \equiv \widetilde{\mathcal{K}}(E_{\text{cm}}) + \widetilde{F}(E, \boldsymbol{P}, L)^{-1}$$

Infinite dimensional matrix –
this is what we must truncate

l=0

$$M(E, \boldsymbol{P}, L) \equiv \mathcal{K}^{(0)}(E_{\text{cm}}) + \overline{F}_{00}(E, \boldsymbol{P}, L)^{-1}$$

$$\mathcal{K}^{(0)}(E_{\text{cm}}) = \frac{16\pi E_{\text{cm}}}{p_{\text{cm}}} \tan \delta_0(E_{\text{cm}})$$

$$\overline{F}_{00}(E, \boldsymbol{P}, L)^{-1} \equiv \frac{16\pi E_{\text{cm}}}{p_{\text{cm}}} \tan \phi(E, \boldsymbol{P}, L)$$

$$\mathcal{C}^{[\ell_{\text{max}}=0]}(E_n(L), L) = \frac{4\pi E^2}{p^3} \left[p \frac{\partial}{\partial p} \delta_0(p) + q \frac{\partial}{\partial q} \phi(q) \right] \Bigg|_{E=E_n(L)}$$

- You might think this is all we need but, due to the finite volume, angular momentum is not conserved
- The kaon+Weak Hamiltonian combo is in the A_1^+ irrep of the octahedral group (this is the good quantum number)

l=4

- A_1^+ couples to l=0, l=4 (l=6,l=8…)

$$M(E,L) \equiv \begin{pmatrix} \mathcal{K}^{(0)}(E) & 0 \\ 0 & \mathcal{K}^{(4)}(E)/p_{\text{cm}}^8 \end{pmatrix} + \begin{pmatrix} \overline{F}_{00}(E,L) & \overline{F}_{04}(E,L) \\ \overline{F}_{40}(E,L) & \overline{F}_{44}(E,L) \end{pmatrix}^{-1}$$

$$\Delta(E_n(L),L) \equiv \frac{\mathcal{C}(E_n(L),L)-\mathcal{C}^{[\ell_{\text{max}}=0]}(E_n(L),L)}{\mathcal{C}^{[\ell_{\text{max}}=0]}(E_n(L),L)}$$

Δ is the relative size of the l=4 contamination

l=4 (cont.)

$$\Delta(E,L) = \underbrace{\left[(2m)^9 \frac{\partial}{\partial E} \frac{\mathcal{K}^{(4)}(E)}{p^8}\right]}_{\text{Dynamics}} \overbrace{\Delta^{[\partial \mathcal{K}(4)]}(E,L)}^{\text{Finite volume effects}} + \underbrace{\left[(2m)^8 \frac{\mathcal{K}^{(4)}(E)}{p^8}\right]}_{\text{Dynamics}} \overbrace{\Delta^{[\mathcal{K}(4)]}(E,L)}^{\text{Finite volume effects}} + \mathcal{O}\left[(\mathcal{K}^{(4)})^2, \mathcal{K}^{(4)}\, \partial \mathcal{K}^{(0)}\right]$$

$$\Delta^{[\mathcal{K}(4)]}(E,L) \equiv \frac{1}{(2m)^8} \frac{1}{\partial_E \overline{F}_{00}(E,L)^{-1}} \frac{\overline{F}_{04}(E,L)^2}{\overline{F}_{00}(E,L)^2} \left[2 \frac{\partial_E \overline{F}_{04}(E,L)}{\overline{F}_{04}(E,L)} - 2 \frac{\partial_E \overline{F}_{00}(E,L)}{\overline{F}_{00}(E,L)} + \frac{m^2}{Ep^2}\right]$$

$$\Delta^{[\partial \mathcal{K}(4)]}(E,L) \equiv \frac{1}{(2m)^9} \frac{1}{\partial_E \overline{F}_{00}(E,L)^{-1}} \frac{\overline{F}_{04}(E,L)^2}{\overline{F}_{00}(E,L)^2}$$

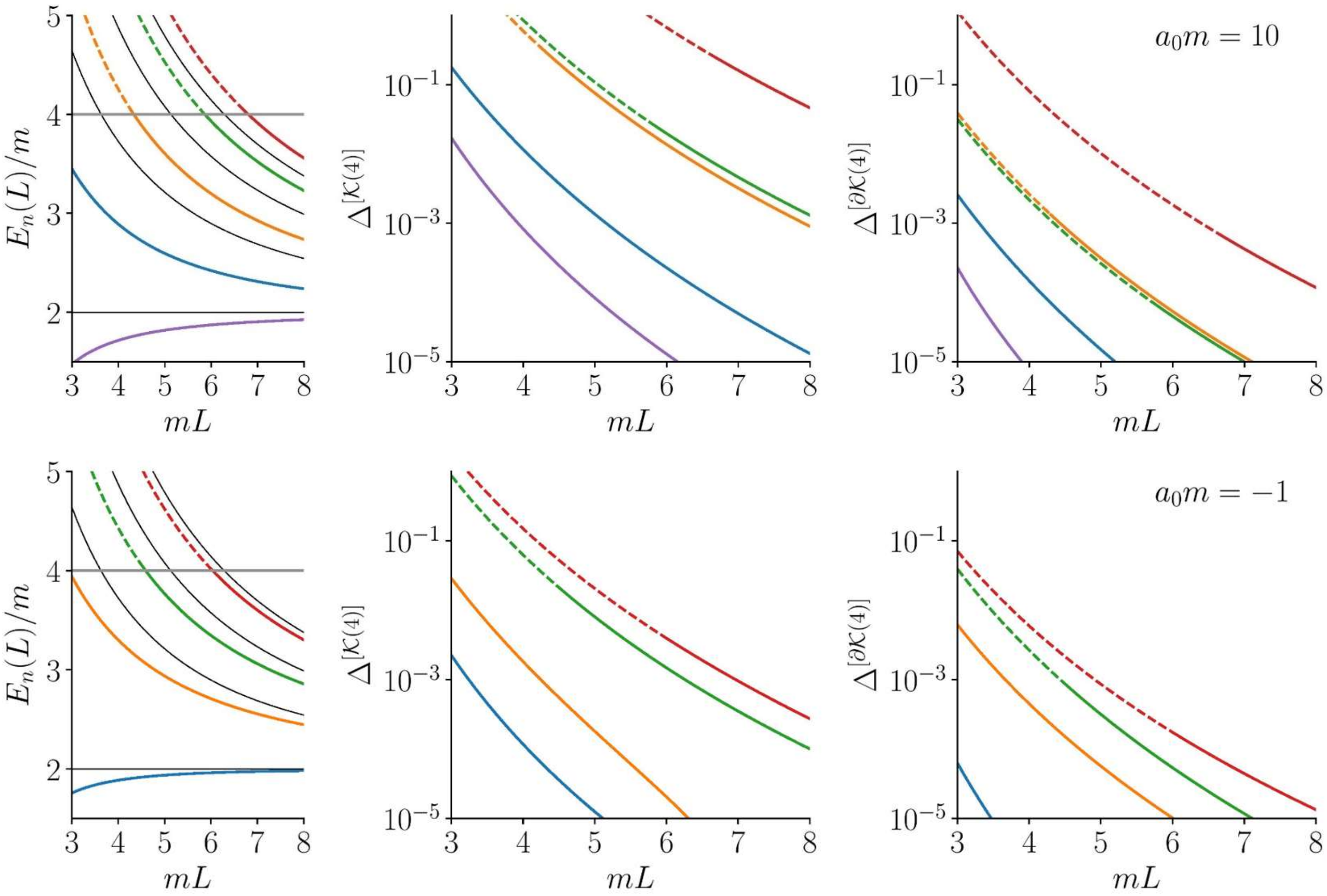
This is what we will plot

Plotting the Δs

- The only inputs we need are the energy level, physical mass and box size
- At leading order we can extract energy levels from $Det(M_{l=0}) = 0$
- We need some dynamical input – We will parameterize $K^{(0)}$ using the scattering length

$$\Delta(E,L) = \left[(2m)^9 \frac{\partial}{\partial E}\frac{\mathcal{K}^{(4)}(E)}{p^8}\right]\Delta^{[\partial\mathcal{K}(4)]}(E,L) + \left[(2m)^8 \frac{\mathcal{K}^{(4)}(E)}{p^8}\right]\Delta^{[\mathcal{K}(4)]}(E,L)$$
$$+\mathcal{O}\left[(\mathcal{K}^{(4)})^2, \mathcal{K}^{(4)}\,\partial\mathcal{K}^{(0)}\right]$$

Results – relative shift from l=0 to l=4



Interlude

- The finite volume causes contamination from higher angular momentum states
- Sometimes the contamination is almost 10% (if $(2\mathrm{m})^8 \frac{K^{(4)}}{p^8} \sim 1$), but sometimes much less
- We haven't found any patterns to predict the size of the effect a-priori

What happens at the 9^{th} energy level?

- Free theory, energy is given by $E = 2\sqrt{n^2\left(\frac{2\pi}{L}\right)^2 + m^2}$

- n^2=9 has 2 inequivalent solutions: (0,0,3) and (2,2,1). (There is no n^2=7)

- At l=0 these give the same energy (even when we include dynamics)

- At l=4 this degeneracy is broken

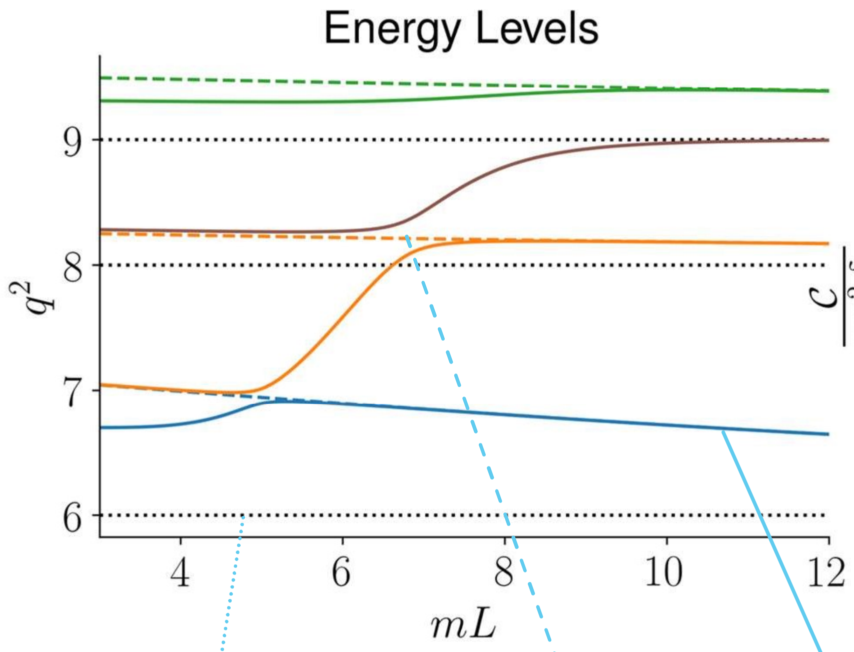
Plotting the accidental degeneracy

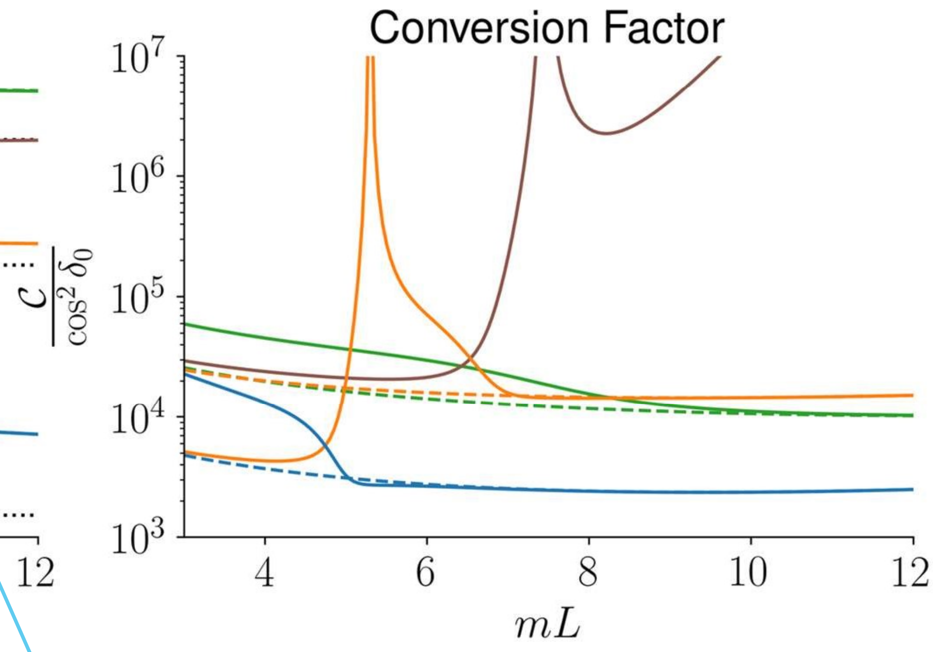
- We extract energy levels from $Det(M_{l=4}) = 0$

- Now we also need a_0, a_4 $\frac{\mathcal{K}^{(\ell)}}{p^{2\ell}} = -16\pi m E a_\ell$

- We plot the full l=4 conversion factor, that is we don't Taylor expand it in any way

Results





Free energy levels

l=0 energy levels

l=4 energy levels

a0= 1
a4=−0.0001

Alternative Form

Any local operator

$$\sqrt{2m_k L^6}\langle n; L|\mathcal{O}|K; L\rangle =$$

$$\mathcal{A}_0\langle \pi\pi; \ell = 0; \infty|\mathcal{O}|K; \infty\rangle + \mathcal{A}_4\underbrace{\langle \pi\pi; \ell = 4; \infty|\mathcal{O}|K; \infty\rangle}_{= 0 \text{ if } \mathcal{O} = \mathcal{H}_W} \ldots$$

If $\mathcal{O} = \mathcal{H}_W$ then no angular momentum inserted

In this case $\mathcal{A}_0 = \frac{1}{\sqrt{\mathcal{C}}}$ and we recover previous result

Everything evaluated at a finite volume energy level

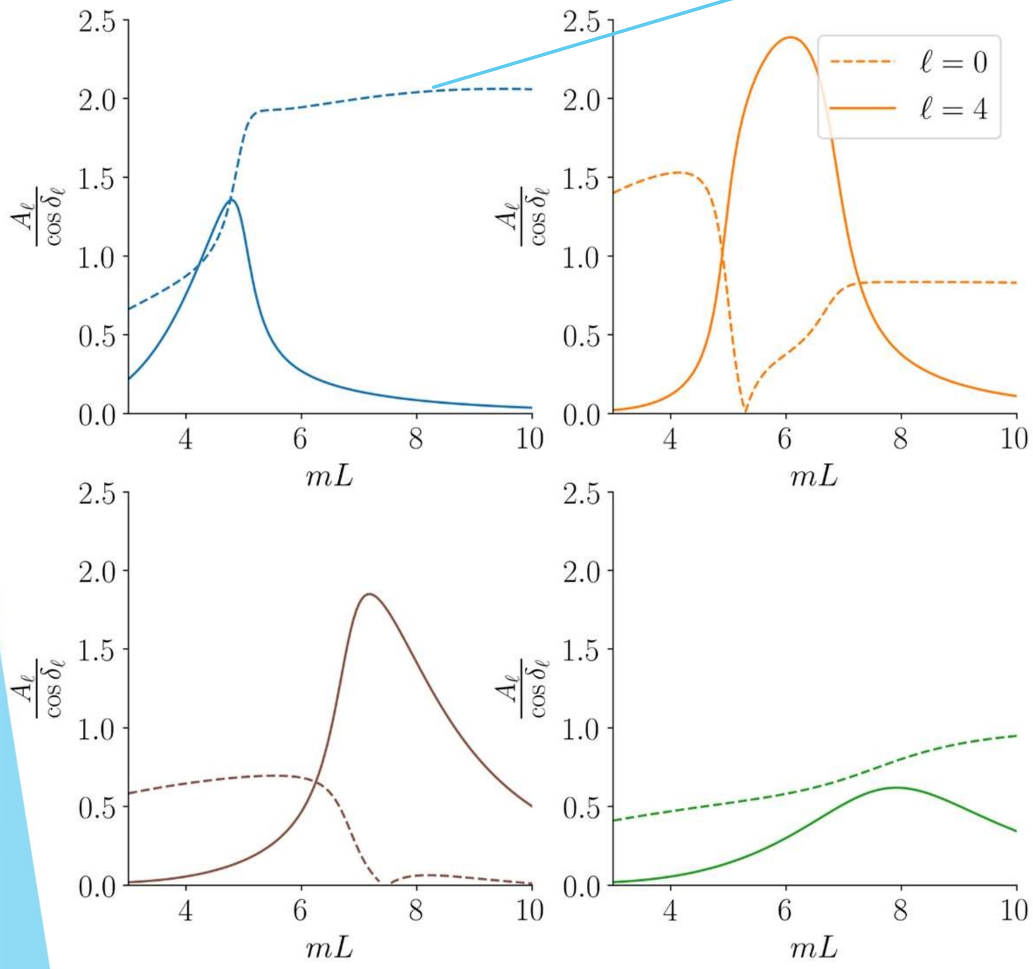
Alternative Form Cont…

$$\sqrt{2m_k L^6}\langle n; L| \text{“}{=}\text{”} \mathcal{A}_0\langle \pi\pi; \ell = 0; \infty| + \mathcal{A}_4\langle \pi\pi; \ell = 4; \infty|$$

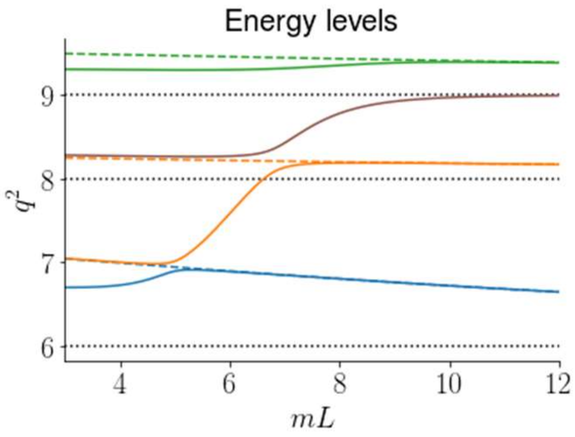
Only true if applied to a local operator and single particle state

Results

Note, scaled up x100 for clarity



$$\sqrt{2m_k L^6}\langle n; L| = \mathcal{A}_0 \langle \pi\pi; \ell = 0; \infty| + \mathcal{A}_4 \langle \pi\pi; \ell = 4; \infty|$$

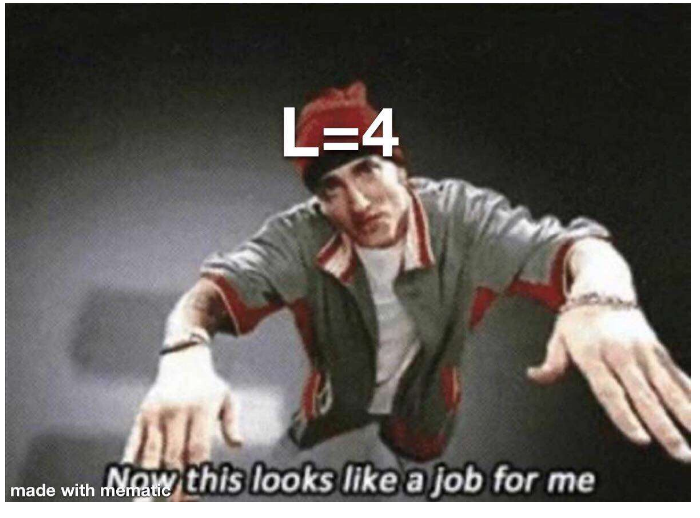


a0= 1
a4=−0.0001

Interlude 2

- Yes, l=4 splits the accidentally degenerate energy level
- When the energy levels line up with the l=0 levels so do the conversion factors
- When the energy levels deviate, the conversion factor grows

An accidentally degenerate 9th energy level:
exists

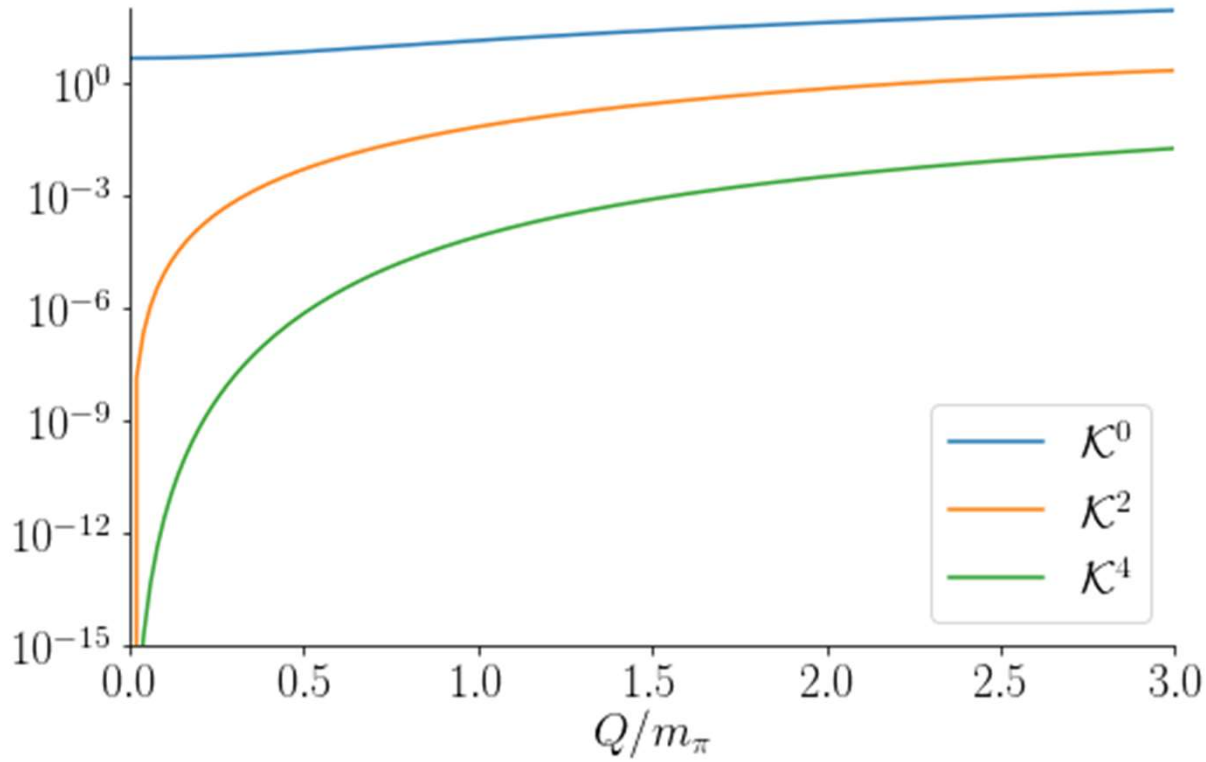


Conclusion

- Yes, higher angular momentum states can contaminate s-wave scattering
- Yes, it breaks the accidental degeneracy
- Sometimes these effects are small
- Sometimes these effects are noticeable
- Hard to predict in advance how big the effect will be

Thank You

Justifying the truncation



Size of the dynamics

- a4=0.0001
- E~3 from plot
- $2^8 * 16\pi * 3 * 10^{-4} \sim 4$