Variations on the Maiani-Testa approach and the inverse problem

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based on *JHEP 06 (2021) 043*, 21xx.yyyy

Lattice 2021, MIT (Virtual)
July 27$^{\mathrm{th}}$, 2021

The problem

"It is of considerable interest to identify the physical quantities, if any, which can be extracted directly from euclidean correlation functions, avoiding analytic continuation" [Maiani, Testa '90]

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Starting point: euclidean correlator $\langle \widetilde{\pi}_{\boldsymbol{q}_1}(t_1)\widetilde{\pi}_{\boldsymbol{q}_2}(t_2)J(0)\rangle$ $(\boldsymbol{q}_1 = -\boldsymbol{q}_2)$

Conclusion: physical scattering only at $\boldsymbol{q} = 0$ [Maiani, Testa '90]

$$\langle \pi, \mathbf{0}|\widetilde{\pi}_{\mathbf{0}}(t_2)J(0)|0\rangle \overset{t_2 \gg 0}{\rightarrow} \mathcal{F}_2(4M_\pi^2)\left[1 + a\sqrt{\frac{M_\pi}{\pi t_2}} + O(t_2^{-3/2})\right]$$

at threshold: $\mathcal{F}_2$+scatt.length a

at $\boldsymbol{q} \neq 0$: unphysical matrix elements

Maiani-Testa: at threshold there is no inverse problem!

analytic control over inverse problem thanks to t_2!

at finite L, $t_2 \Delta E \ll 1$, with ΔE level spacing

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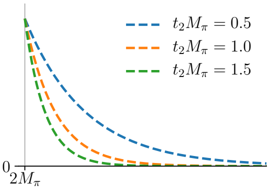
Inverse problem

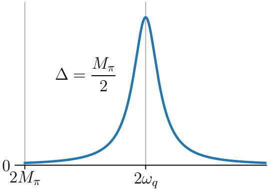


[Maiani, Testa '90]

$$\langle \pi, \boldsymbol{q}_1 | \widetilde{\pi}_{\boldsymbol{q}_2}(0)\, e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2}\, J(0)|0\rangle$$

physical scattering at $\boldsymbol{q}_1 = \boldsymbol{q}_2 = 0$

exponentials mimick "half" $\delta(E - 2M_\pi)$

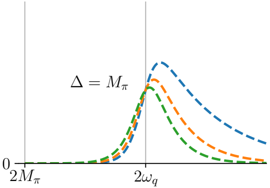


[Bulava, Hansen '18]

$$\langle \pi, \boldsymbol{q}_1 | \widetilde{\pi}_{\boldsymbol{q}_2}(0)\, \delta(\hat{H} - 2\omega_{\boldsymbol{q}}, \Delta)\, J(0)|0\rangle$$

physical scattering at $E = 2\omega_{\boldsymbol{q}}$

ordered double-limit $\lim_{\Delta\to 0}\lim_{V\to\infty}$



[Bruno, Hansen '21]

$$\langle \pi, \boldsymbol{q}_1 | \widetilde{\pi}_{\boldsymbol{q}_2}(0)\, \Theta(\hat{H} - 2\omega_{\boldsymbol{q}}, \Delta) e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2}\, J(0)|0\rangle$$

physical scattering at pole $E = 2\omega_{\boldsymbol{q}}$

physical scattering at fixed Δ

Generalized Maiani-Testa

$$\langle \pi, \mathbf{0} | \widetilde{\pi}_{\mathbf{0}}(0) e^{-(\hat{H}-2\omega_{\mathbf{0}})t_2} J(0) | 0 \rangle$$ [Maiani, Testa '90]

$$\overset{t_2 \gg 0}{\to} \mathcal{F}_2(4M_\pi^2) \left[1 + a \sqrt{\frac{M_\pi}{\pi t_2}} + O(t_2^{-3/2}) \right]$$

$$\langle \pi, \boldsymbol{q} | \widetilde{\pi}_{-\boldsymbol{q}} \Theta(\hat{H} - 2\omega_{\boldsymbol{q}}, \Delta) e^{-(\hat{H}-2\omega_{\boldsymbol{q}})t_2} J(0) | 0 \rangle$$ [Bruno, Hansen '21]

$$\to \ \mathrm{Re}\left[\mathcal{F}_2(4\omega_{\boldsymbol{q}}^2)\right] + \sum_{n=0} g_n \mathcal{J}^{(n)}(t_2, \omega_{\boldsymbol{q}}, \Delta)$$

Generalization of Maiani-Testa work

- $\mathcal{J}^{(n)}$ known kinematic functions
- unitarity relations imply $g_0 \simeq \mathrm{Im}\left[\mathcal{F}_2\right]$

$$2i\mathrm{Im}\left[\mathcal{F}_2\right] = \mathrm{disc}[\mathcal{F}_2] = 2\pi i \sum_n \int d\Phi_n \mathcal{M}_{2n}^* \mathcal{F}_n \delta(2\omega_{\boldsymbol{q}} - \sum_i \omega_{\boldsymbol{p}_i})$$

- valid for any energy/open channels

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Further extensions

Four-point functions for $2 \to 2$ scattering [Bruno, Hansen '21]

at threshold: $\langle \pi | \widetilde{N}(t) N(0) | \pi \rangle \to a_{N\pi} t + c a_{N\pi}^2 \sqrt{t} + O(t^0)$

$a_{N\pi}$ scatt. length, similar expression for $\pi\pi \to \pi\pi$

exploratory study in ϕ^4 [M. Garofalo's talk]

at $\sqrt{s} > 0$: from $\langle \pi | \widetilde{\pi}(t) \Theta(\hat{M} - 2\omega_q, \Delta) \pi(0) | \pi \rangle \to \mathrm{Re}\mathcal{M}_2, \mathrm{Im}\mathcal{M}_2$

Other interesting observables w/ smooth Θ [Gambino, Hashimoto '20]
[Fukaya, Hashimoto, Kaneko, Ohki '20]

But what about the simpler two-point functions?

Spectral reconstruction

$$G(t) = \int d^3\boldsymbol{x}\,\langle J(t,\boldsymbol{x})J(0)\rangle = \langle \widetilde{J}_{\boldsymbol{q}}(t)J(0)\rangle = \int_{2m}^{\infty} d\omega\, e^{-\omega t}\,\rho(\omega^2)$$

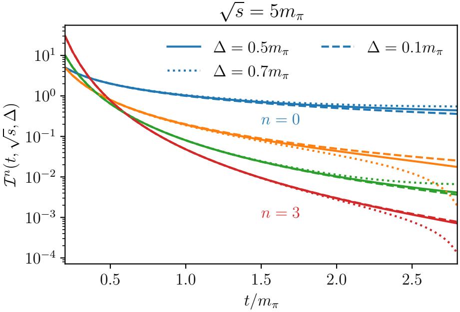
1. re-use idea of smooth Θ function

$$G^{\Theta}(t|\mathcal{E}) = \langle \widetilde{J}_{\boldsymbol{0}}(0)\, e^{-(\hat{H}-\mathcal{E})t}\, \Theta(\hat{H}-\mathcal{E},\Delta)\, J(0)\rangle$$

2. perform large t expansion to localize "right shoulder"

$$G^{\Theta}(t|\mathcal{E}) = \sum_n r_n \mathcal{I}_n(t,\mathcal{E},\Delta)$$

[in prep.]



$\mathcal{I}_n$ pure analytic functions asymptotic series

r_n free fit parameters

$r_0 \equiv \rho(s = \mathcal{E}^2)$

r_n all physical $\leftrightarrow \partial_s^n \rho$

The G^Θ correlator

1. build G^Θ from original G
   i. GEVP w/ several operators $\rightarrow$ exact reconstruction of G^Θ
   ii. numerical reconstruction of G^Θ
      Backus-Gilbert-like methods [Hansen, Lupo, Tantalo '19]
      Chebyshev polynomials [Bailas, Hashimoto, Ishikawa '20]

2. fit G^Θ, $t \in [t_{\min}, t_{\max}]$ using $\mathcal{I}_n$ basis functions

   similar numerical complexity of syst. error from excited states

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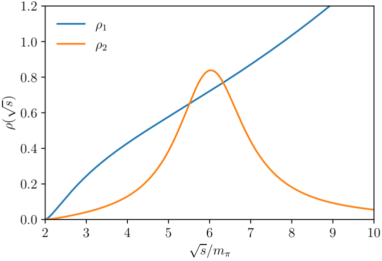
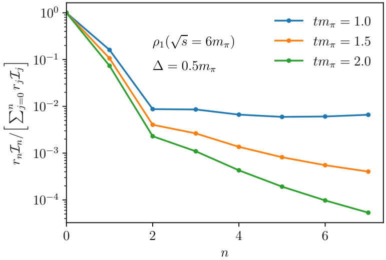
Pratical simulations finite L

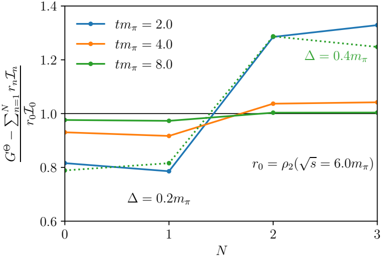
$\Delta L \gg 1$ should lead to $O(e^{-\Delta L})$ FV corrections
e.g. $\Delta = 0.5 m_\pi$, $m_\pi L = 6 \;\rightarrow\; e^{-\Delta L} \simeq 5.0\ \%$
e.g. $\Delta = 1.0 m_\pi$, $m_\pi L = 6 \;\rightarrow\; e^{-\Delta L} \simeq 0.2\ \%$

but finite statistics too, and we should worry about that more...

Toy model - I







Analytic control over G^Θ and r_n

Infinite volume ρ

$\rho_1 \approx \pi\pi \; I = 0, 2$

$\rho_2 \approx \pi\pi \; I = 1$

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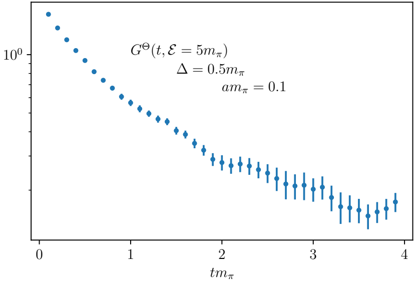
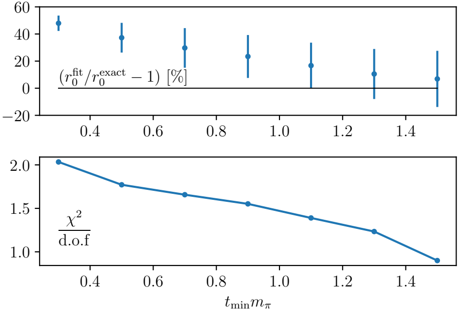
Toy model - II



Synthetic correlated data for $G^\Theta(t, \mathcal{E})$
$\to$ rel. error $1 \to 20\%$



fits with $r_0 \mathcal{I}_0 + r_1 \mathcal{I}_1$

"excited state" analysis

prediction of r_0 w/ 20% stat. err.

χ^2 analysis agrees w/ exact prediction

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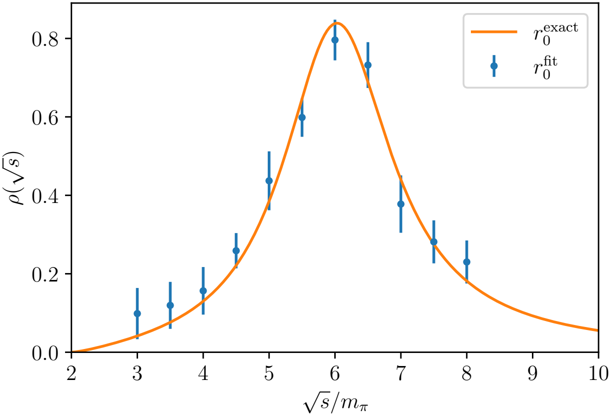
Toy model - III

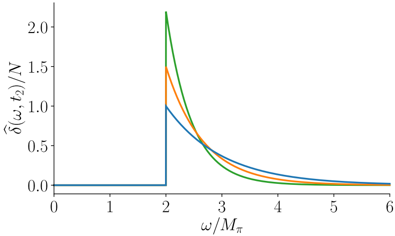
Plot: each point fit to G^{Θ} at that energy, w/ fixed $t_{\min}$, $\Delta = 0.5 m_\pi$
stat. error only, syst. errors expected of same size

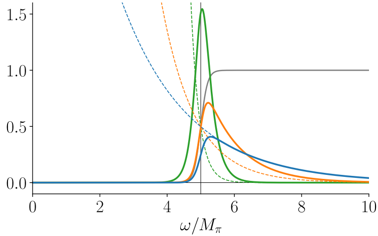


In practical cases errors grow for larger $\sqrt{s}$
Toy model unrealistic but defines goal for stat. accuracy

Conclusions

Connection of Maiani-Testa result to inverse problem [Bruno, Hansen '21]





smeared $\Theta \to$ left shoulder

exponentials $e^{-\omega t} \to$ right shoulder

Extension of Maiani-Testa result [Bruno, Hansen '21]

- $0, 1, 2 \to 2$ processes away from threshold
- derivation of $2 \to 2$ at threshold, relevant for $a_{N\pi}$

Outlook

Extension of Θ-idea to two-point function [in prep.]

in $0, 1, 2 \to 2$ higher-order terms non physical
in two-point function all r_n physical $\to$ analytic study

smooth spectral density, e.g. $\pi\pi$ $I = 0, 2$
excellent convergence of series, $O(< 1\%)$ syst. errors
high stat. accuracy required
multi-level simulations ideal [Cé, Schaefer, Giusti][Giusti's talk]

resonant spectral functions, encouraging results at 10-20 %

finite volume errors to be addressed
master-field should work even better (if accurate data) [Cé's talk]

Thanks for your attention

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Toy model - IV

Spectral reconstruction of ρ_1, w/ fixed $t_{\min}$, $\Delta = 0.5 m_\pi$