Finite volume effects and meson scattering in the 2-flavour Schwinger model

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Introduction and motivation Schwinger model

Exploring 2-flavour Schwinger model in canonical formulation

- Similarities with QCD ⇒ Toy model
- Physics at fixed number of particles
- Ground-state energies of multi-meson states
- Physics at non-zero quark chemical potential μ_q.
  Dimensional reduction ⇒ Factorize out μ_q-dependence from determinant of Wilson-Dirac matrix

$$\det M[U;\mu_q] = \sum_{n_q=-L_x}^{L_x} \det_{n_q} M[U] e^{\mu_q L_t n_q}$$

- New sampling methods? (Sign problem at finite density μ_q)

Grand canonical gauge theories

- 1-flavour Schwinger model with chemical potential μ_q

$$Z_{\mathrm{GC}}(\mu) = \int \mathcal{D}U\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi\; e^{-S_g[U]-\overline{\psi}M[U;\mu_q]\psi}$$

- Integrate out fermionic (quark) degrees of freedom

$$\int \mathrm{d}\psi_{i_j} = 0, \qquad \int \psi_{i_j}\mathrm{d}\psi_{i_j} = 1$$

- General result for N_f mass-degenerate flavours

$$Z_{\mathrm{GC}}(\mu) = \int \mathcal{D}U\, e^{-S_g[U]}(\det \overbrace{M[U;\mu_q]}^{\text{Wilson-Dirac operator}})^{N_f},$$

Dimensional reduction of $N_f = 1$ Schwinger model

- After performing the dimensional reduction on determinant of Wilson-Dirac matrix

$$\underbrace{\det M[U;\mu_q]}_{\substack{\text{Determinant of}\\ \text{matrix of size } (2L_x L_t)^2}} = \sum_{n_q=-L_x}^{L_x} \underbrace{\det_{n_q} M[U]}_{\substack{\text{Projection on}\\ \text{canonical sector } n_q}} e^{\mu_q L_t n_q}$$

- Canonical determinant $\det_{n_q} M[U]$

$$\det_{n_q} M[U] \propto \sum_{A,|A|=k+L_x} \det(\mathcal{T}^{AA})$$

where $\mathcal{T} = \prod_{i=1}^{L_t} \mathcal{T}_i$ is a product of transfermatrices, size $(2L_x)^2$

- Canonical determinant $\det_{n_q} M[U]$ → net-quark number

$N_f = 2$ Schwinger model in $d = 2$

- Physics in the 2-flavour model is more interesting
  - denote the quark flavours by u and d
  - isospin chemical potential generates multi-meson states
- Number of u- and d-quarks restricted:

$$\text{charge } Q = n_u + n_d = 0 \quad \Leftrightarrow \quad \text{Gauss' law}$$
$$\text{isospin } I = (n_u - n_d)/2 \quad \text{arbitrary}$$

- Corresponding canonical partition functions (with $n_u = -n_d$):

$$Z_{n_u,n_d} = \int \mathcal{D}U \, e^{-S_g[U]} \det{}_{n_u} M_u[U] \det{}_{n_d} M_d[U]$$

- Vacuum sector is described by $Z_{0,0}$

Calculating the pion energy

- The flavour-triplet meson (pion) $|\pi\rangle = |\overline{\psi}\gamma_5\tau^a\psi\rangle$ has quantum numbers

$$\begin{aligned} Q &= 0 \quad \text{total quark number} \\ I &= 1 \quad \text{isospin} \end{aligned}$$

  and is mass degenerate $m_\pi = m_{\pi^+} = m_{\pi^-} = m_{\pi^0}$
- State with maximal $I_z = 1$: $|\pi^+\rangle = |u\bar{d}\rangle$
  is groundstate of system with $n_u = +1, n_d = -1$:

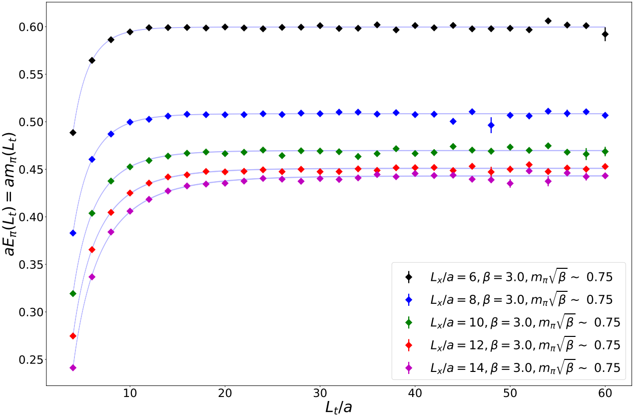
$$Z_{+1,-1} = \int \mathcal{D}U\, e^{-S_g[U]} \det{}_{+1} M_u[U] \det{}_{-1} M_d[U]$$

- The free energy difference to the vacuum at $T \to 0$ defines the pion mass:

$$E_\pi(L) = m_\pi(L) = -\lim_{L_t\to\infty} \frac{1}{L_t} \log \frac{Z_{+1,-1}(L_t)}{Z_{0,0}(L_t)}$$

Computation $m_\pi(L)$ different volumes, $T \to 0$

$$E_\pi(L) = m_\pi(L) = -\lim_{L_t\to\infty} \frac{1}{L_t} \log \frac{Z_{+1,-1}(L_t)}{Z_{0,0}(L_t)}, \qquad T = \frac{1}{L_t} \to 0 \Leftrightarrow \frac{L_t}{a} \to \infty$$



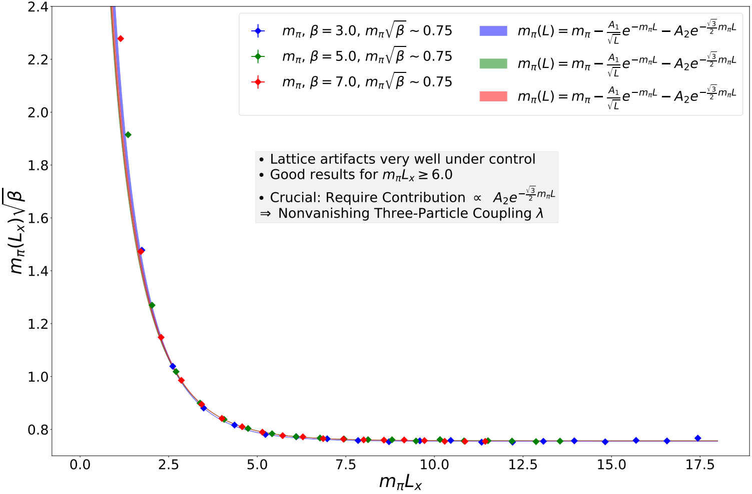
Finite volume effects (1 pion in the box)

- Use canonical formulation to compute $m_\pi(L)$
- Finite volume effects described by [Lüscher, 1984]

$$m_\pi(L) = m_\pi - \frac{1}{\sqrt{m_\pi L}} \underbrace{\left(\frac{F(0)}{\sqrt{2\pi} 4 m_\pi}\right)}_{A_1} e^{-m_\pi L} - \underbrace{\left(\frac{\lambda^2}{4\sqrt{3} m_\pi^3}\right)}_{A_2} e^{-\frac{\sqrt{3}}{2} m_\pi L}$$

- Infinite volume Pion-mass m_π, effective three-pion coupling λ, $F(0)$ forward scattering amplitude

Pion mass as a function of the volume $m_\pi\sqrt{\beta}$ fixed



Arbitrary n-pion ground-states

- **Before:** 1 **Pion in the box** $\rightarrow$ n **Pions in the box**
- Groundstate energy of the system with $n_u = +n, n_d = -n$:

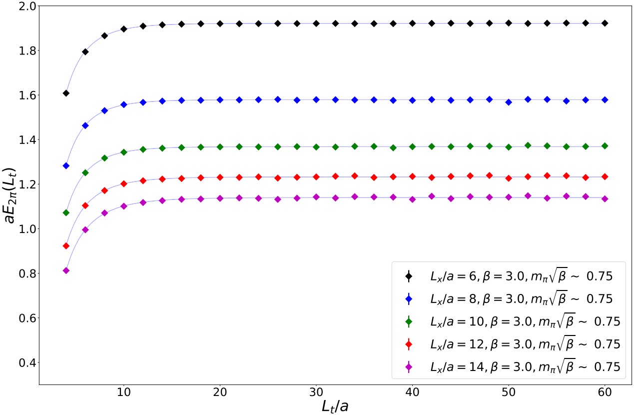
$$Z_{+n,-n} = \int \mathcal{D}U\, e^{-S_g[U]} \det{}_{+n} M_u[U] \det{}_{-n} M_d[U]$$

- The free energy difference to the vacuum at $T \rightarrow 0$ defines the energy of the n-pion system:

$$E_{n\pi}(L) = -\lim_{L_t \rightarrow \infty} \frac{1}{L_t} \log \frac{Z_{+n,-n}(L_t)}{Z_{0,0}(L_t)}$$

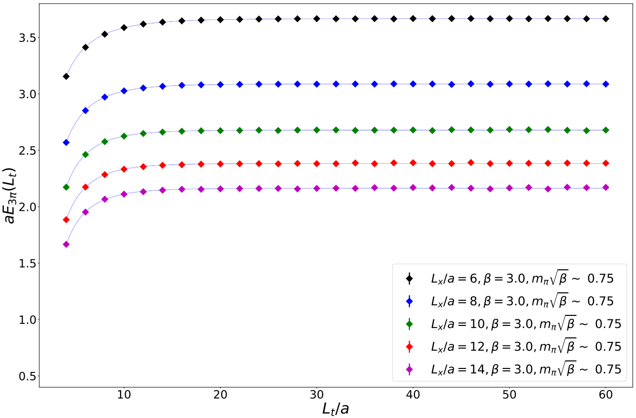
Computation $E_{2\pi}(L)$ diff. volumes, $T \to 0$

$$E_{2\pi}(L) = -\lim_{L_t\to\infty} \frac{1}{L_t} \log \frac{Z_{+2,-2}(L_t)}{Z_{0,0}(L_t)}$$



Computation $E_{3\pi}(L)$ diff. volumes, $T \to 0$

$$E_{3\pi}(L) = -\lim_{L_t \to \infty} \frac{1}{L_t} \log \frac{Z_{+3,-3}(L_t)}{Z_{0,0}(L_t)}$$



Two-particle scattering and scattering phase shifts

- Bosonic dispersion relation **on lattice** ⇒ 2-Pion energy

$$E_{2\pi}(L) = 2\text{arccosh}(\cosh(m_\pi) + 1 - \cos(k(L))), \qquad (1)$$

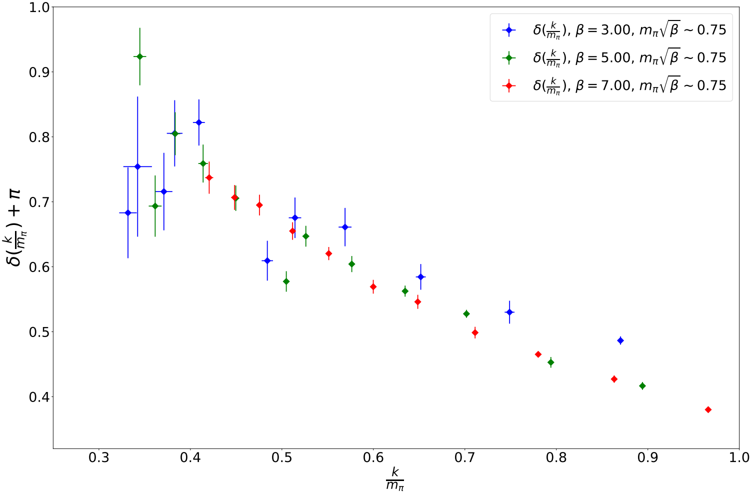
with $k = \frac{p_1 - p_2}{2}$ relative momentum (center of mass)

- Two pions in box → relative wavefunction requires correction at the boundaries → scattering-phase shift δ [Lüscher, 1986]
- Quantization condition for the relative momentum k dependent on the scattering-phase shift δ

$$\delta(k(L)) = -\frac{k(L)L}{2} \equiv \delta(L)$$

- Strategy: Compute $\delta(L)$ by using $k(L)$ given by equation (1)

Scattering phase shifts $\delta(k/m_\pi)$



Three-particle scattering and 3-pion energy 1/3

- 3 pion ground state energy **on the lattice**

$$E_{3\pi}(L) = \sum_{i=1}^{3} \text{arccosh}(\cosh(m_\pi) + 1 - \cos(p_i(L))),$$

with $\sum_{i=1,2,3} p_i = 0$ in center of mass system

- For three-particle scattering assume pairwise scattering → quantization conditions related to the scattering-phase shift δ

- Step 1: Fit $\delta(k)$ for the Schwinger Model using an Ansatz

Three-particle scattering and 3-pion energy 2/3

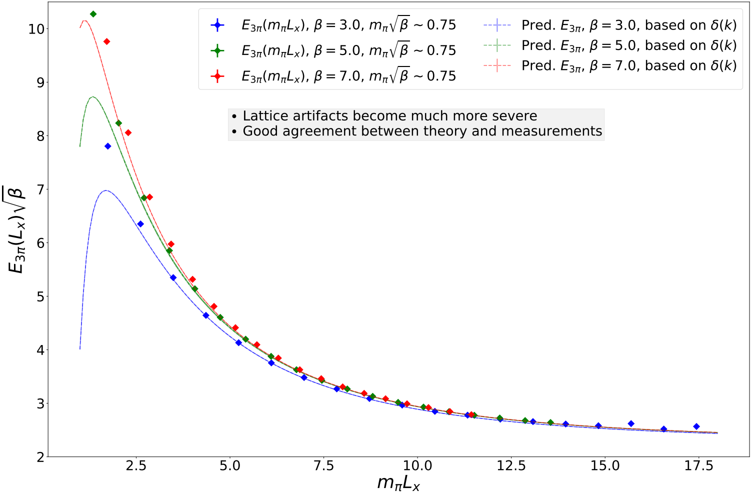
- Step 2: In the center of mass system ($P = p_1 + p_2 + p_3 = 0$), quantization conditions read [Guo, Morris, 2018]

$$\cot(\delta(-q_{31}) + \delta(q_{12})) + \cot(\frac{p_1 L}{2}) = 0$$
$$\cot(\delta(-q_{23}) + \delta(q_{12})) - \cot(\frac{p_2 L}{2}) = 0,$$

with $q_{ij} = \frac{p_i - p_j}{2}$

- Step 3: Compute p_1, p_2, make predictions for $E_{3\pi}(m_\pi, p_1, p_2, p_3, P)$
- Step 4: Compare results to directly measured $E_{3\pi}(L_x)$

Three-particle scattering and 3-Pion energy 3/3



Summary and outlook

- **Predicted** 3**-pion ground state energies** $E_{3\pi}(L_x)$ **based on finite-volume corrections of the** 2**-pion ground state energy** $E_{2\pi}(L_x)$ **and** $m(L_x)$**, via the scattering phase shift** $\delta(\frac{k}{m_\pi})$

Canonical formalism is generally applicable: Outlook

- Can also be applied to QCD [Wenger, Alexandru, 2010]
- Phase structure of Schwinger model
- Transfermatrix $\mathcal{T}$ shows very interesting features by itself (⇒ current topic of investigation)

Thank you very much for your attention

BU-Slide: Dimensional reduction Nf=1 Schwinger Model

- Consider the Wilson Dirac matrix for a single quark with chemical potential μ (Lattice of size $L_x \times L_t$):

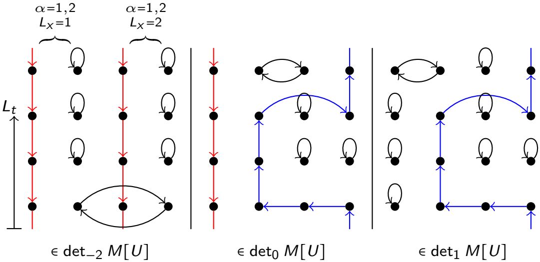
$$M[U;\mu] = \begin{pmatrix} B_1 & -2\kappa P_+ A_1^+ & & & 2\kappa P_- A_{L_t}^- \\ -2\kappa P_- A_1^- & B_2 & -2\kappa P_+ A_2^+ & & \\ & -2\kappa P_- A_2^- & B_3 & \ddots & \\ & & \ddots & \ddots & \\ & & & B_{L_t-1} & -2\kappa P_+ A_{L_t-1}^+ \\ 2\kappa P_+ A_{L_t}^+ & & & -2\kappa P_- A_{L_t-1}^- & B_{L_t} \end{pmatrix}$$

  - $M[U;\mu]$ matrix of size $(2L_xL_t)^2$, building blocks of size $(2L_x)^2$
  - B_t are (spatial) Wilson Dirac operators on time-slice t,
  - Dirac projectors $P_\pm = \frac{1}{2}(\mathbb{I} \mp \sigma_z)$,
  - temporal hoppings are

$$A_t^+ = e^{+\mu} \cdot \mathbb{I}_{2\times 2} \otimes U_t = \left(A_t^-\right)^{-1}$$

BU-Slide: Canonical Determinant

Canonical determinant $\det_k M[U]$ → net-fermion number



Assume canonical Partition Function can be written as

$$Z_{(n_u,n_d)}(T) = \sum_{k=0}^{\infty} n_k^{(n_u,n_d)} e^{-E_k^{(n_u,n_d)}/T},\ n_0^{(0,0)} = 1$$

$$\begin{aligned}
F_{(n_u,n_d)} - F_{(0,0)} &= -T\log\left(\frac{Z_{(n_u,n_d)}}{Z_{(0,0)}}\right) \\
&= E_0^{(N_u,N_d)} - E_0^{(0,0)} - T\log(n_0^{(N_u,N_d)}) \\
&- T\log\left(\frac{1+\sum_{k=1}^{\infty}\frac{n_k^{(N_u,N_d)}}{n_0^{(N_u,N_d)}} e^{-(E_k^{(N_u,N_d)}-E_0^{(N_u,N_d)})/T}}{1+\sum_{k=1}^{\infty} n_k^{(0,0)} e^{-(E_k^{(0,0)}-E_0^{(0,0)})/T}}\right) \\
&\approx E_0^{(N_u,N_d)} - E_0^{(0,0)} - T\log(n_0^{(N_u,N_d)}) \\
&- T\left(\frac{n_1^{(N_u,N_d)}}{n_0^{(N_u,N_d)}} e^{-(E_1^{(N_u,N_d)}-E_0^{(N_u,N_d)})/T} - n_1^{(0,0)} e^{-(E_1^{(0,0)}-E_0^{(0,0)})/T}\right)
\end{aligned}$$

$$\lim_{T\to 0}(F_{(n_u,n_d)} - F_{(0,0)}) = E_0^{(N_u,N_d)} - E_0^{(0,0)} = E_{n\pi},\ n = N_u$$

BU-Slide: Ansatz

- Fit-ansatz given by

$$Z_{(n_u,n_d)}(T) = \sum_{k=0}^{\infty} n_k^{(n_u,n_d)} e^{-E_k^{(n_u,n_d)}/T}, n_0^{(0,0)} = 1, \text{mit } n = n_u = -n_d \text{ gilt}$$

$$E_{n\pi}(L_t) \approx E_{n\pi,\infty} - T\log(n_0^{(n,-n)}) - T\left(\frac{n_1^{(n,-n)}}{n_0^{(n,-n)}} e^{-(E_1^{(N_u,N_d)} - E_0^{(N_u,N_d)})/T}\right)$$

- $n_0^{(n,-n)}$ multiplicity for the ground-state of sector $(n,-n)$
- $n_1^{(n,-n)}$ multiplicity for the 1.st excited state of sector $(n,-n)$
- Problem: Have several potential models I could use to extract $E_{n\pi,\infty}$, however I want to have sth. modelindependent.

BU-Slide: Bayesian Fitting[Jay, Neil, 2020]

- Formulate Models $M_1, M_2, \ldots, M_{n_m}$, which differ by multiplicity $n_j^{(n,-n)}$ and starting-point of the Fit $L_{t,min}$.
- Assign a weight to each model

$$pr(M_i|D) \propto \exp(-\frac{1}{2}(\chi^2 + 2k + 2N_{cut})), \quad \text{s.t.} \quad \sum_i pr(M_i|D) = 1$$

where χ^2 is the normal $\chi^2 = \sum_{i=1}^{n_{data}} \left(\frac{y_i - f(x_i, \vec{a})}{sy_i}\right)^2$, k number of fitparameters, N_{cut} number of datapoints which have been cut-off.

- Result: Averaged value

$$\langle E_{n\pi,\infty} \rangle = \sum_{i=1}^{n_m} pr(M_i|D) \langle E_{n\pi,\infty} \rangle_i$$

BU-Slide: Spectroscopy via correlation functions

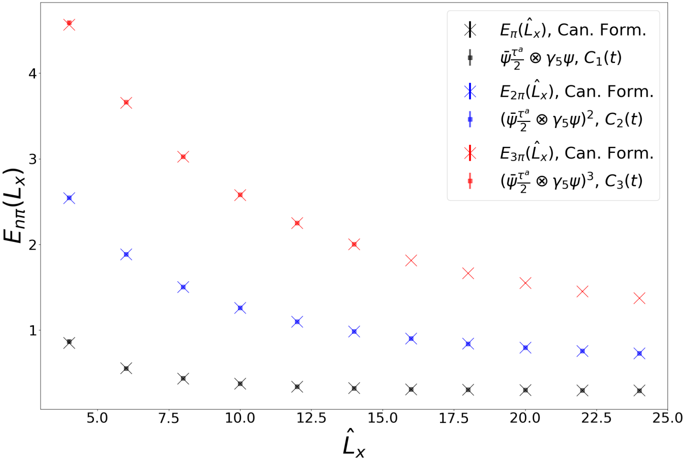
- n-pion ground state energies $E_{n\pi}(L)$ via Correlation Functions
- Define connected contribution T_j and using fermion propagator G

$$T_j = \mathrm{Tr}[\Pi^j] \quad , \text{ with } \quad \Pi = \sum_{x,y} G(x,t;y,t_0)G^{\dagger}(x,t;y,t_0)$$

- Correlation function become increasingly more difficult

$$\begin{aligned}
|\pi\rangle \rightarrow \quad & C_1(t) \propto T_1 \\
|\pi^2\rangle \rightarrow \quad & C_2(t) \propto T_1^2 - T_2 \\
|\pi^3\rangle \rightarrow \quad & C_3(t) \propto T_1^3 - 3T_1T_2 + 2T_3 \\
\vdots \\
|\pi^6\rangle \rightarrow \quad & C_6(t) \propto T_1^6 - 15T_2T_1^4 + 40T_3T_1^3 + 45T_2^2T_1^2 - 90T_4T_1^2 + 40T_3^2 \\
& \qquad - 120T_2T_3T_1 + 144T_5T_1 - 15T_2^3 + 90T_2T_4 - 120T_6
\end{aligned}$$

BU-Slide: Comparison Canonical Formulation and Spectroscopy, n-meson energies $E_{n\pi}(\hat{L}_x), \beta = 5.0, am_\pi = 0.2925(3)$



BU-Slide: Dispersion Relation

- Continuum Dispersion Relations for n-particle -states:

$$E_n(\mathbf{p}) = \sum_{\mathbf{i=1}}^{\mathbf{n}} \sqrt{\mathbf{m^2} + \mathbf{p_i}^2}$$

- Lattice Dispersion Relations for Single particle:

$$\left(2\sinh\left(\frac{E(\mathbf{p})}{2}\right)\right)^2 = \left(2\sinh\left(\frac{m}{2}\right)\right)^2 + \left(2\sin\left(\frac{\mathbf{p}}{2}\right)\right)^2$$

$$\Leftrightarrow E(\mathbf{p}) = \operatorname{arccosh}(\cosh(m) + 1 - \cos(\mathbf{p}))$$

- Lattice Dispersion Relations for n-particle states:

$$E_n(\mathbf{p}) = \sum_{\mathbf{i=1}}^{\mathbf{n}} \operatorname{arccosh}(\cosh(\mathbf{m}) + \mathbf{1} - \cos(\mathbf{p_i}))$$

- Fit $\delta(k)$ for the Schwinger Model using an Ansatz
- Ansatz is modified version of Scattering Phase Shift for Sine-Gordon

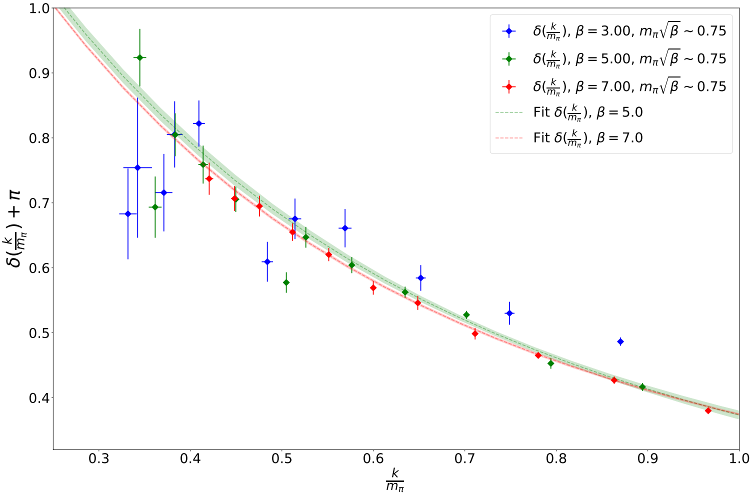
$$\delta(\frac{k}{m_\pi}) = \frac{1}{2i}\log\left(\frac{\sinh(2\sinh^{-1}(A\frac{k}{m_\pi})) + i\sin(B\frac{\pi}{3})}{\sinh(2\sinh^{-1}(A\frac{k}{m_\pi})) - i\sin(B\frac{\pi}{3})}\right)$$

- Go into center of mass system of third particle (i.e. $p_3 = 0$), and assume $p_2 = -p_1$, s.t. quantization conditions reduce to

$$\delta(\frac{p_1}{2}, m) + \delta(p_1, m) + \frac{p_1 L}{2} - \pi = 0,$$

  now solve for p_1
- Once we have p_1, we can make predictions for $E_{3\pi}(m_\pi, p_1, p_2, p_3, P)$, and compare them to the experimentally measured results from the canonical formalism

BU-Slide: Scattering phase shifts $\delta(k/m_\pi)$ with fits



Meson-number

$$\langle N\rangle(\mu_I) = \frac{1}{Z}\int \mathcal{D}U e^{-S_g[u]} \sum_{k=-L_x}^{L_x} k|\det_k M[U]|^2 e^{\mu L_t k} \qquad \mu_I = (F(N) - F(N-1)) = -\frac{1}{L_t}\log\left(\frac{Z_{N,-N}}{Z_{(N-1),-(N-1)}}\right)$$