Lattice determination of the pion mass difference $M_{\pi^+} - M_{\pi^0}$ at order $\mathcal{O}(\alpha_{em})$ and $\mathcal{O}((m_d - m_u)^2)$ including disconnected diagrams.

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The charged/neutral pion mass difference and ℓ_7

$$[M_{\pi^+} - M_{\pi^0}]_{exp.} = 4.5936(5)\ \text{MeV}$$

- Due to $q_u \neq q_d$ and $m_u \neq m_d$.
- The mass splitting is a tiny effect of $\sim 3\%$ due to the smallness of

$$\alpha_{em} = e^2/4\pi \sim (m_d - m_u)/\Lambda_{QCD} \sim \mathcal{O}(1\%)$$

- QED + isospin breaking effects can be evaluated on the lattice using the **RM123 approach**.

Seminal papers:

"*Isospin breaking effects due to the up-down mass difference in Lattice QCD*", [JHEP 1204 (2012)]

"*Leading isospin breaking effects on the lattice*", [PRD87 (2013)]

The RM123 expansion

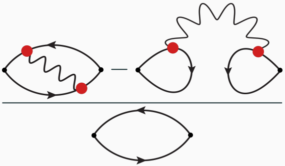
$$\mathcal{L} = \mathcal{L}_{isoQCD} + \mathcal{L}_{photon}[A] + \underbrace{\mathcal{L}^{I}_{QED}}_{\propto e} + \underbrace{\mathcal{L}_{IB}}_{\propto m_d - m_u}$$

- $\mathcal{L}^{I}_{QED}$ and $\mathcal{L}_{IB}$ treated as a perturbation.

$$\delta\langle\mathcal{O}\rangle = \left\langle \langle \mathcal{O} \left[\frac{1}{2} \left(\int \mathcal{L}^{I}_{QED} \right)^2 - \int \mathcal{L}_{IB} \right] \rangle_{isoQCD} \right\rangle_A + \ldots$$

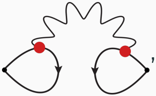
- At leading order $\mathcal{O}(\alpha_{em}, m_d - m_u)$, the pion mass splitting is a **pure electromagnetic effect** (no $\mathcal{O}(m_d - m_u)$ contrib.)

$$\Delta M_\pi \equiv M_{\pi^+} - M_{\pi^0} = \frac{e^2}{2}(q_u - q_d)^2 \partial_t$$



Evaluating the disconnected contribution

- We employ $N_f = 2+1+1$ isoQCD gauge configurations generated by the ETM Collaboration using twisted mass (TM) Wilson fermions.

- The disconnected diagram "" arises from the neutral pion sector, which is notoriously noisy in TM QCD (Neglected in our previous analysis: *arXiv:1704.06561*).

- **Solution**: In the continuum limit perform a $\pi/4$ rotation of the u and d quark fields and discretize the rotated Lagrangian (RTM scheme: *arXiv:2106.07107*)

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} u+d \\ d-u \end{pmatrix}$$

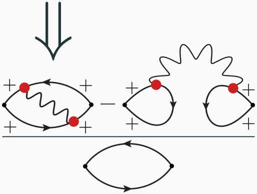
The RTM scheme

TM QCD

$$\psi_\ell = \begin{pmatrix} u \\ d \end{pmatrix}, \quad r_u = -r_d = +1$$

$$J^3_\mu = \bar{\psi}_\ell \gamma_\mu \tau^3 \psi_\ell$$

$$\Delta M_\pi \propto Z_V^2 \partial_t$$

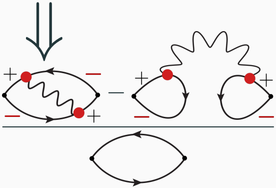


RTM scheme

$$\psi'_\ell = \begin{pmatrix} u' \\ d' \end{pmatrix}, \quad r_{u'} = -r_{d'} = +1$$

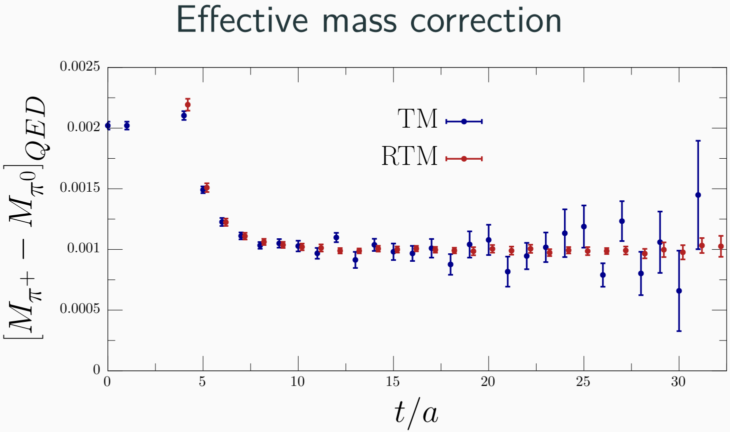
$$J^3_\mu = J'^1_\mu = \bar{\psi}'_\ell \gamma_\mu \tau^1 \psi'_\ell$$

$$\Delta M_\pi \propto Z_A^2 \partial_t$$



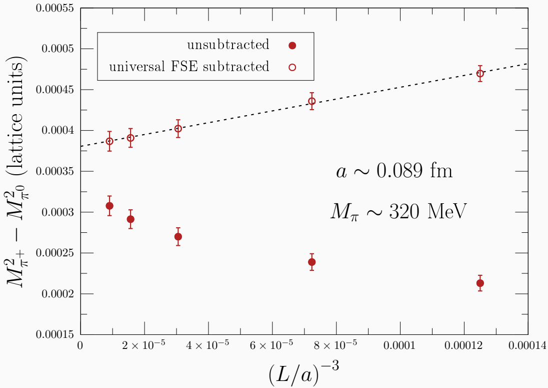
- Diagrams in TM and RTM differ by $\mathcal{O}(a^2)$ effects.
- In the RTM scheme, **only rotated charged pion propagates** in disconnected diagram (opposite Wilson parameter $r = \pm 1$ in quark lines).
- Z_A and Z_V are the RCs of the axial and vector current.

Comparison RTM vs TM



- Remarkable improvement of the statistical accuracy in the RTM scheme.
- Data refer to simulations on a $32^3 \times 64$ lattice at $a \sim 0.089$ fm, $M_\pi \sim 320$ MeV.

Finite size effects

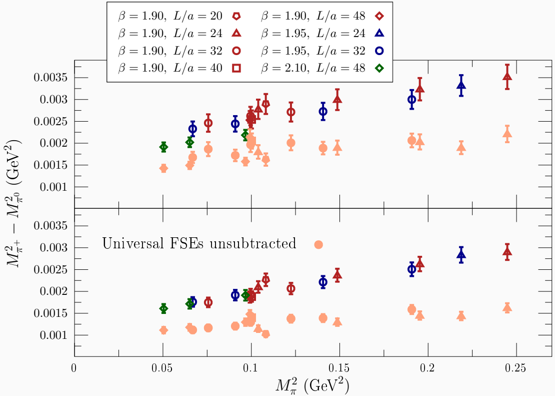


- Power-law finite size effects (FSEs) due to QED.
- For a pseudoscalar meson of charge Q and mass M_{PS}, $\mathcal{O}(1/L)$ and $\mathcal{O}(1/L^2)$ corrections are universal

$$M^2_{PS}(L) - M^2_{PS}(\infty) = -Q^2 \alpha_{em} \frac{\kappa}{L^2} \left(2 + M_{PS} L\right), \quad \kappa \sim 2.837$$

- After subtraction, much smaller residual FSEs of order $\mathcal{O}(1/L^3)$.

Numerical results



$\Longleftarrow$ connected only

$\Longleftarrow$ connected + disconnected

- We computed both connected and disconnected diagrams on the $N_f = 2+1+1$ ETMC ensembles.
- The ensembles correspond to three lattice spacings 0.089, 0.082, 0.062 fm and $M_\pi \simeq 210 - 450$ MeV.
- Strange and charm quark masses are set to their physical values.

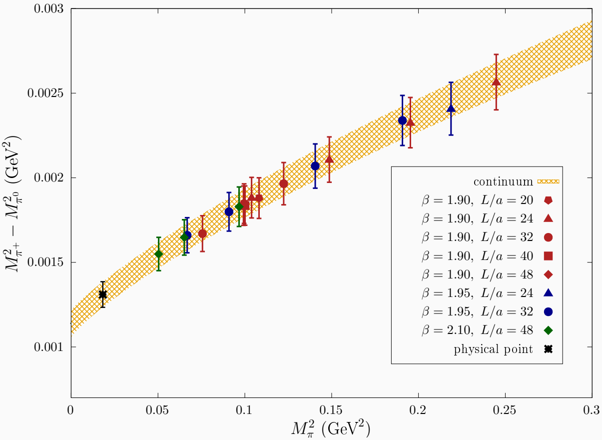
Continuum and physical point extrapolation

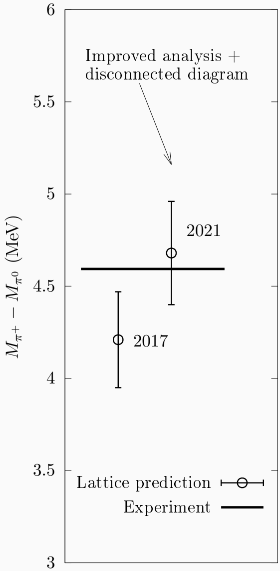
Fitting ansatz (Refined w.r.t. previous analysis):

$$\left[M_{\pi^+}^2 - M_{\pi^0}^2\right]_{\text{lat.}} = a^2\Big\{e^2 f_\pi^2\Big[C - \overbrace{C_1\frac{M_\pi^2}{(4\pi f_\pi)^2}\log\left(\frac{M_\pi}{4\pi f_\pi}\right)^2 + A_1^\pi\frac{M_\pi^2}{16\pi^2 f_\pi^2} + A_2^\pi\frac{M_\pi^4}{(4\pi f_\pi)^4}}^{\text{pion mass dependence}}\Big] + \underbrace{K\frac{e^2}{3}\frac{M_\pi}{L^3}}_{FSEs} + \underbrace{D^\pi a^2 + D_m^\pi a^2 M_\pi^2}_{\mathcal{O}(a^2)\ \text{artifacts}}\Big\}$$

- $\left[M_{\pi^+}^2 - M_{\pi^0}^2\right]_{\text{lat.}}$ is the squared pion mass difference in lattice units after the subtraction of the universal FSEs.
- $C, C_1, A_1^\pi, A_2^\pi, K, D^\pi, D_m^\pi$ are free fit parameters.
- Pion mass dependence inspired by ChPT at NLO and NNLO.
- C_1 not set to the $N_f = 2$ ChPT prediction $C_1 = 3 + 4C$ (as in previous analysis).

Final result (Preliminary)





- Mild $\mathcal{O}(a^2)$ within accuracy.
- Final result: $M_{\pi^+} - M_{\pi^0} = 4.68(28)$ MeV.

QCD contribution to the pion mass difference and ℓ_7

Pion mass difference at $\mathcal{O}((m_d - m_u)^2)$ and ℓ_7

- $[M_{\pi^+} - M_{\pi^0}]_{QCD}$ relevant for the extraction of the scale invariant ChPT low-energy constant ℓ_7 (ℓ_7 has a $> 50\%$ uncertainty).
- ℓ_7 parametrizes isospin-breaking effects in the ChPT Lagrangian at **NLO**. Important for e.g. axion physics [Di Cortona et al. *arXiv:1511.02867*, Di Luzio et al. *arXiv:2101.10330*].

Two methods to extract ℓ_7

[Frezzotti et al. *arXiv:2107.11895*, Gasser and Leutwyler *Annals Phys.*, 1984].

Mass method:

$$[M_{\pi^+} - M_{\pi^0}]_{QCD} = (m_u - m_d)^2 \frac{M_\pi^3}{2m_l^2 f_\pi^2} \ell_7$$

Matrix element method:

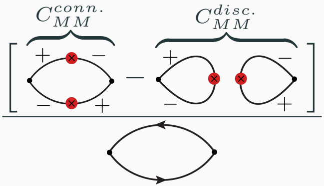
$$Z_{P^0\pi^0} \equiv \langle 0|P^0|\pi^0\rangle = \frac{1}{\sqrt{2}}\langle 0|\bar{u}\gamma_5 u + \bar{d}\gamma_5 d|\pi^0\rangle = -(m_u - m_d)\frac{M_\pi^4}{f_\pi m_\ell^2} \ell_7$$

Evaluating l_7 using the RM123 approach

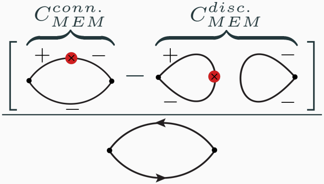
Expand $Z_{P^0\pi^0}$ and $[M_{\pi^+} - M_{\pi^0}]_{QCD}$ in powers of $\Delta m = \frac{m_d - m_u}{2}$.

In the rotated basis $\psi'_\ell = \begin{pmatrix} u' \\ d' \end{pmatrix}$: $\mathcal{L}_{IB} \propto \bar{\psi}'_\ell \, \tau^1 \, \psi'_\ell$

$[M_{\pi^+} - M_{\pi^0}]_{QCD} \propto (\Delta m)^2 \partial_t$

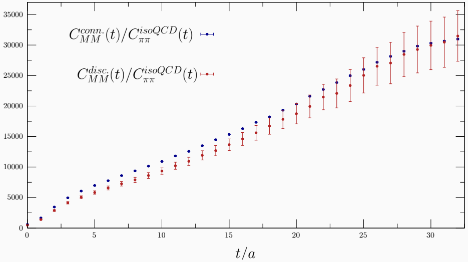


$Z_{P^0\pi^0} \propto \Delta m$

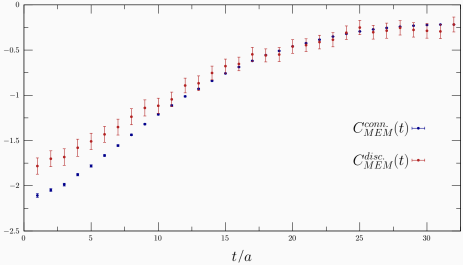


Evaluation of the connected and disconnected diagrams

mass method



matrix element method



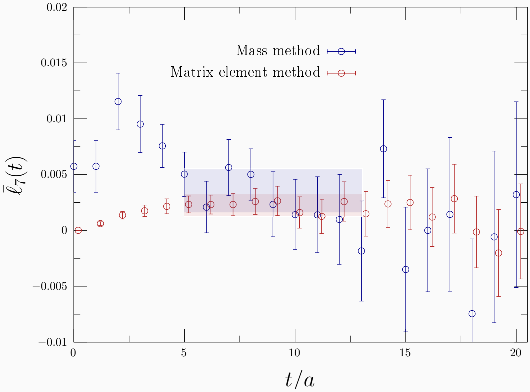
- Pilot study on a single ETMC Wilson-clover ensemble (cA211.30.32) at $a \simeq 0.095$ fm and $M_\pi \simeq 260$ MeV.
- Analyzed 1.2×10^3 gauge configurations.
- **Strong cancellation** between connected and disconnected diagrams in both methods $\implies$ RTM scheme fundamental to increase the statistical accuracy.

Numerical results for ℓ_7

We have built the following estimators:

$$\bar{\ell}_7(t) = \left(\frac{Z_S}{Z_P}\right)^2 \cdot \frac{\hat{f}_\pi^2 \hat{m}_\ell^2}{\hat{M}_\pi^3} \cdot \partial_t \left[\frac{C_{MM}^{conn.}(t) - C_{MM}^{disc.}(t)}{C_{\pi\pi}^{\text{isoQCD}}(t)}\right] \qquad \text{(mass method)}$$

$$\bar{\ell}_7(t) = -\left(\frac{Z_S}{Z_P}\right) \cdot \frac{\hat{f}_\pi \hat{m}_\ell^2}{\hat{M}_\pi^4} \cdot \hat{Z}_{\pi\pi} \cdot \left[\frac{C_{MEM}^{conn.}(t) - C_{MEM}^{disc.}(t)}{C_{\pi\pi}^{\text{isoQCD}}(t)}\right] \qquad \text{(matrix element method)}$$



$$\left\{ \begin{array}{l} \ell_7^{MM} = 3.5(2.0) \times 10^{-3} \\ \ell_7^{MEM} = 2.3(1.0) \times 10^{-3} \end{array} \right\}$$

$\Downarrow$ combined

$$\ell_7 = 2.5(1.4) \times 10^{-3}$$

To be compared with:

$\ell_7^{*} = 7(4) \times 10^{-3}$

$\ell_7^{**} = 6.5(3.8) \times 10^{-3}$

*[NLO SU(3) ChPT]

**[RBC-UKQCD: *arXiv:1511.01950*]

Summary

- We computed the disconnected diagram appearing in the expansion of the charged/neutral pion mass difference $M_{\pi^+} - M_{\pi^0}$ at $\mathcal{O}(\alpha_{em})$.
- After extrapolating to the continuum and infinite volume limit, and to the physical point we obtain a value for $M_{\pi^+} - M_{\pi^0}$ that is consistent with the experimental determination.
- We evaluated the $\mathcal{O}((m_d - m_u)^2)$ contribution to $M_{\pi^+} - M_{\pi^0}$ and the coupling of the isoscalar operator P^0 to the neutral pion at $\mathcal{O}(m_d - m_u)$, from which we determined the ChPT low-energy constant ℓ_7.
- In the future we plan to perform a more systematic study of ℓ_7 in order to extrapolate its value to the continuum and chiral limit.

Thank you for your attention!