Infinite volume, three-body scattering formalisms in the presence of bound states

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38th International Symposium on Lattice Field Theory
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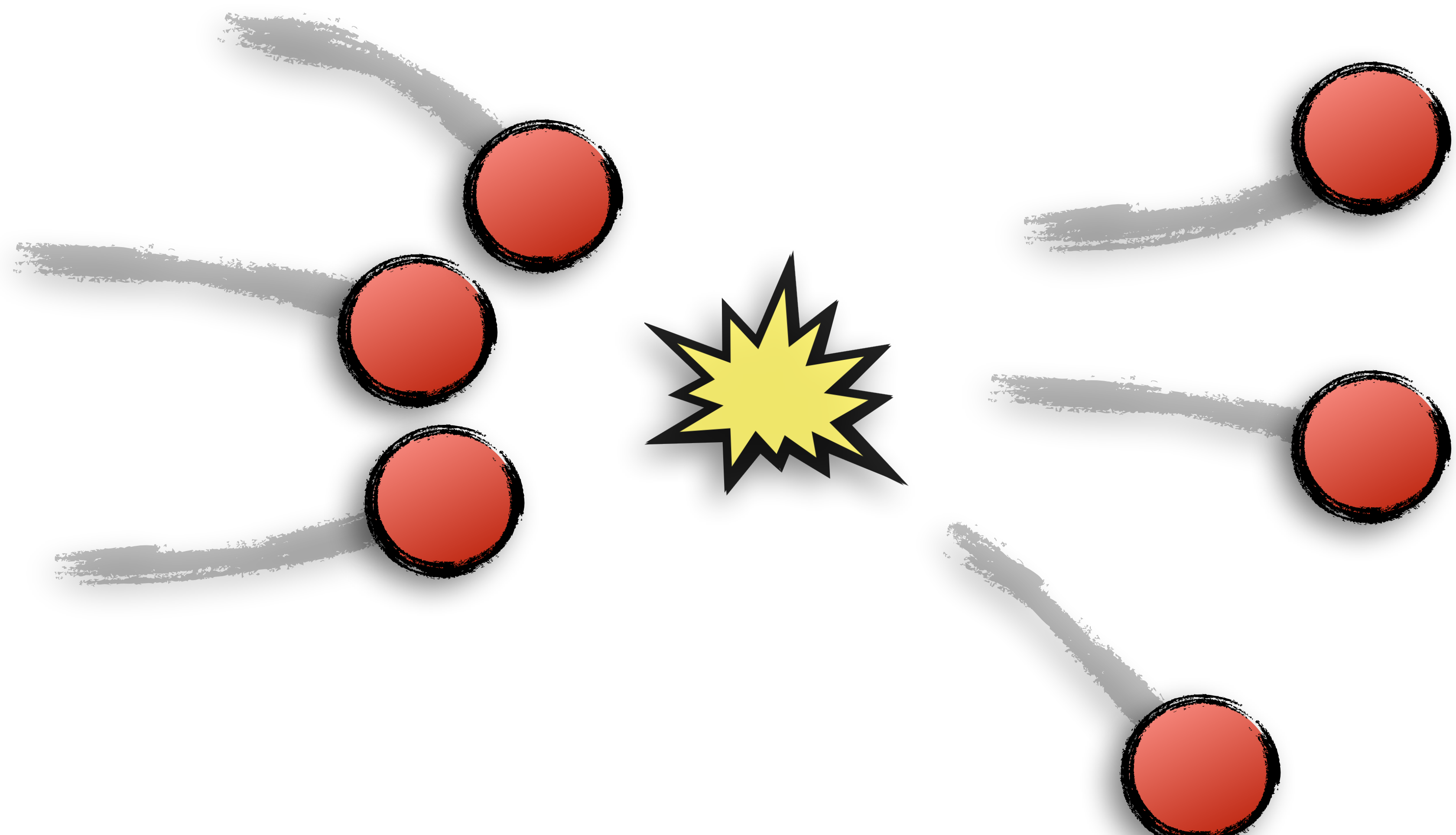
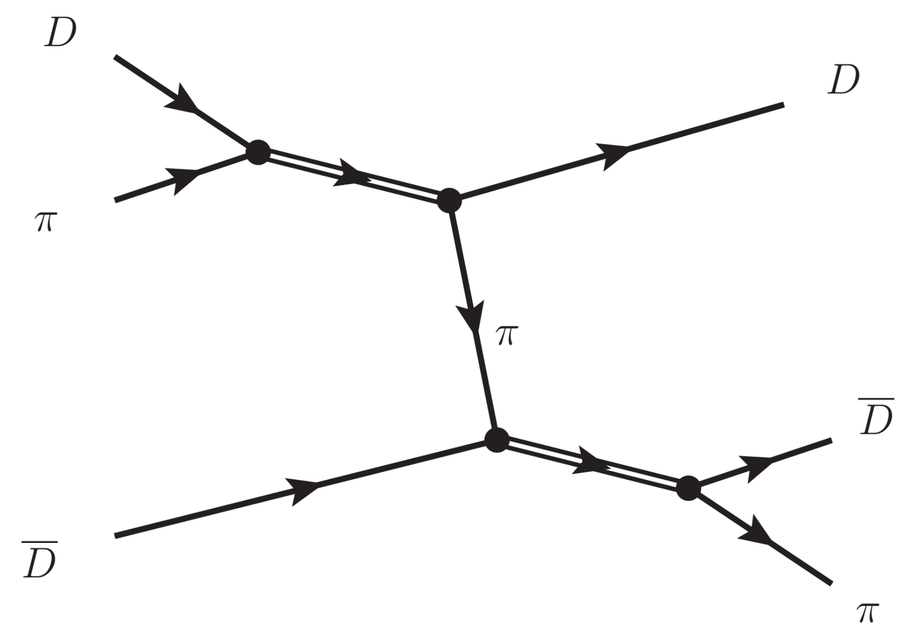


LATTICE21
JULY 26-30 2021, ZOOM/GATHER@MIT

Indiana University Bloomington

Three-body problems

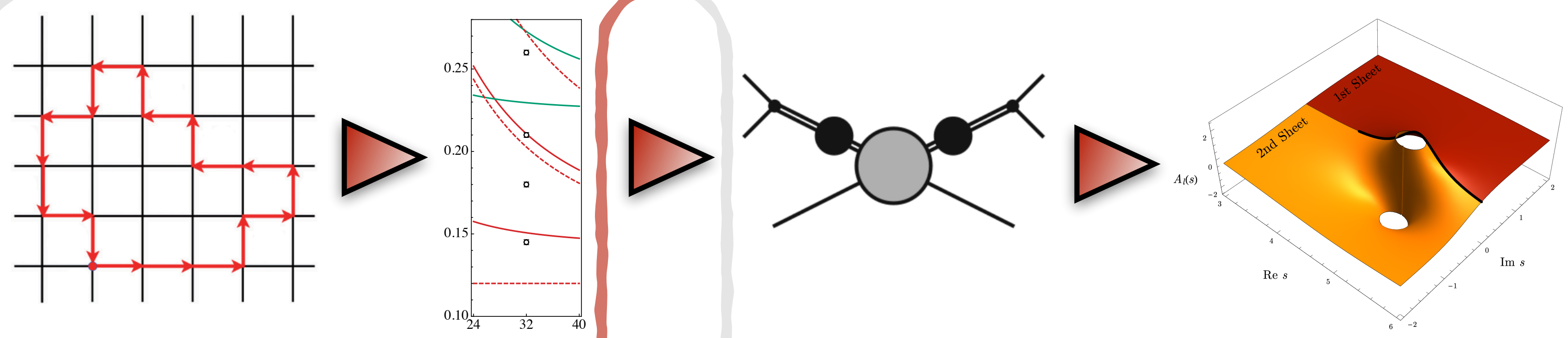
- Exotic resonances decay to three-particle final states
  - X(3872), N*(1440), a_1(1260), ...
- Interpretations of X(3872)
  - molecule
  - charmonium-molecule hybrid
  - diquark-antidiquark
  - kinematical effect
- Roper resonance N*(1440)



Pion-nucleon scattering in the Roper channel from lattice QCD,
Lang, Leskovec, Padmanath, Prelovsek, Phys. Rev. D 95 (2017) 1, 014510

> Combining experimental $N\pi$ phase shifts with elastic approximation and the Lüscher formalism suggests in the spectrum an additional energy level near the Roper mass $m_R = 1.43$ GeV for our lattice. We do not observe any such additional energy level, which implies that $N\pi$ elastic scattering alone does not render a low-lying Roper resonance. The current status indicates that the $N^*(1440)$ might arise as dynamically generated resonance from coupling to other channels, most notably the $N\pi\pi$.

Relativistic formalisms



Lattice QCD | FV Spectrum | Amplitudes | Particle properties

This talk

- Finite volume spectrum ⟶ Quantization Condition ⟶ Three-body K-matrix
- K-matrix + two-body subprocesses ⟶ integral equations ⟶ 3-body amplitudes
- Final amplitudes analytically continued to the unphysical Riemann sheets

Two main frameworks

- Relativistic Effective Field Theory
  - generic scalar EFT,
  - summation of 2PI, 3PI diagrams,

  (references → prof. Sharpe's presentation)
- First results – three pions at I=3

  Blanton et al., Phys. Rev. Lett. 124 (2020) 3, 032001
  Hansen et al., Phys. Rev. Lett. 126 (2021) 012001

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Jackura et al. Phys.Rev.D 100 (2019) 3, 034508

Blanton, Sharpe, Phys.Rev.D 102 (2020) 5, 054515

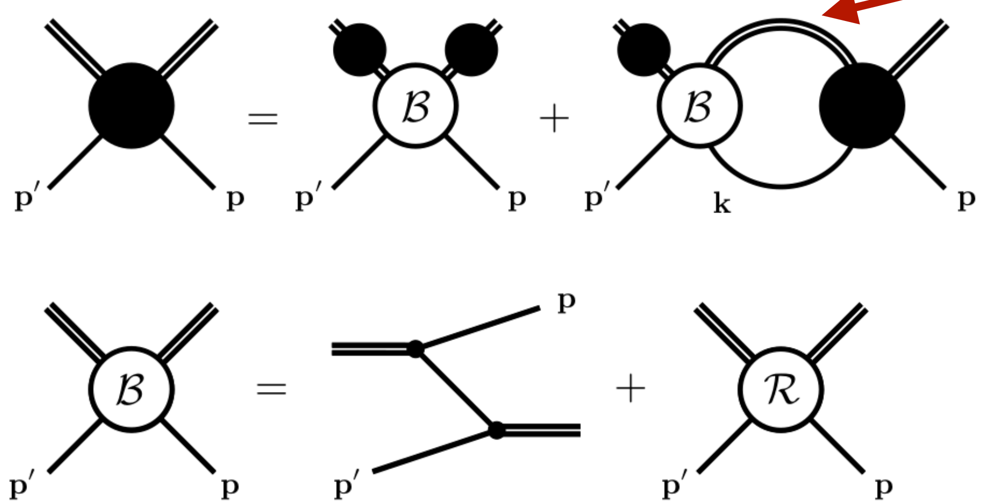
- Unitarity-based framework
  - parametrization based on the S matrix unitarity,

  (references → prof. Sharpe's presentation)
- First results – three pions at I=3

  Mai, Döring, Phys. Rev. Lett. 122, 062503 (2019)
  Culver et al., Phys. Rev. D 101 (2020) 11, 114507

The B-matrix approach

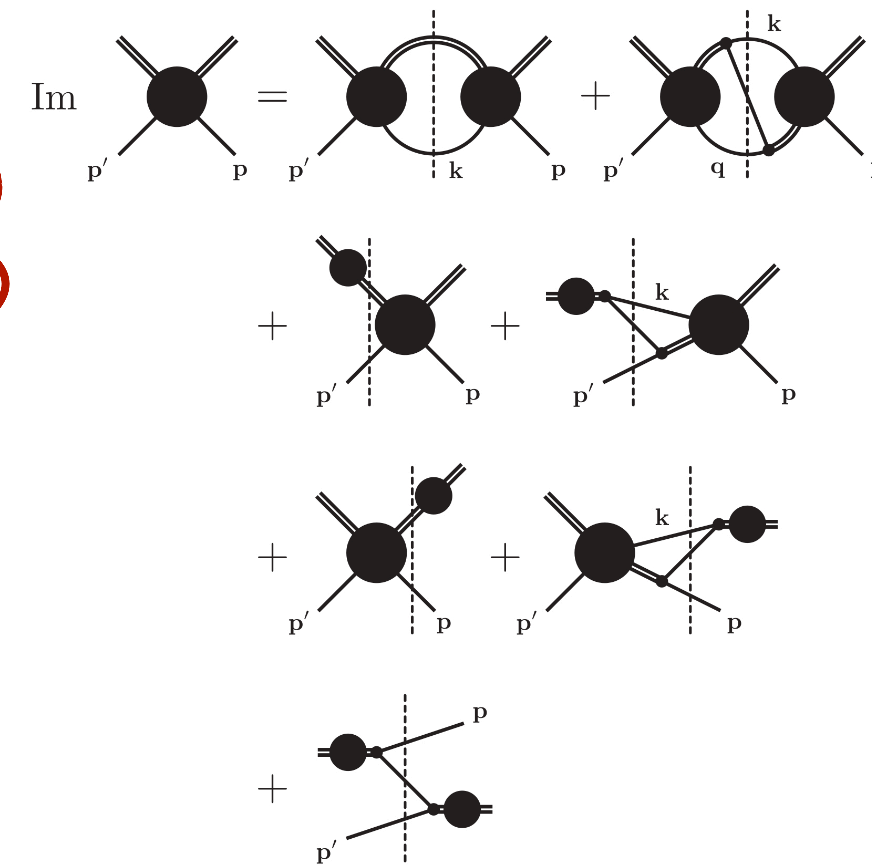
- Starting with the S matrix unitarity
- Physical degrees of freedom (domain of integration)
- Simple parametrization with clear interpretation



One Particle Exchange

Short Range Interactions

$$\mathcal{A} = \mathcal{M}_2 \mathcal{B} \mathcal{M}_2 + \mathcal{M}_2 \int \mathcal{B} \mathcal{A}$$



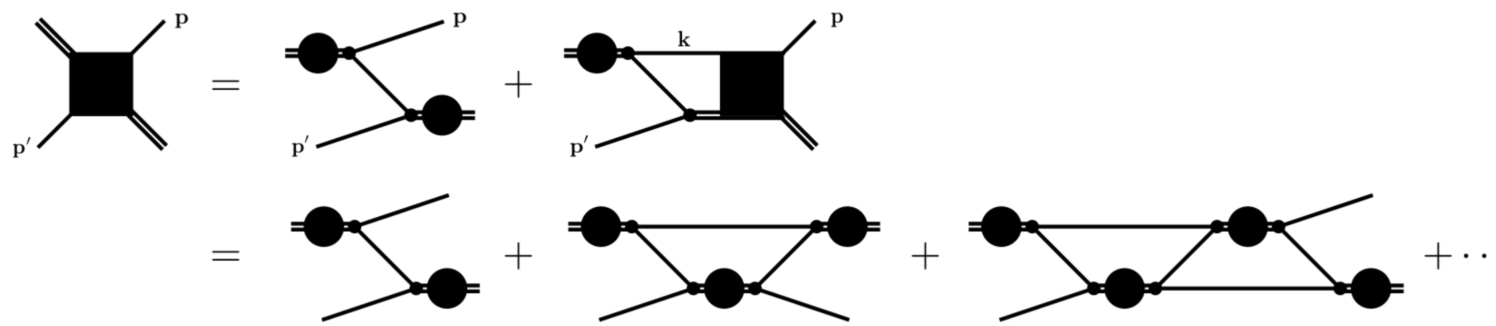
Three-body amplitude

$$\mathcal{A}_{\ell m_\ell ; \ell' m_\ell}(\sigma', s, \sigma)$$

- pair-spectator,
- partial waves,
- symmetrization,

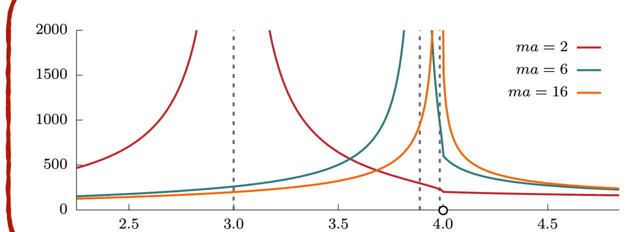
Solving the EFT three-body ladder equation

- Ladder approximation, B = G + (R=0)



- Numerical solution of the three-body EFT equations

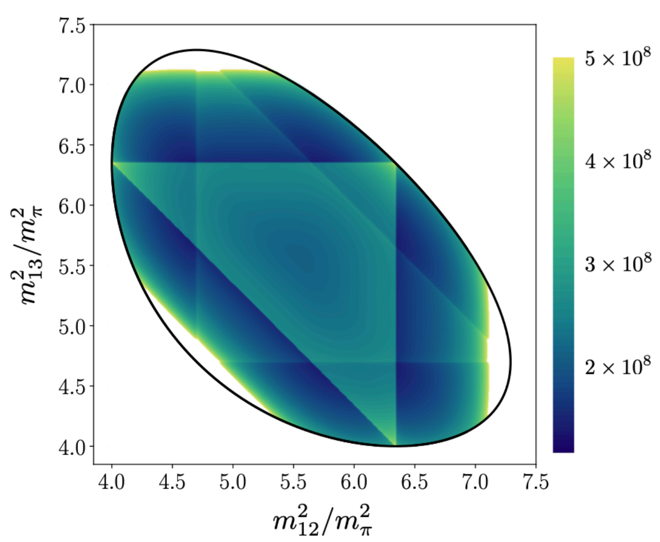
A. Jackura, R. Briceño, S. Dawid, M. H. E Islam, C. McCarty, Phys.Rev.D 104 (2021) 1, 014507



$$\mathcal{M}_2^{-1} \sim -\frac{1}{a} - i\rho_2$$

$$\lim_{\sigma,\sigma'\to\sigma_b} \mathcal{A}_3 = \frac{g}{\sigma' - \sigma_b}\,\mathcal{A}_2\,\frac{g}{\sigma - \sigma_b}$$

- Similar studies



- weakly interacting system in the $\pi^+\pi^+$ and $\pi^+\pi^+\pi^+$

  Hansen et al., Phys. Rev. Lett. 126 (2021), 012001

- decay $a_1(1260) \to \rho^0\pi^- \to \pi^-\pi^+\pi^-$

  Sadasivan et al., Phys.Rev.D 101 (2020) 9, 094018

Numerical procedure

- Discretization of the integral equation $\longrightarrow$ N linear equations (Matrix equation)
- Regulation of the bound-state pole via ϵ-prescription

$$\mathcal{A}_2(s) = \lim_{\epsilon \to 0^+} \lim_{N \to \infty} \mathcal{A}_2(s; N, \epsilon)$$

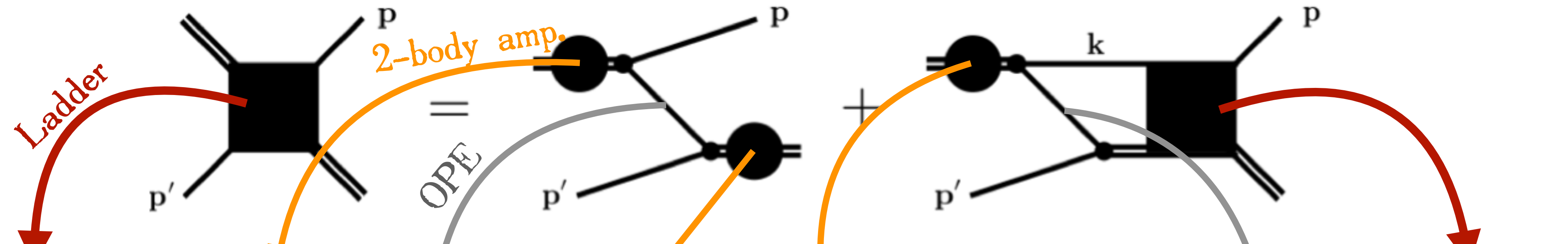
- Systematics
  - **Unitarity test:** $\mathrm{Im}\mathcal{A}_2(s) = \rho_2(s)|\mathcal{A}_2(s)|^2 \longrightarrow \Delta\rho_2 = 100 \times \left| \frac{\mathrm{Im}\,\mathcal{A}_2^{-1}(s; N, \epsilon) + \rho_2(s)}{\rho_2(s)} \right|$
  - **Convergence test:** $\Delta_N \mathcal{A}_2 = 2 \times \left| \frac{\mathcal{A}_2(s; N+1, \epsilon) - \mathcal{A}_2(s; N, \epsilon)}{\mathcal{A}_2(s; N+1, \epsilon) + \mathcal{A}_2(s; N, \epsilon)} \right|$

- Methods

“Brute force” | Explicit pole removal | Spline-based quadratures

Hansen et al., Phys. Rev. Lett. 126 (2021), 012001
A. Jackura et al, Phys.Rev.D 104 (2021) 1, 014507

Brute force method



$$\mathcal{D}_{\mathsf{s}}^{(u,u)}(p,k) = -\mathcal{M}_2(E^\star_{2,p})G_{\mathsf{s}}(p,k,\epsilon)\mathcal{M}_2(E^\star_{2,k}) - \mathcal{M}_2(E^\star_{2,p})\int_0^{k_{\max}} \frac{k'^2\,dk'}{(2\pi)^2\omega_{k'}} G_{\mathsf{s}}(p,k',\epsilon)\mathcal{D}_{\mathsf{s}}^{(u,u)}(k',k),$$

Discretization

$$\boldsymbol{D}(N,\epsilon) = -\boldsymbol{\mathcal{M}}\cdot\boldsymbol{G}(\epsilon)\cdot\boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{M}}\cdot\boldsymbol{G}(\epsilon)\cdot\boldsymbol{P}\cdot\boldsymbol{D}(N,\epsilon)$$

Amputation

$$\mathcal{D}^{(u,u)}(\mathbf{p},\mathbf{k}) \equiv \mathcal{M}_2(p)d^{(u,u)}(\mathbf{p},\mathbf{k})\mathcal{M}_2(k)$$

Matrix inversion

$$\boldsymbol{D}(N,\epsilon) = -\left[\mathbb{I} + \boldsymbol{\mathcal{M}}\cdot\boldsymbol{G}(\epsilon)\cdot\boldsymbol{P}\right]^{-1}\cdot\boldsymbol{\mathcal{M}}\cdot\boldsymbol{G}(\epsilon)\cdot\boldsymbol{\mathcal{M}}$$

Limits

$$\mathcal{D}_{\mathsf{s}}^{(u,u)}(p,k) = \lim_{\epsilon\to 0}\lim_{N\to\infty}\boldsymbol{D}_{pk}(N,\epsilon)$$

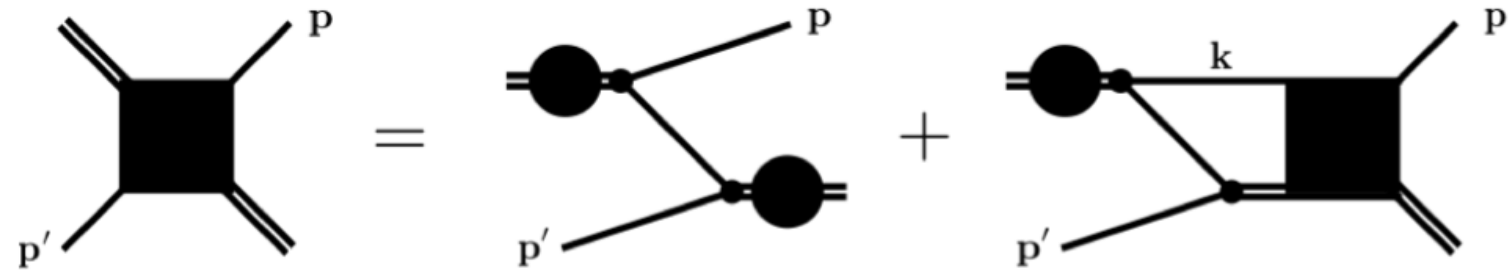
Double ordered limit ⟶ single limit

$$\epsilon \propto \eta/N$$

Splines-based method

W. Glöckle, G. Hasberg, and A. R. Neghabian, Z. Phys. A 305, 217 (1982)

A. Jackura et al, Phys.Rev.D 104 (2021) 1, 014507

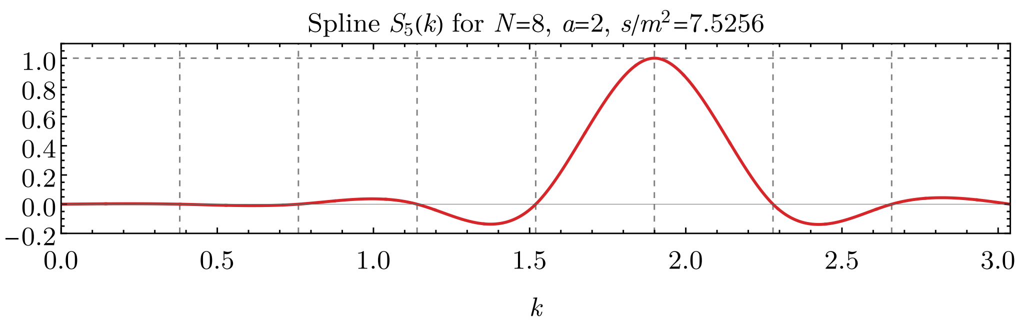


$$d_S^{(u,u)}(p,k) = -G_S(p,k) - \int_0^\infty \frac{dk' k'^2}{(2\pi)^2 \omega_{k'}} G_S(p,k') \mathcal{M}_2(k') d_S^{(u,u)}(k',k)$$

Quadratures

$$d_S^{(u,u)}(s_{2k}, s_{2p}) = -G_S(s_{2k}, s_{2p}) - \sum_{n=0}^{N_s} \omega_n(s_{2k}, s) d_S^{(u,u)}(s_{2q,n}, s_{2p}).$$

$$d_S^{(u,u)}(s_{2k}, s_{2p}) \approx \sum_{n=0}^{N_s} S_n(s_{2k}) d_S^{(u,u)}(s_{2q,n}, s_{2p}).$$



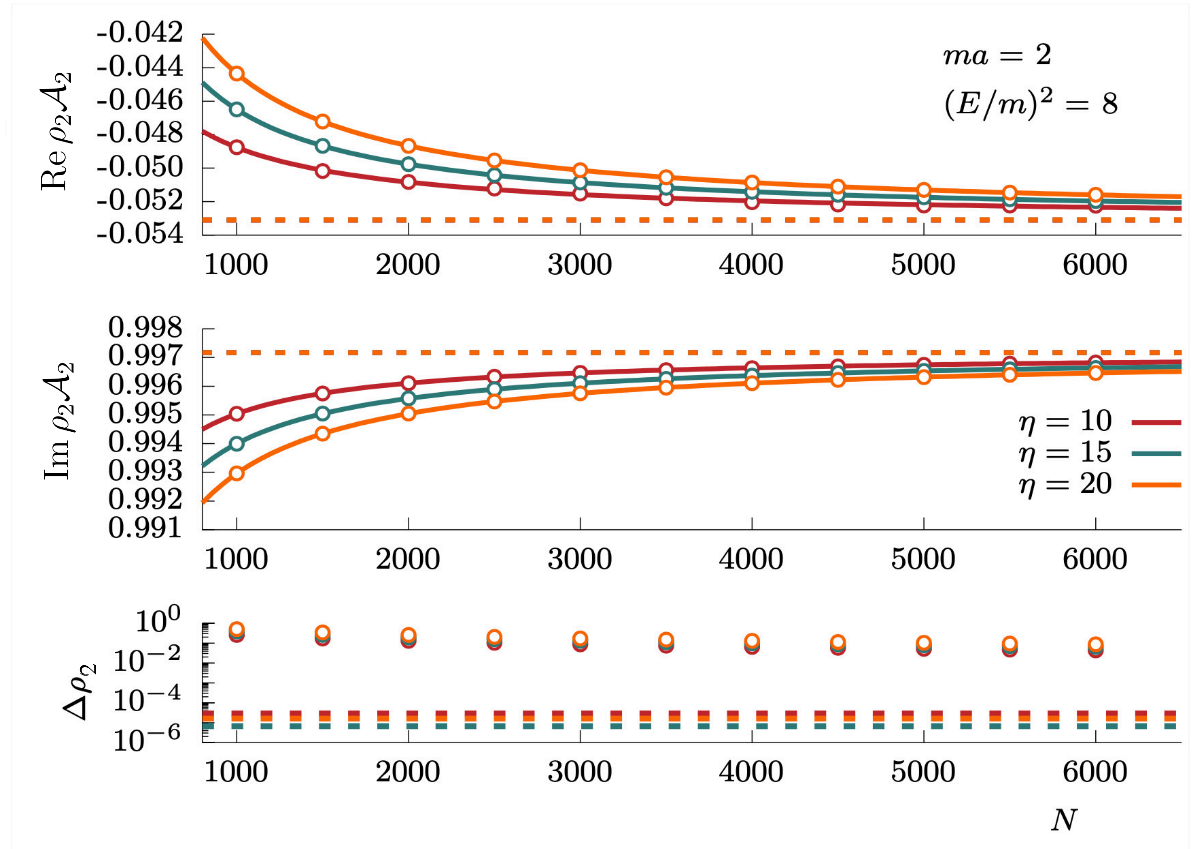
Possible improvement
analytic removal of ϵ

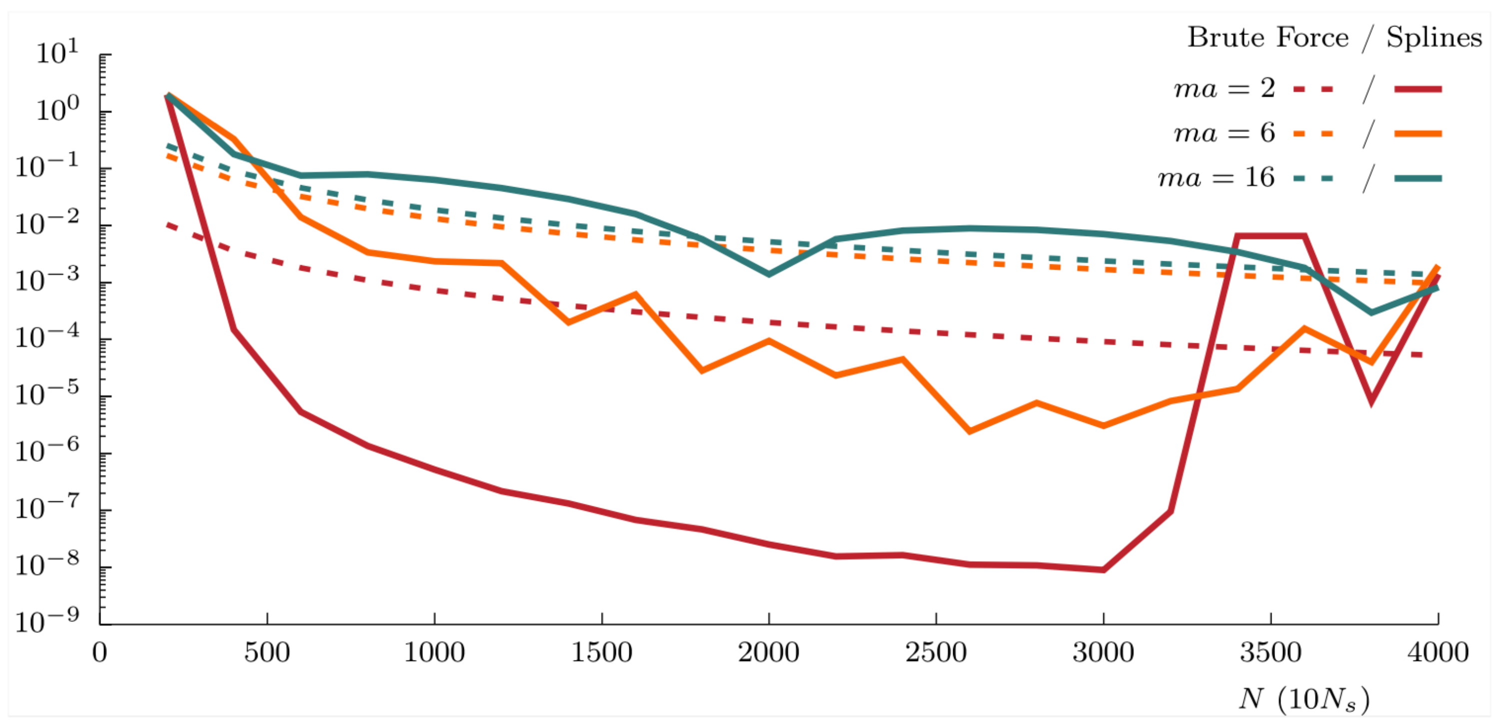
$$\int_0^{(E-m)^2} \frac{ds_{2q}}{\pi} G_S(s_{2k}, s_{2q}) \frac{\lambda^{1/2}(E^2, s_{2q}, m^2)}{16\pi E^2} S_i(s_{2q}) = \sum_{n=0}^{N_s} \omega_n(s_{2k}) S_i(s_{2q,n})$$

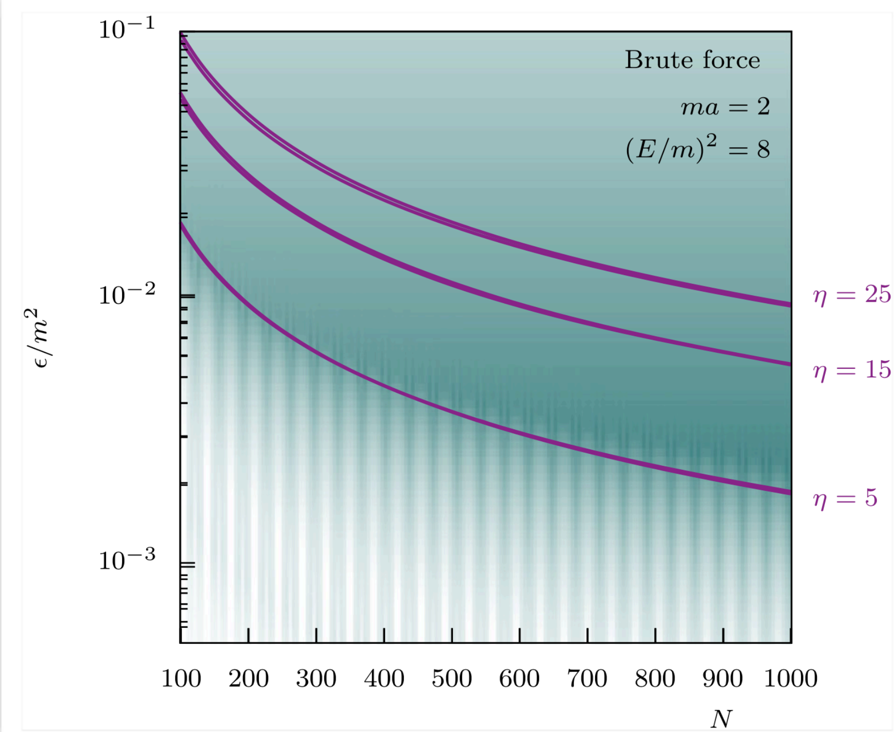
Extrapolations of the results

- Expansion in 1/N
- Epsilon regulator fixed to

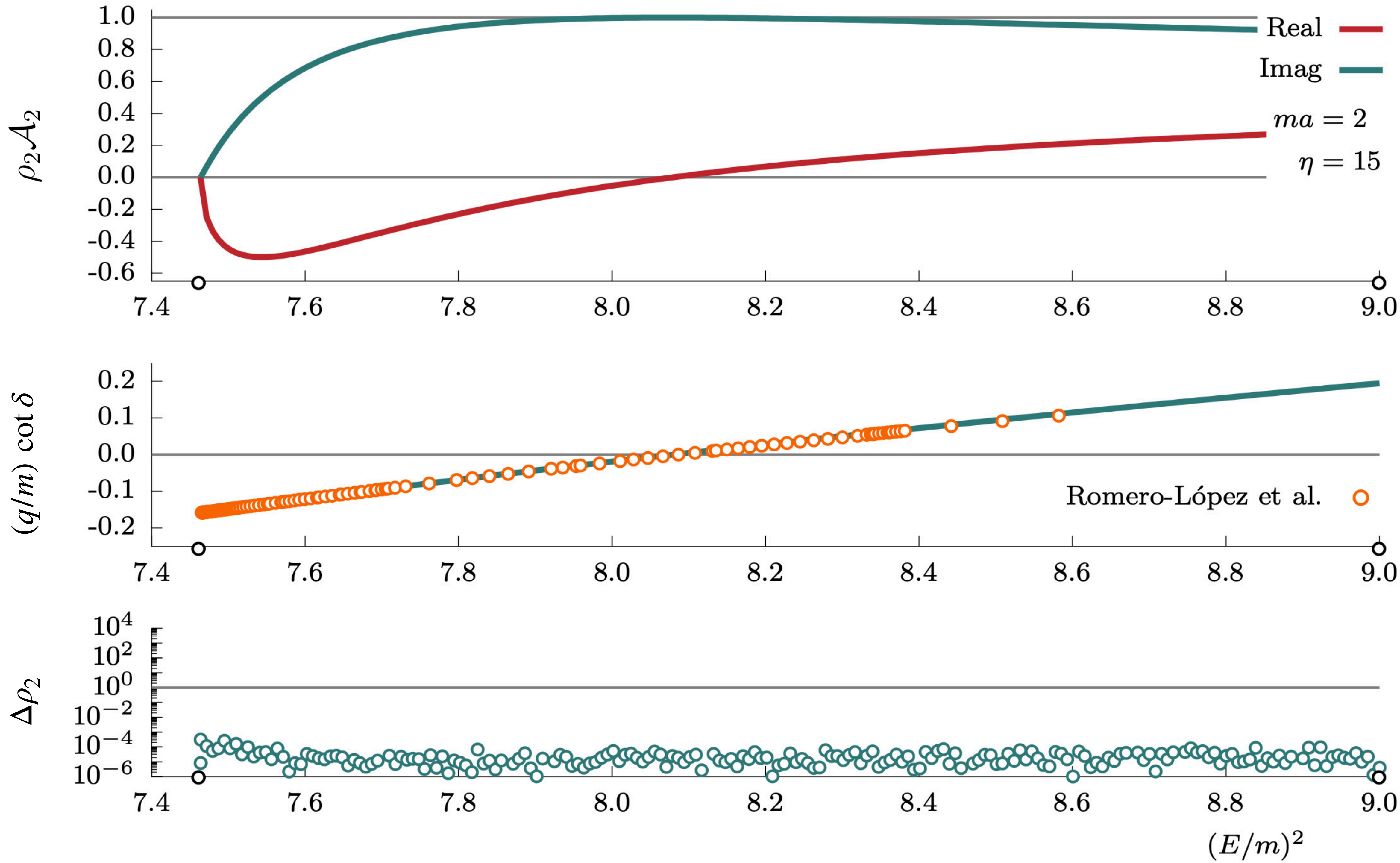
$$\epsilon \propto \eta / N$$



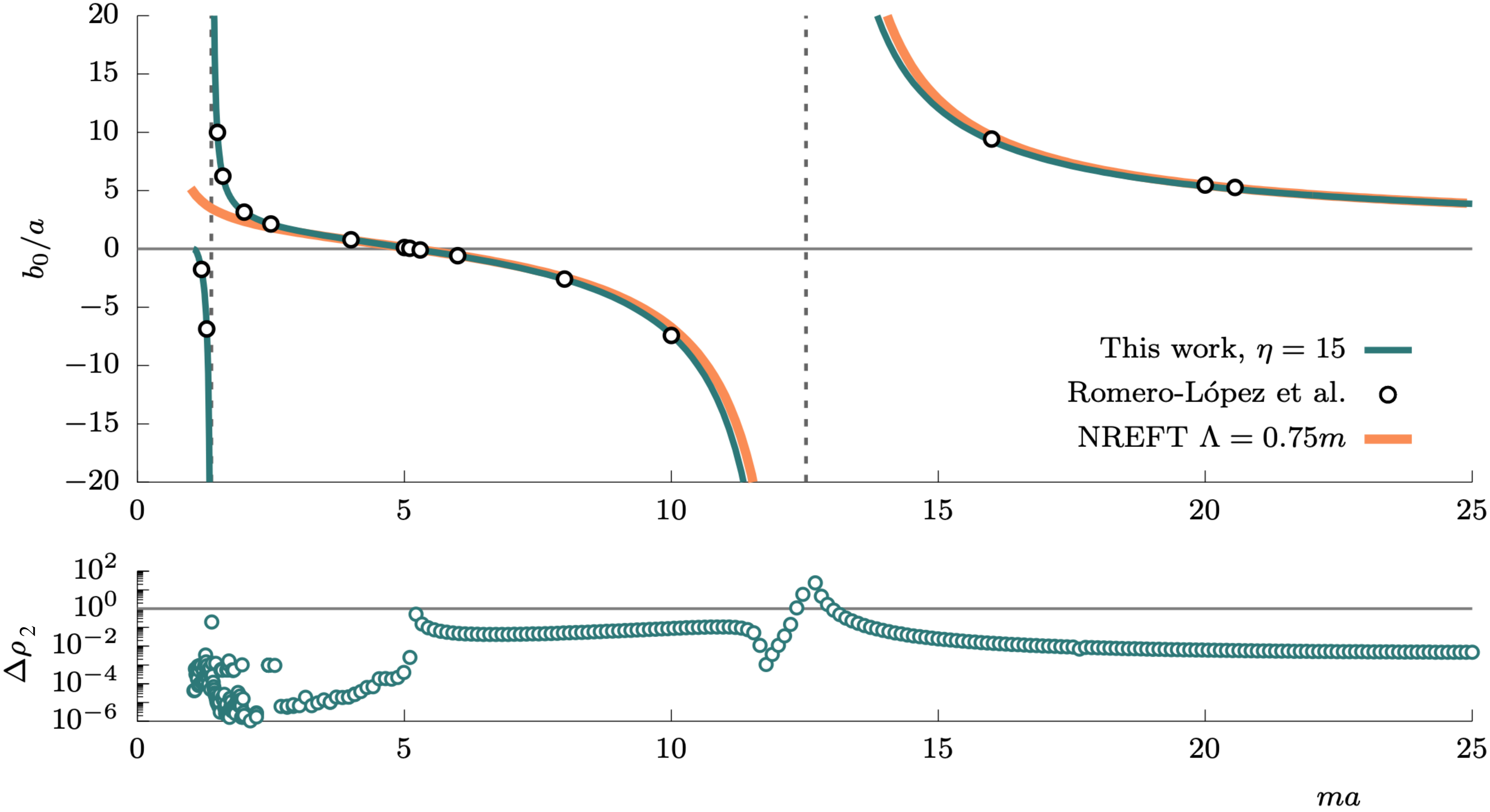




Example results, $M^2 = 3m^2$



Example result, three-body bound state



Romero-Lopez et al., JHEP 10 (2019) 007

Bedaque et al., Nucl. Phys. A 646 (1999) 444

Conclusions

- Systematic procedure for solving the integral equations
- Agreement with previous studies

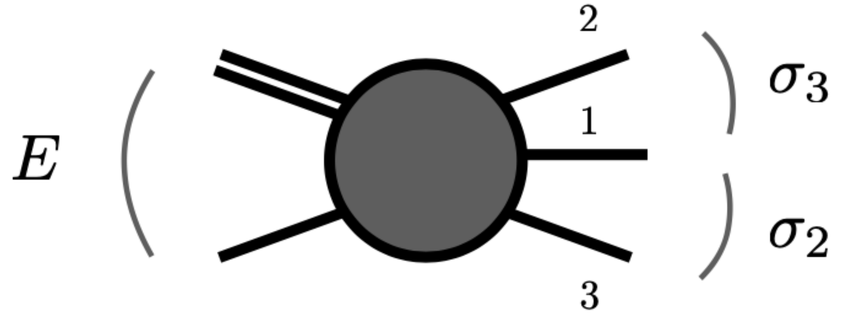
Future prospects

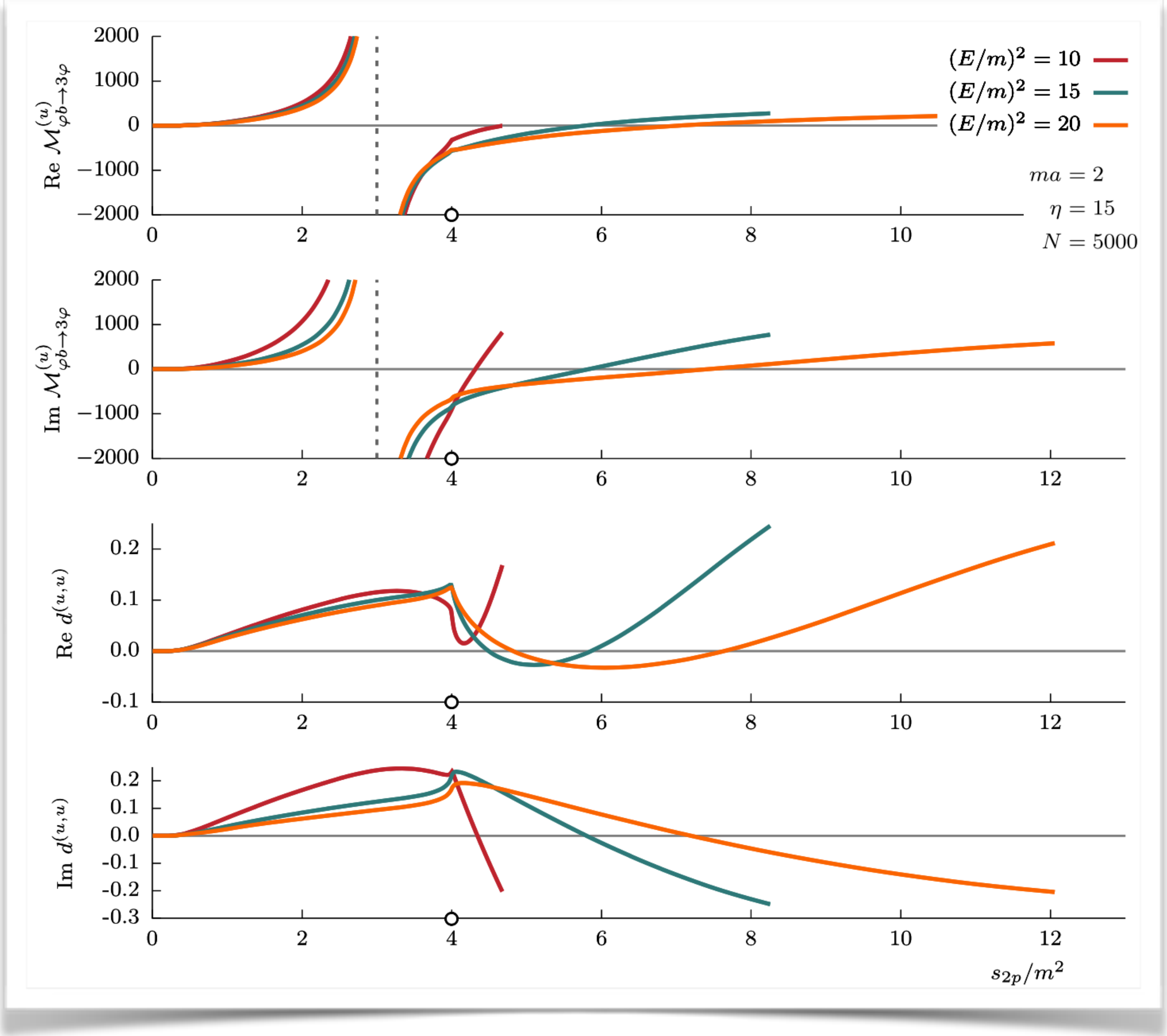
- Continuing below the two-body threshold and to complex energies
- Efimov physics?
- Controlling the scheme/ cutoff dependence

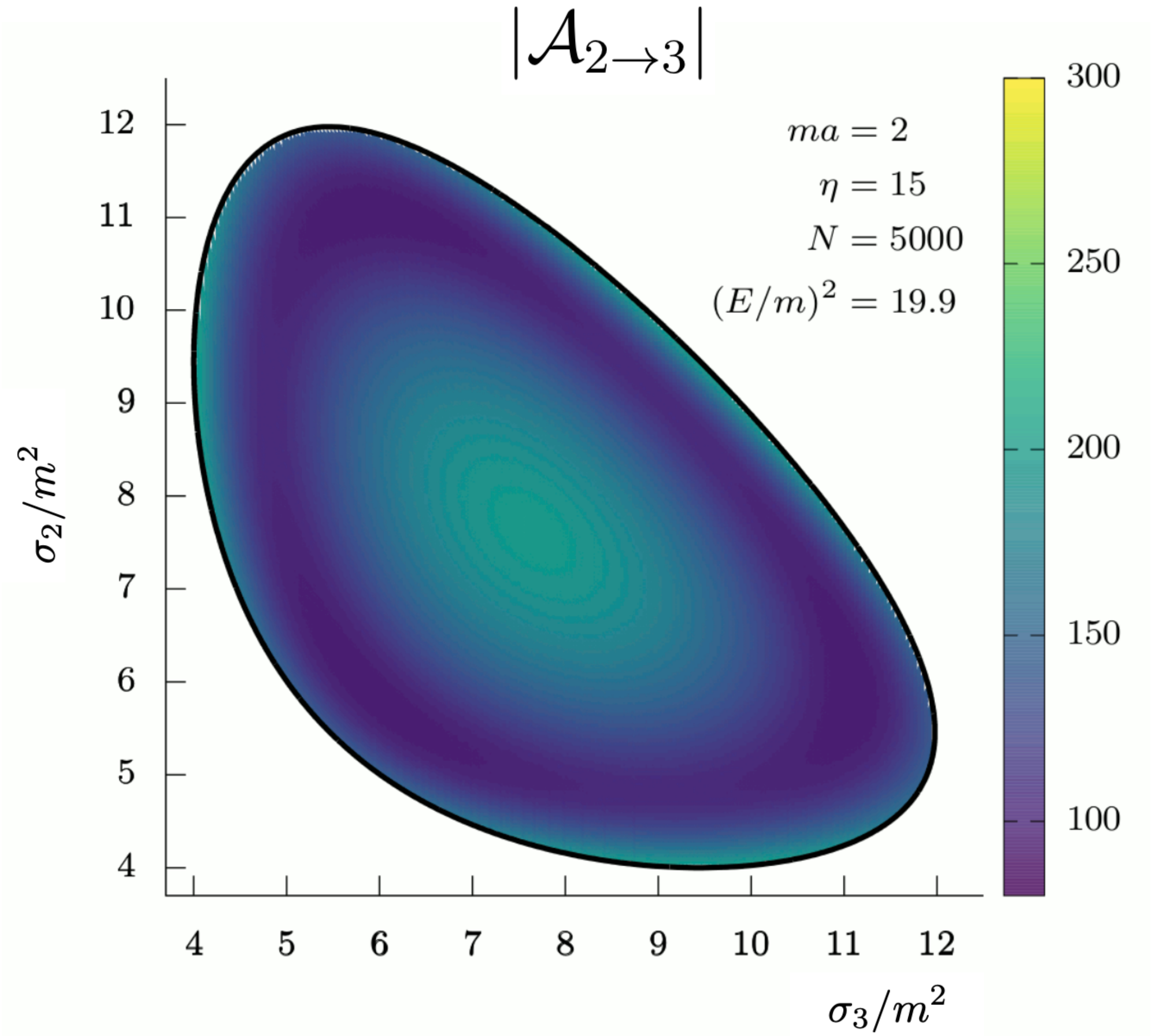
DO I HAVE SOME TIME LEFT?

Example results, 2⟶3 amplitude

- We are not limited below the three-body threshold,



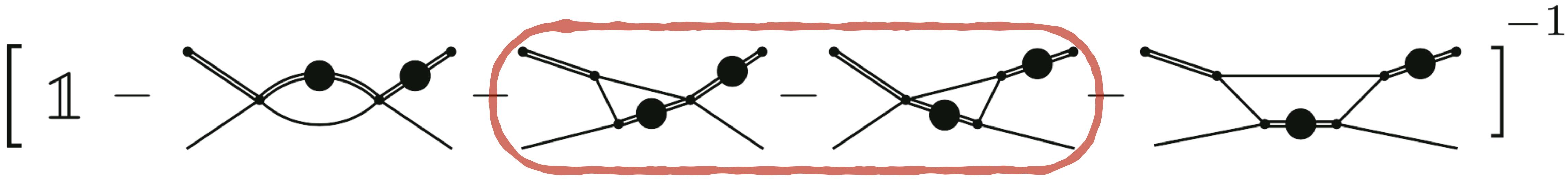


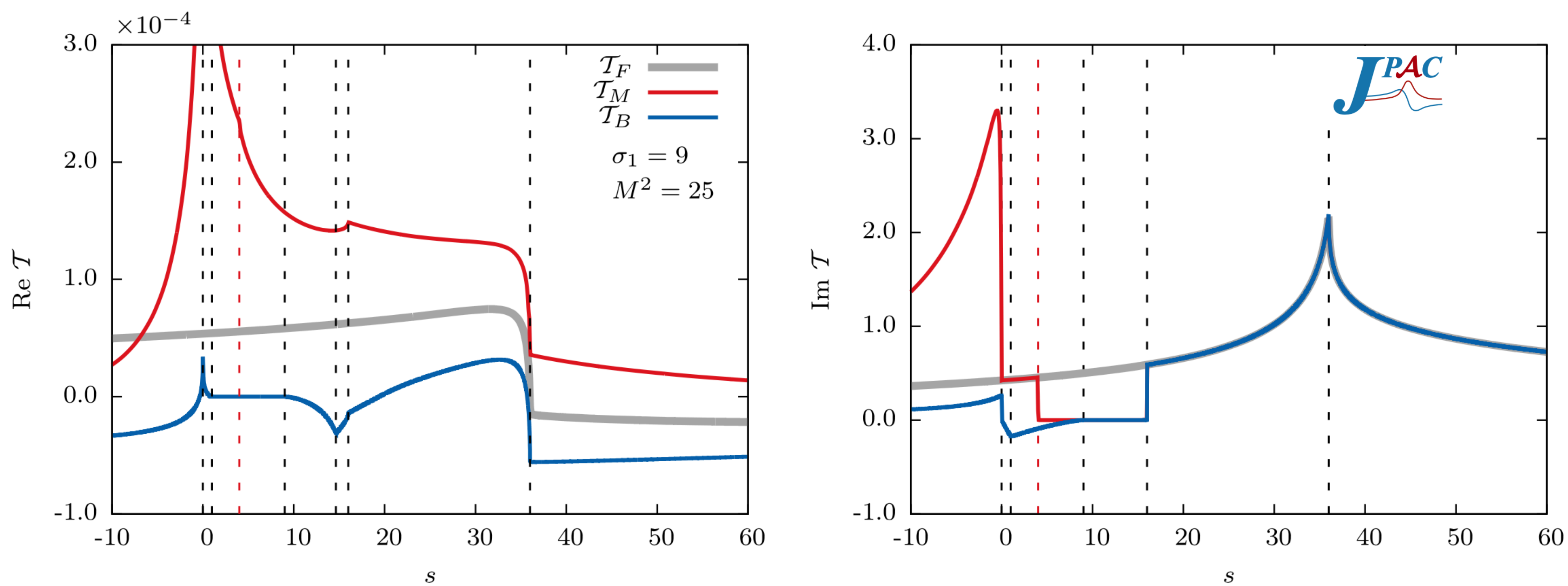


Bound-state–particle scattering

- Model study—formation of the three-body bound states
- Analytic properties of the B-matrix formalism

S. Dawid, A. Szczepaniak, Phys.Rev.D 103 (2021) 1, 014009

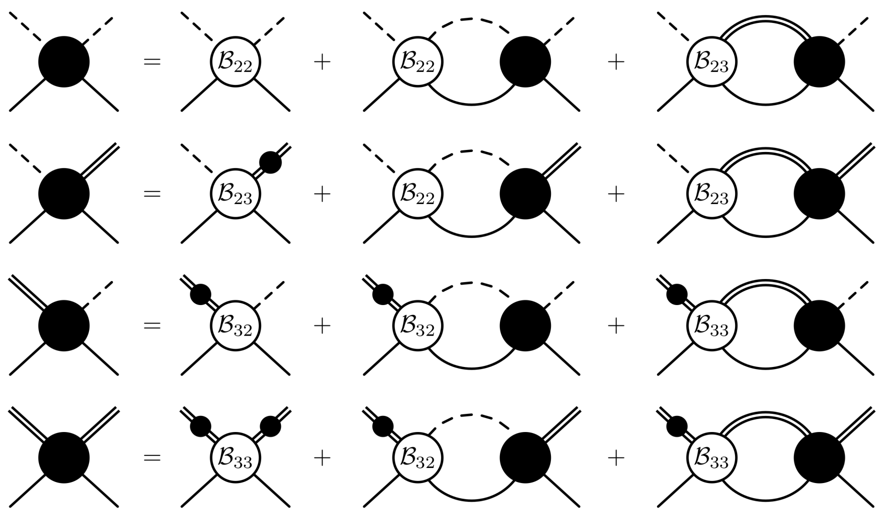




Jackura et al [JPAC], Eur. Phys. J C 79, no. 1, 56 (2019)

Generalization of the B-matrix equations

- Bound-state energies outside of the physical region
- Multi-hadron scattering requires generalization to all channels



$$\mathcal{A}_{22} = \mathcal{B}_{22} + \int_{\hat{\boldsymbol{k}}} \mathcal{B}_{22} i\rho_2 \mathcal{A}_{22} + \int_{\boldsymbol{q}} \mathcal{B}_{23,\boldsymbol{q}}\, \mathcal{A}_{32,\boldsymbol{q}},$$

$$\mathcal{A}_{23,\boldsymbol{p}} = \mathcal{B}_{23,\boldsymbol{p}}\, \mathcal{F}_{\boldsymbol{p}} + \int_{\hat{\boldsymbol{k}}} \mathcal{B}_{22} i\rho_2 \mathcal{A}_{23,\boldsymbol{p}} + \int_{\boldsymbol{q}} \mathcal{B}_{23,\boldsymbol{q}}\, \mathcal{A}_{33,\boldsymbol{q}\boldsymbol{p}},$$

$$\mathcal{A}_{32,\boldsymbol{p}'} = \mathcal{F}_{\boldsymbol{p}'}\, \mathcal{B}_{32,\boldsymbol{p}'} + \int_{\hat{\boldsymbol{k}}} \mathcal{F}_{\boldsymbol{p}'} \mathcal{B}_{32,\boldsymbol{p}'} i\rho_2 \mathcal{A}_{22} + \int_{\boldsymbol{q}} \mathcal{F}_{\boldsymbol{p}'}\, \mathcal{B}_{33,\boldsymbol{p}'\boldsymbol{q}}\, \mathcal{A}_{32,\boldsymbol{q}},$$

$$\mathcal{A}_{33,\boldsymbol{p}'\boldsymbol{p}} = \mathcal{F}_{\boldsymbol{p}'}\, \mathcal{B}_{33,\boldsymbol{p}'\boldsymbol{p}}\, \mathcal{F}_{\boldsymbol{p}} + \int_{\hat{\boldsymbol{k}}} \mathcal{F}_{\boldsymbol{p}'}\, \mathcal{B}_{32,\boldsymbol{p}'} i\rho_2 \mathcal{A}_{23,\boldsymbol{p}} + \int_{\boldsymbol{q}} \mathcal{F}_{\boldsymbol{p}'}\, \mathcal{B}_{33,\boldsymbol{p}'\boldsymbol{q}}\, \mathcal{A}_{33,\boldsymbol{q}\boldsymbol{p}}\,.$$

- Satisfies unitarity above the three-particle threshold
- **Approximation:** all multi-particle interactions are constant and real (couplings g_{ij})

$$a_{33}(s) = \frac{g_{33}}{1 - g_{22} i\rho_2 - g_{33}\,\mathcal{I}(s)},$$

$$a_{22}(s) = \frac{g_{22}}{1 - g_{22} i\rho_2 - g_{33}\,\mathcal{I}(s)}.$$

Analyticity of the B-matrix equations

- Solutions do not satisfy unitarity below the three-body threshold
- Spurious singularities start arbitrarily close to the two-body threshold

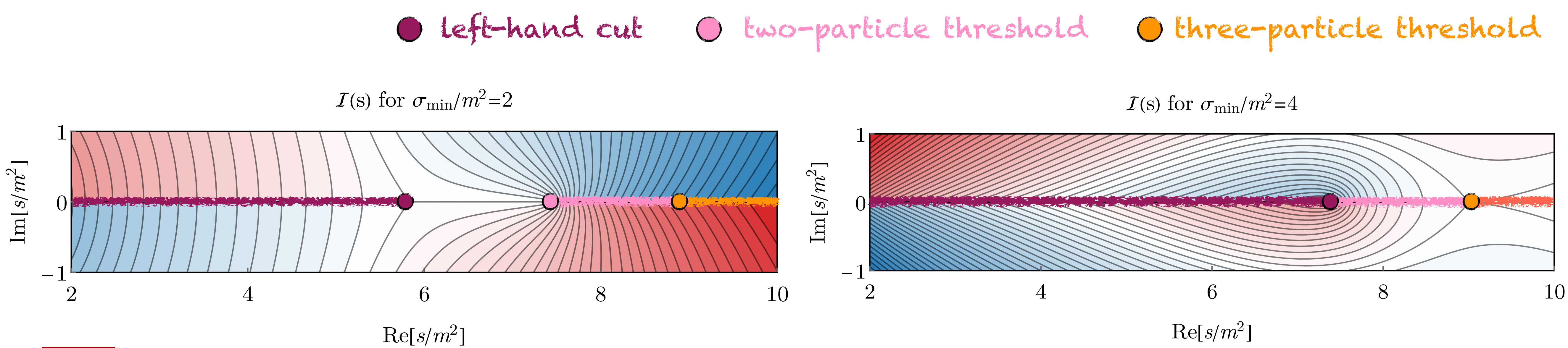
Kernel suffering from non-physical left-hand cuts:

$$\mathcal{I}(s) = \int_{\sigma_{\min}}^{(\sqrt{s}-m)^2} \frac{d\sigma_{\boldsymbol{q}}}{2\pi} \, \tau(s,\sigma_{\boldsymbol{q}}) \, \mathcal{F}(\sigma_{\boldsymbol{q}})$$

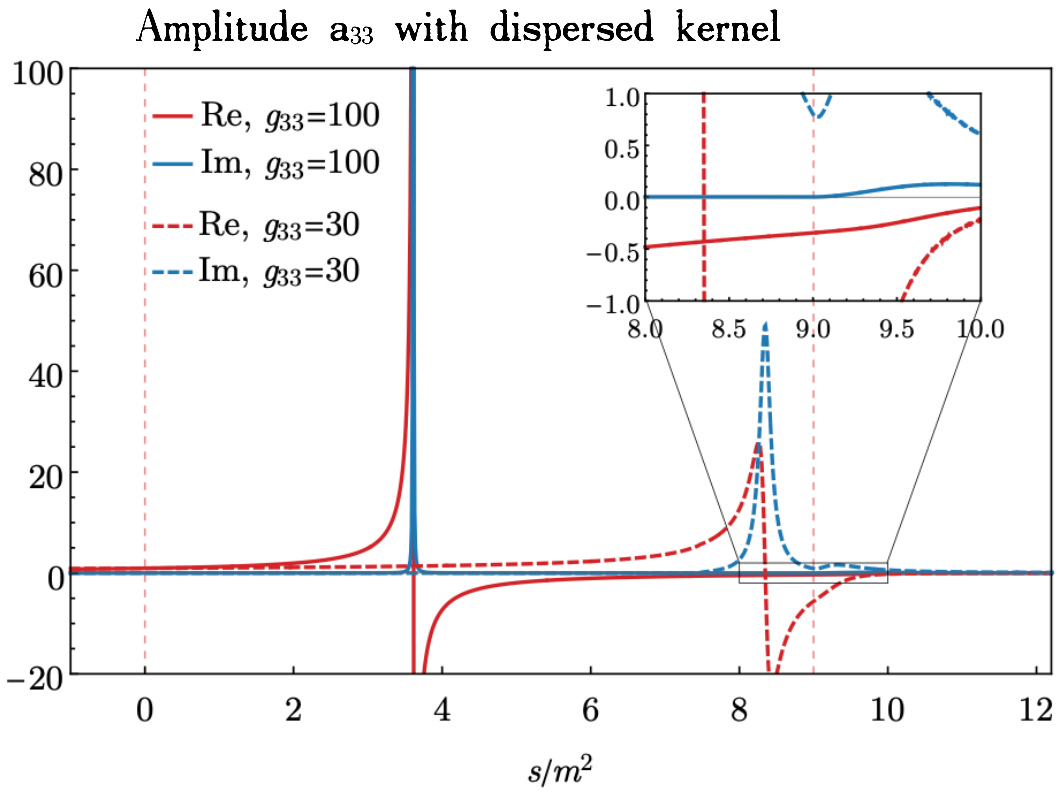
- Dispersion procedure ensures analyticity

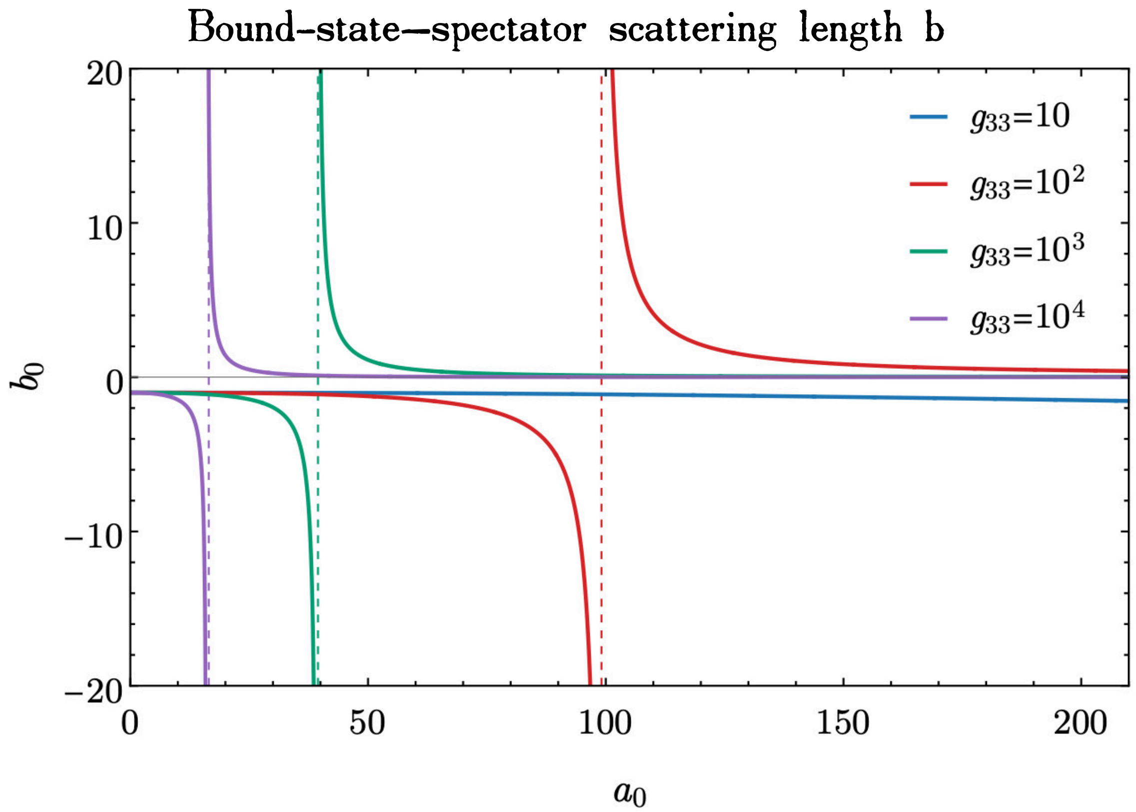
Kernel free from those problems:

$$\mathcal{I}_d(s) = \frac{(s-s_s)^2}{\pi} \int_{(3m)^2}^{\infty} ds' \, \frac{\mathrm{Im}\, \mathcal{I}(s')}{(s'-s-i\epsilon)(s'-s_s-i\epsilon)^2}$$



Results





$$b_0 = -\frac{g_{22}}{\left(1 - g_{33}\,\mathcal{I}_d(s_{\mathrm{th},2})\right)}$$

- Spurious singularities pushed to non-physical Riemann sheets, can study physics,
- General dispersion procedure for the three-body unitarity formalism is needed,

THANK YOU