Investigations of decuplet baryons from meson–baryon interactions in the HAL QCD method

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- Introduction
- HAL QCD method
- Setups for $N\pi$ and $\Xi\bar{K}$ interactions
- Results

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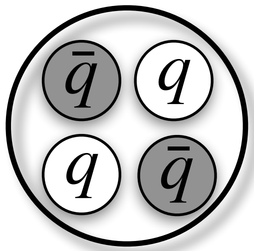
Introduction

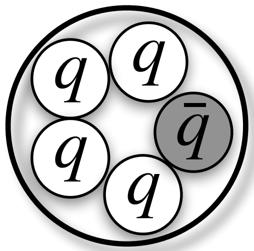
Motivation

- Most of all hadrons can be explained in the quark model

- Some exceptions have been found in experiments:
  Exotic hadrons

  unstable particles (resonances)
  living in **high energy** region





- **QCD describes all hadrons** although it needs non-perturbative calculations

Lattice QCD studies of hadron resonances are important for identifying exotic hadrons

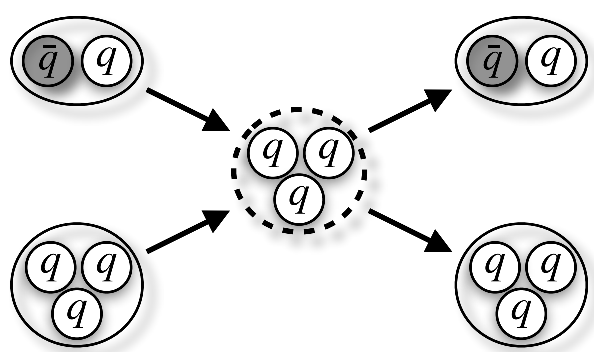
How to see hadron resonances in lattice QCD

- **Masses** can be estimated from 2-point functions

$$\langle 0 | O(t+t_0) O^\dagger(t_0) | 0 \rangle \simeq | \langle H, \mathbf{p}=\mathbf{0} | O(0) | 0 \rangle |^2 \, e^{-m_H t}$$

- To know **decay rates**, we need to see **hadron scatterings**

Hadron scatterings in lattice QCD

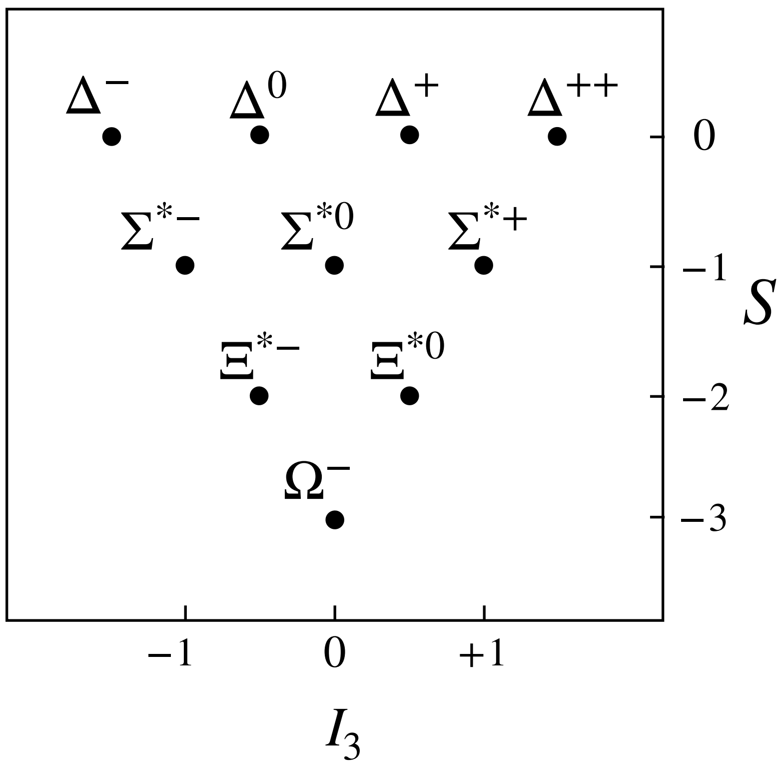


- Finite volume method [Lüscher 1991]
  - cheap numerical cost
  - difficult for systems including baryons
- HAL QCD method [Ishii, Aoki, Hatsuda 2007]
  - efficient for systems including baryons
  - expensive numerical cost
- …

- baryon-baryon
- meson-baryon

Decuplet baryons

- Flavor-SU(3) symmetric, spin 3/2 baryons, living in low-energy region
- All of them are resonances except for Ω baryon



$\Delta(1232)$ (resonance)
$\cdots$decay into $N\pi$

Ω^- (stable particle)
$\cdots$ bound state of $\Xi\bar{K}$?

- see the difference from $N\pi$ **and $\Xi\bar{K}$ interactions**
- use **HAL QCD method** to extract interactions
- use heavier quarks where Δ exists as a bound state

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HAL QCD method

Idea of HAL QCD method

[Ishii, Aoki, Hatsuda 2007]

- we get **interaction potential** from Schrödinger equation

$$\boxed{\int d^3r'\; U(\mathbf{r},\mathbf{r}')\Psi^W(\mathbf{r}') = \left(\frac{k^2}{2\mu} - H_0\right)\Psi^W(\mathbf{r})}$$

non-local potential

$(W = \sqrt{k^2+m_1^2} + \sqrt{k^2+m_2^2})$

$\Psi^W(\mathbf{r}) = \langle 0 | O_1(\mathbf{x}+\mathbf{r},0) O_2(\mathbf{x},0) | 2H, W \rangle$: NBS wave function

- derivative expansion

$$U(\mathbf{r},\mathbf{r}') = \sum_{k=0}^{\infty} V_k(\mathbf{r})(\nabla)^k \delta^{(3)}(\mathbf{r}-\mathbf{r}')$$

- Naive way to get NBS w.f.: use **4-point function** $F(t,\mathbf{r})$

$$F(t,\mathbf{r}) = \langle 0 | O_1(\mathbf{x}+\mathbf{r},t) O_2(\mathbf{x},t)\; \bar{J}(0) | 0 \rangle \overset{t\to\infty}{\to} \Psi^{W_0}(\mathbf{r})\langle 2H, W_0 | \bar{J}(0) | 0 \rangle\; e^{-W_0 t}$$

difficult when we consider baryons

- exponentially suppressed S/N ratio
- fake plateau [Iritani et al. 2016]

Time-dependent HAL QCD method [Ishii et al. 2011]

- R-correlator

$(\Delta W_n = W_n - m_1 - m_2)$

$$R(t,\mathbf{r}) = \frac{F(t,\mathbf{r})}{C_1(t)C_2(t)} \simeq \sum_n \underline{A_n\ \Psi^{W_n}(\mathbf{r})\ e^{-\Delta W_n t}} + (inelastic)$$

2-point function

elastic part

- Each elastic term satisfies the Schrödinger equation

$$\left(\frac{k_n^2}{2\mu} - H_0\right) \underline{A_n \Psi^{W_n}(\mathbf{r})\ e^{-\Delta W_n t}} = \int d^3r'\ U(\mathbf{r},\mathbf{r}')\ \underline{A_n \Psi^{W_n}(\mathbf{r})\ e^{-\Delta W_n t}}$$

$$= \Delta W_n + \frac{1+3\delta^2}{8\mu}\Delta W_n^2 - \frac{\delta^2}{2m_1 m_2}\Delta W_n^3 + O(\Delta W_n^4) = -\frac{\partial}{\partial t} + \frac{1+3\delta^2}{8\mu}\frac{\partial^2}{\partial t^2} + \frac{\delta^2}{2m_1m_2}\frac{\partial^3}{\partial t^3} + O(\Delta W_n^4)$$

$\left(\delta = \frac{m_1 - m_2}{m_1 + m_2}\right)$

$$\boxed{\int d^3r'\ U(\mathbf{r},\mathbf{r}')R(\mathbf{r}',t) \simeq \left(-\frac{\partial}{\partial t} + \frac{1+3\delta^2}{8\mu}\frac{\partial^2}{\partial t^2} + \frac{\delta^2}{2m_1m_2}\frac{\partial^3}{\partial t^3} - H_0\right)R(\mathbf{r},t) + O(\Delta W_n^4)}$$

- We do not need to pick up only ground state

➡ **applicable to baryons**

Contents

Setups for $N\pi$ and $\Xi\bar{K}$ interactions

Target

- $I = 3/2,\ J^P = 3/2^+\ N\pi$ $\cdots$ channel to Δ baryon
- $S = -3,\ I = 0,\ J^P = 3/2^+\ \Xi\bar{K}$ $\cdots$ channel to Ω baryon

4-point functions

$$F^{N\pi}_{\alpha j_z}(\mathbf{r}, t) = \langle \pi^+(\mathbf{r} + \mathbf{x}, t) p_\alpha(\mathbf{x}, t) \bar{\Delta}^{++}_{j_z}(t_0) \rangle$$

3-quark type operators

$$\bar{\Delta}^{++} \propto \sum_{\mathbf{y}} \bar{u}(\mathbf{y})\bar{u}(\mathbf{y})\bar{u}(\mathbf{y})$$

$$F^{\Xi\bar{K}}_{\alpha j_z}(\mathbf{r}, t) = \langle (\bar{K}(\mathbf{r} + \mathbf{x}, t) \Xi_\alpha(\mathbf{x}, t))_{I=0}\ \bar{\Omega}^-_{j_z}(t_0) \rangle$$

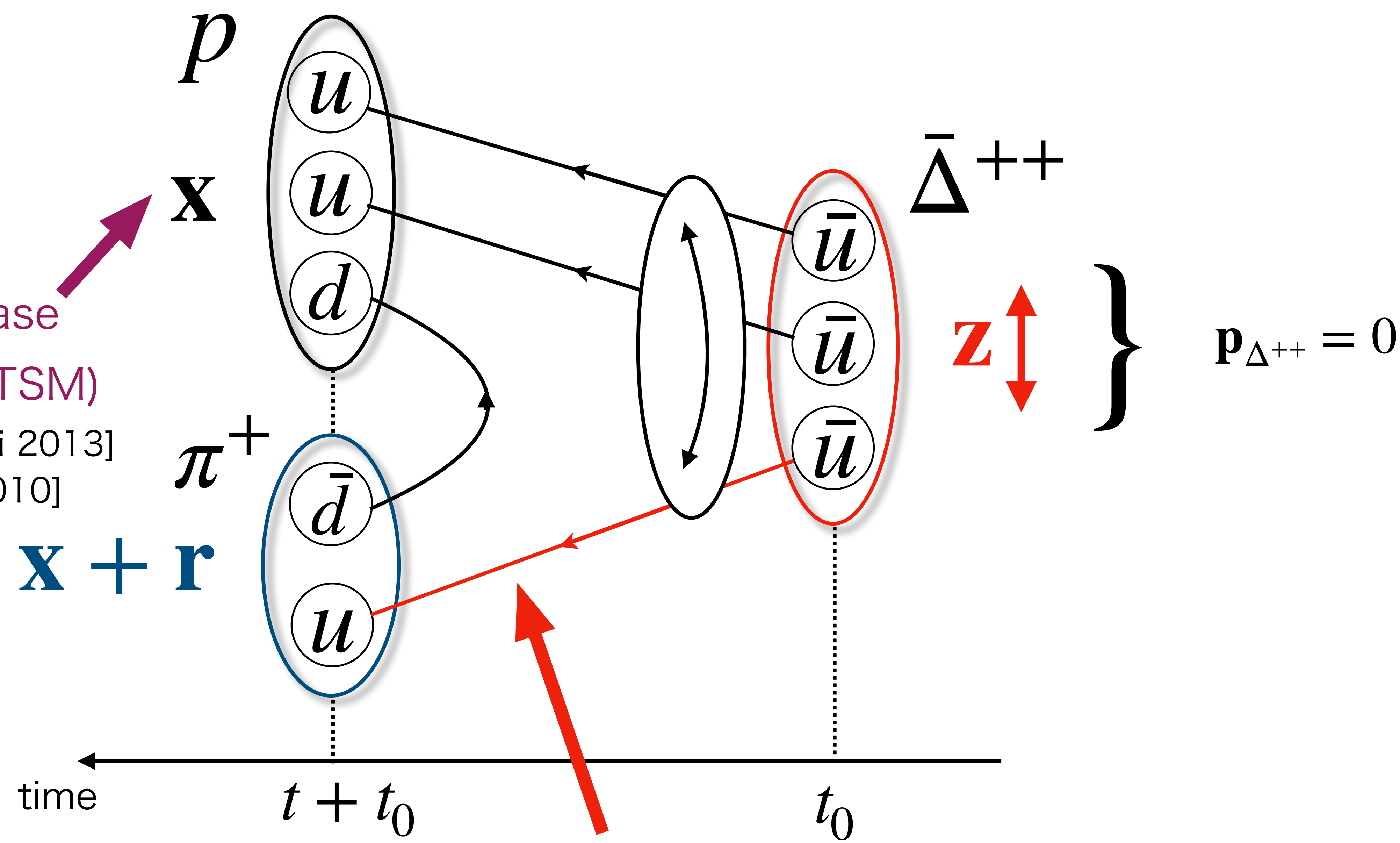
3-quark type operators

$$\bar{\Omega}^- \propto \sum_{\mathbf{y}} \bar{s}(\mathbf{y})\bar{s}(\mathbf{y})\bar{s}(\mathbf{y})$$

Quark contraction for $N\pi$

move $\mathbf{x}$ to increase statistics (CAA + TSM)

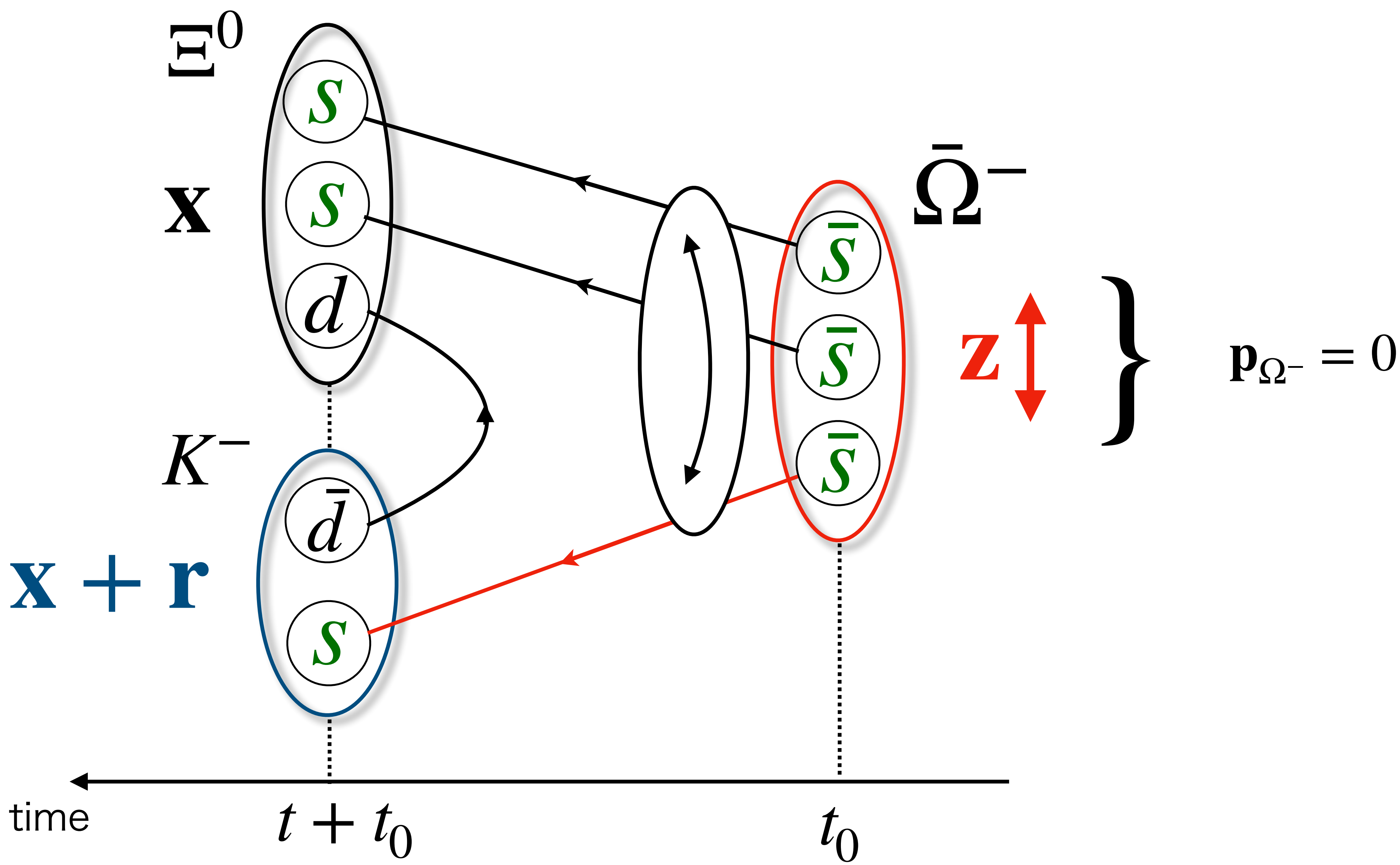
[Blum, Izubuchi, Shintani 2013]
[Bali, Collins, Schäfer 2010]



all-to-all propagator calculation
using stochastic method

$\mathbf{z}$: summed, $\mathbf{x}$: fixed, $\mathbf{r}$:argument of 4-point functions

Quark contraction for $\Xi\bar{K}$



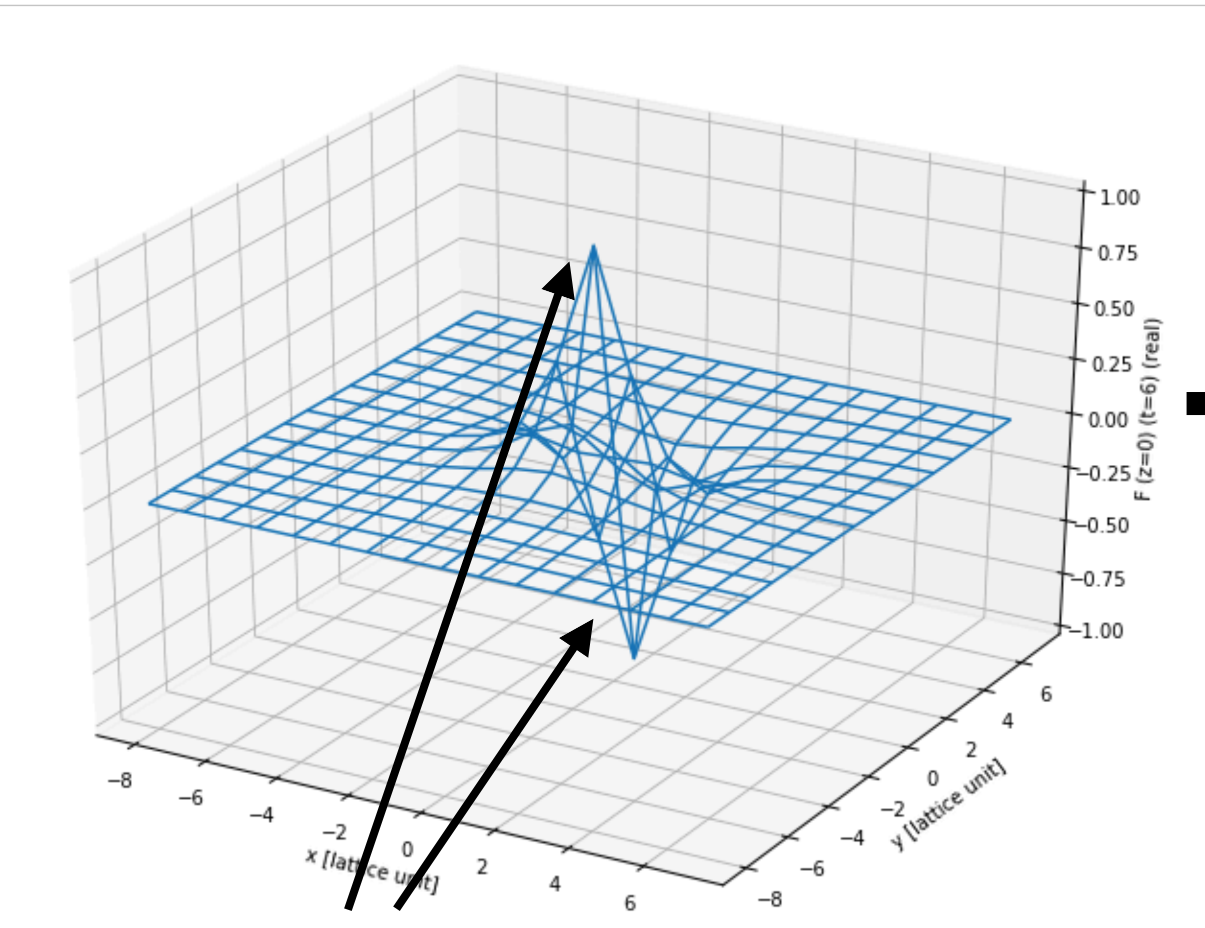
calculate in the same way as that for $N\pi$

Problems due to short-range structure

- quarks at the sink: point-like

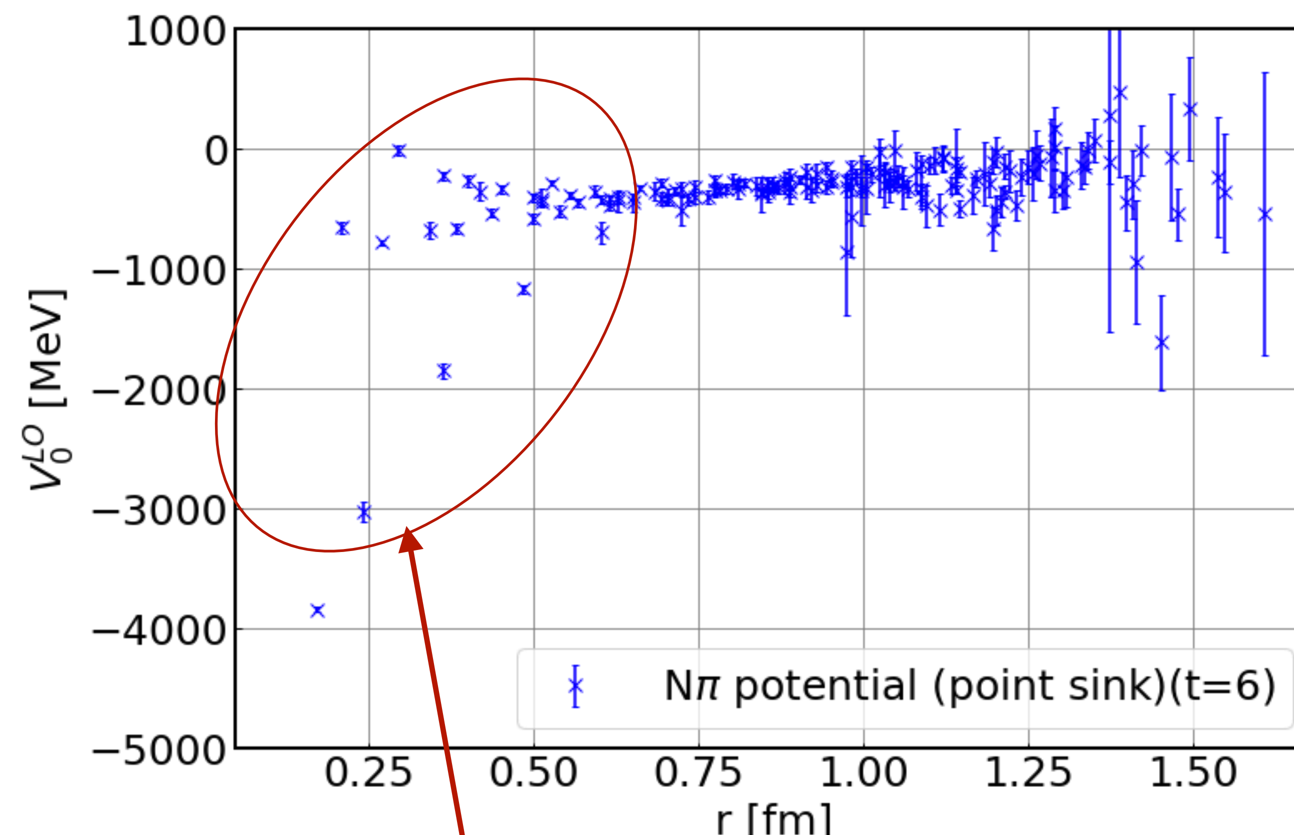
(These are the test simulations on small volume)

$F^{N\pi}(x, y, z = 0, t = 6)$



sharp structure

$N\pi$ LO potential (t=6)



impossible to fit this potential

$$F^{N\pi}(\mathbf{r}) \sim \langle u(\mathbf{r})u(\mathbf{r})\ \underline{d(\mathbf{r})\bar{d}(\mathbf{0})}\ u(\mathbf{0})\ \bar{J}\rangle$$
$$\underset{r\to 0}{\propto} \frac{1}{r^3} Y_{1,m}(\Omega)$$

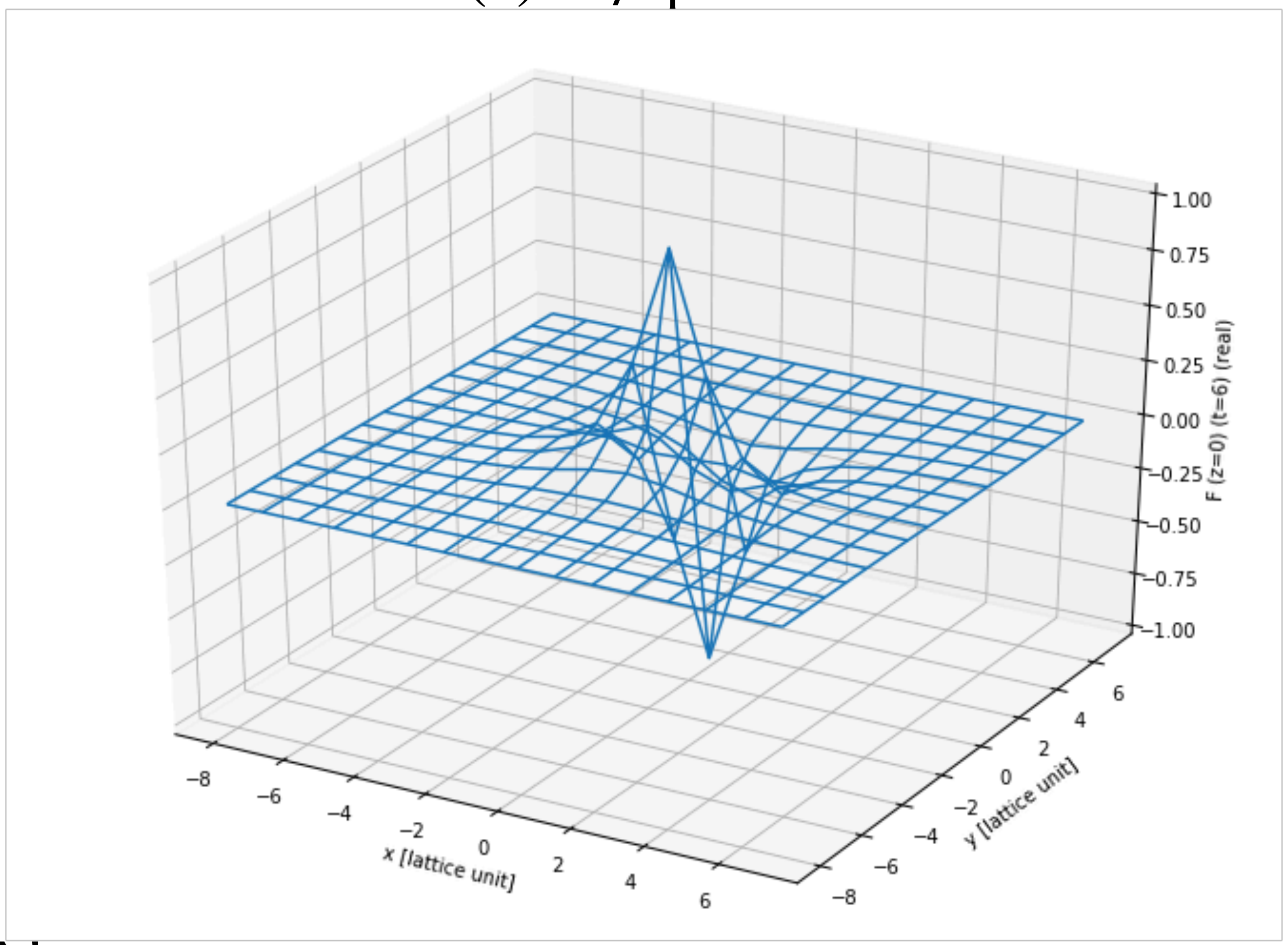
(The same thing happens in I=1 P-wave $\pi\pi$ system)

(cf. Y. Akahoshi's talk)

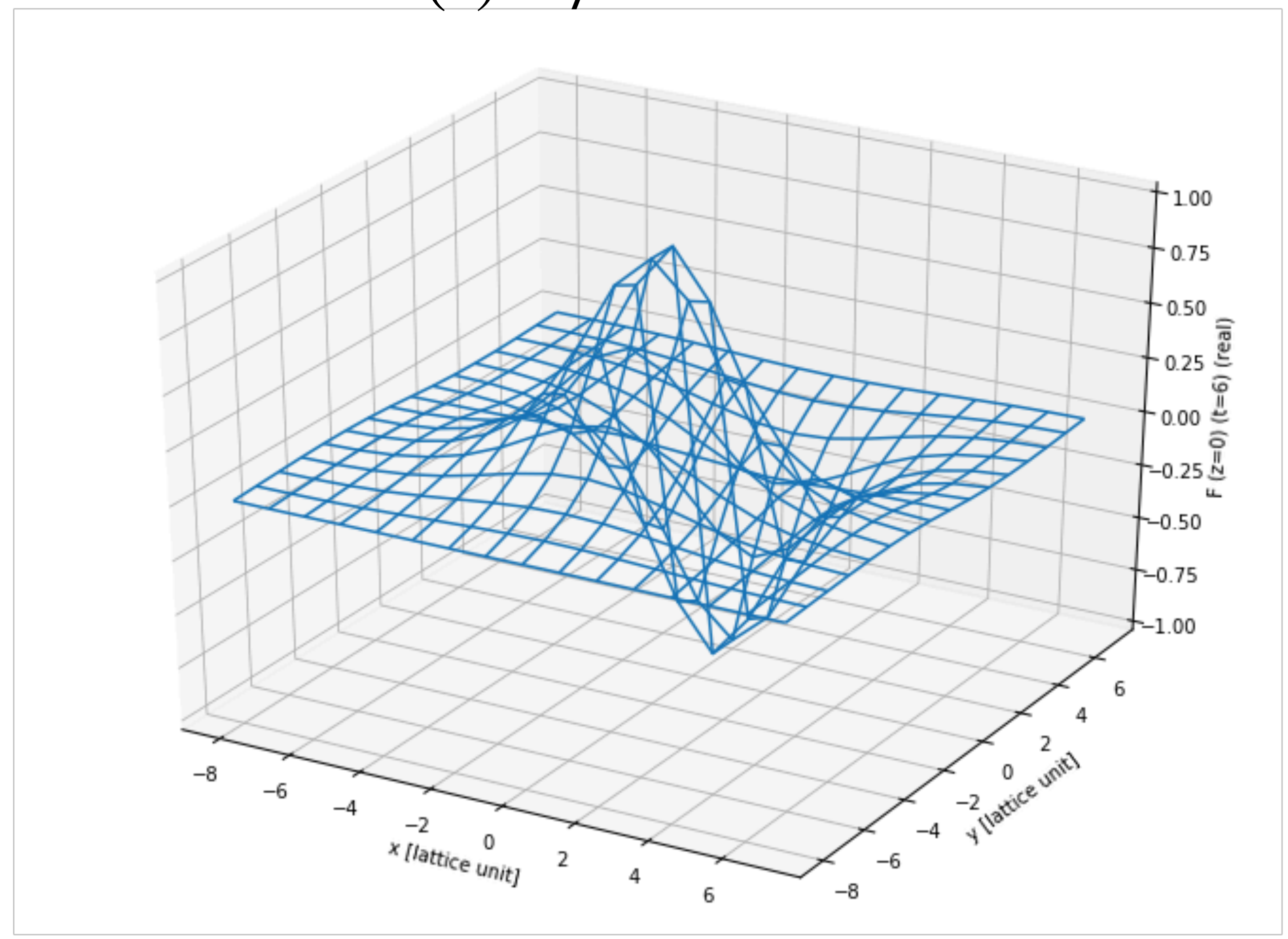
Solution to the fitting problem

- One of the solutions to this problem: **smeared sink**

$F^{N\pi}(\mathbf{r})$ w/ point sink



$F^{N\pi}(\mathbf{r})$ w/ smeared sink



Note:

- Too much smeared sink may enhance the contribution from non-locality of potentials

☑ suppressed in low-energy region in the similar systems

- $I = 1$ S-wave NK w/ $m_s = m_l$
- $I = 1$ S-wave $\Xi\bar{K}$

Numerical setup

- PACS-CS, (2+1)-flavor conf.: [PACS-CS Collab., 2009]
  Iwasaki gauge action + Wilson-clover quark action
  (gauge fixing, 450 confs.)
- $a = 0.0907$ fm on $32^3 \times 64$ lattices
- 16 timeslice at the sink t_0
- smearing quarks both at the source and **sink**
- use 64 spatial points to increase statistics
  $\mathbf{x} = (0,0,0), (0,0,8) \ldots (24,24,24)$

	mass (MeV)
m_π	411
m_K	635
m_N	1215
m_Ξ	1503
m_Δ	1513
m_Ω	1840

- LO analysis in the time-dependent HAL QCD method
- $m_\pi + m_N > m_\Delta,\ m_{\bar{K}} + m_\Xi > m_\Omega$ ➡ $\Delta,\ \Omega \cdots$ **bound states**

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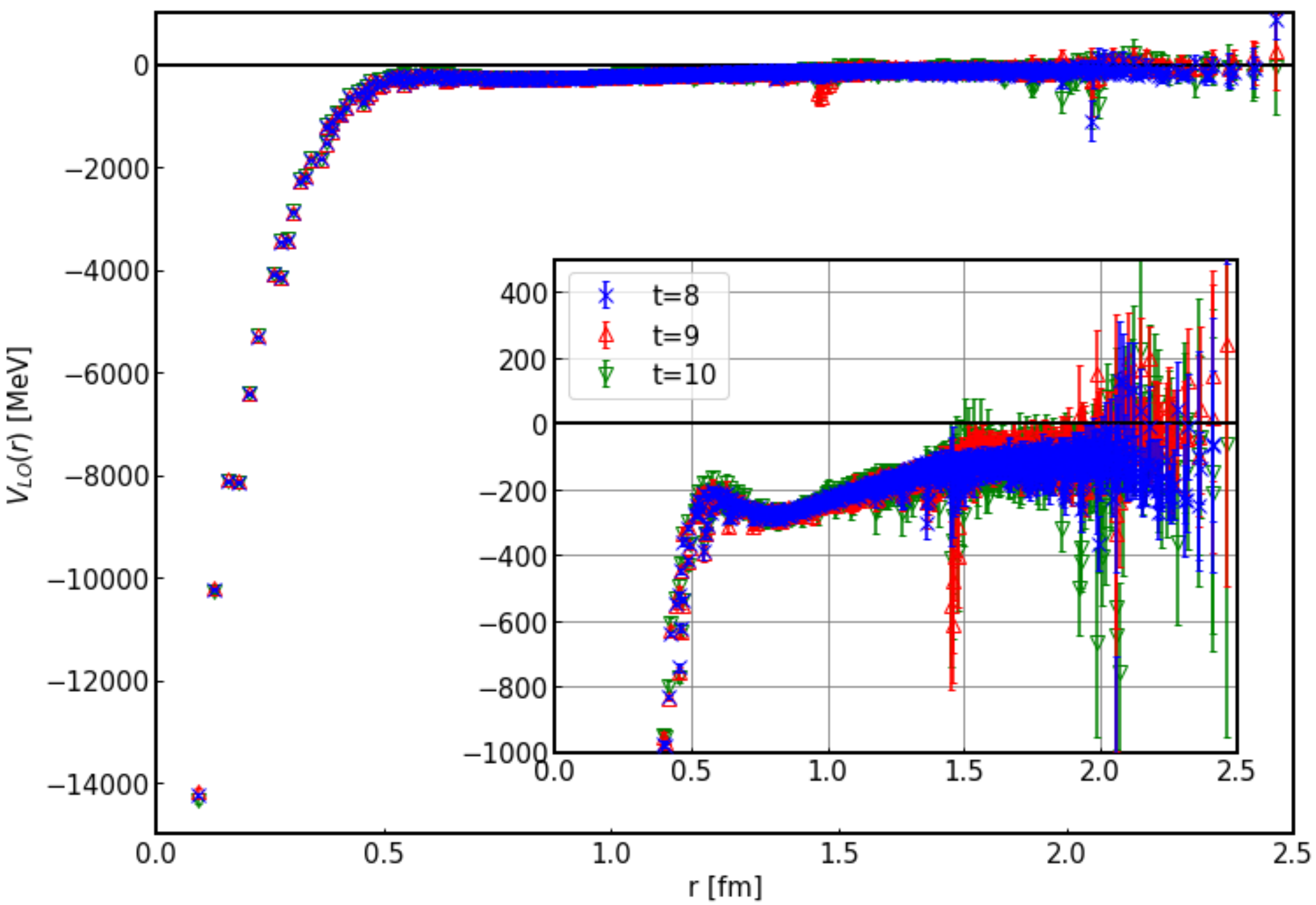
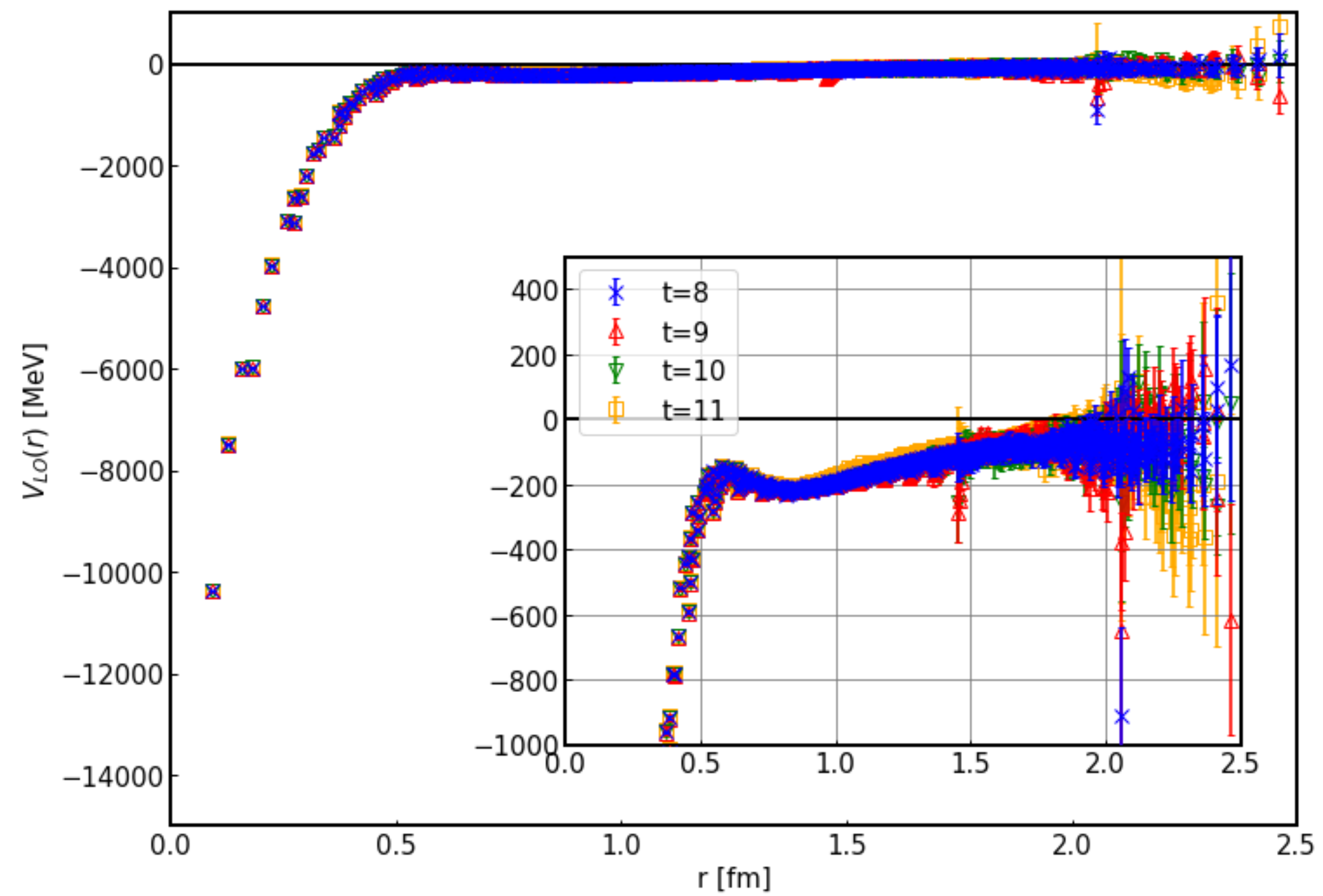
Results

Potentials

$N\pi$



$\Xi\bar{K}$



- Strong attractions to create bound states, horns at $r \approx 0.5$ fm
- Two potentials are **quite similar**

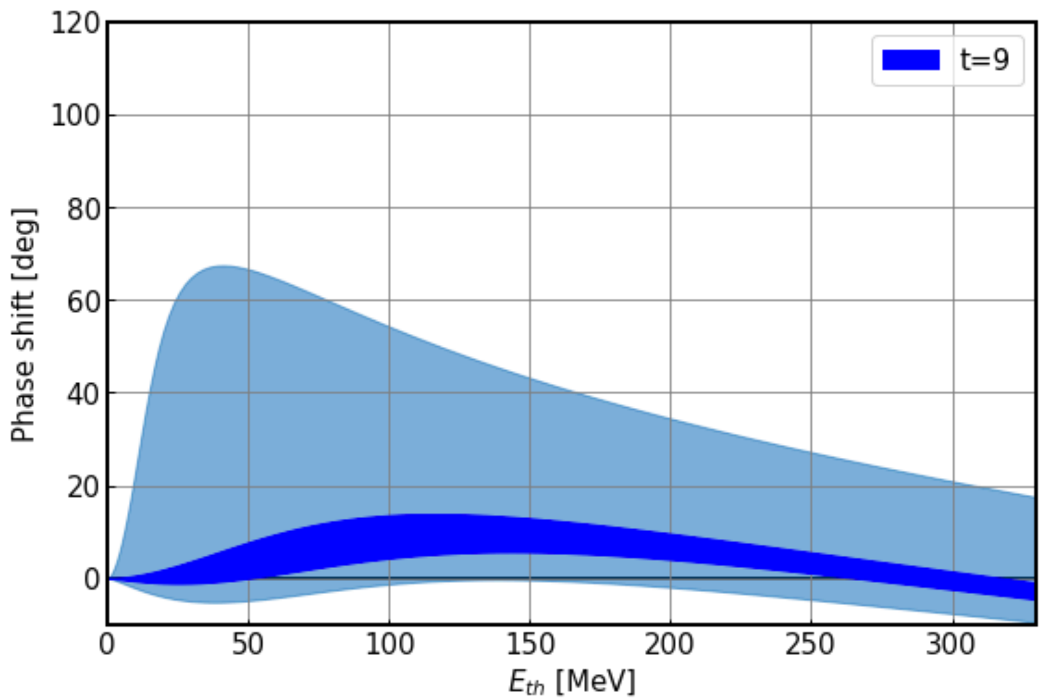
➡ • contribution from quark-antiquark pair is dominant

• difference between Δ and Ω masses comes only from the reduced masses of $N\pi$ and $\Xi\bar{K}$

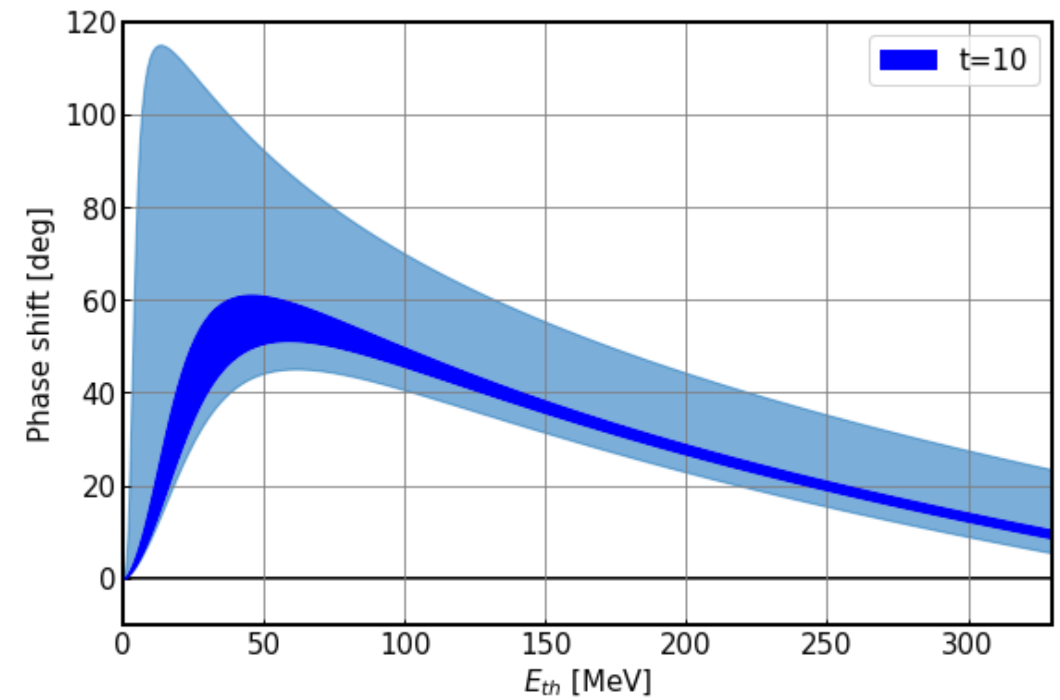
Phase shifts and binding energies

(The systematic error is estimated so that the **lattice artifact in short range**, **finite volume effect**, and **timeslice dependence** are taken into account.)

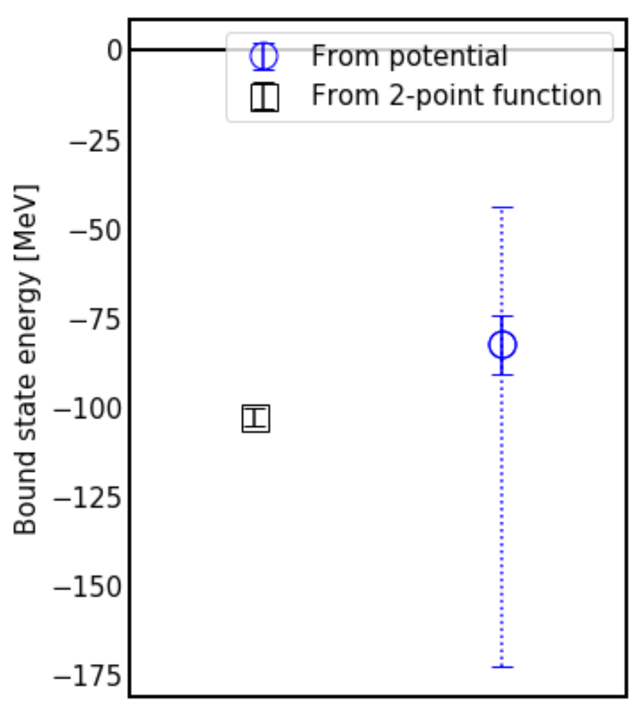
$N\pi$



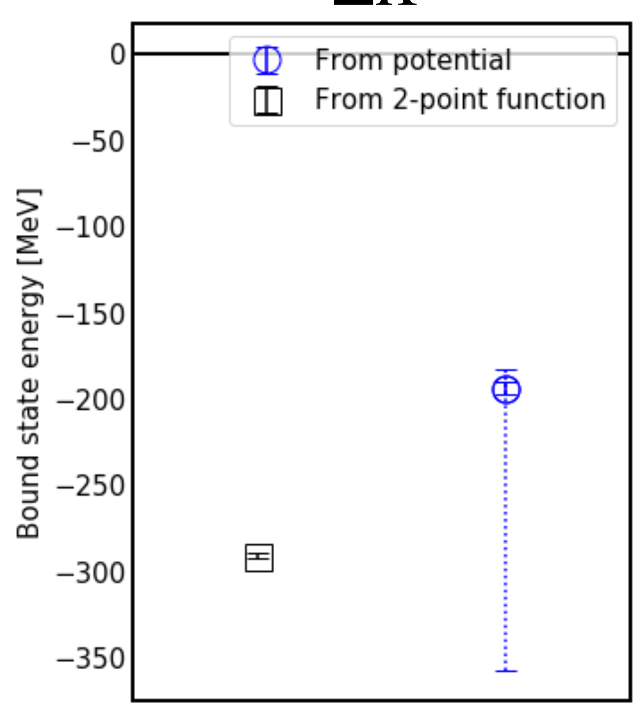
$\Xi\bar{K}$



$N\pi$



$\Xi\bar{K}$



- large systematic error but consistent with the energies estimated from 2pt functions

Summary

- We analyze **P-wave I=3/2 $N\pi$ and I=0 $\Xi\bar{K}$ interactions in the HAL QCD method** at heavy pion mass, where Δ and Ω baryons exist as **bound states.**

- We use **3-quark-type source operators with zero momentum.**

- The **two similar potentials** indicate that only the kinematics of $N\pi$ and $\Xi\bar{K}$ contribute to the difference between Δ and Ω.

Future Works

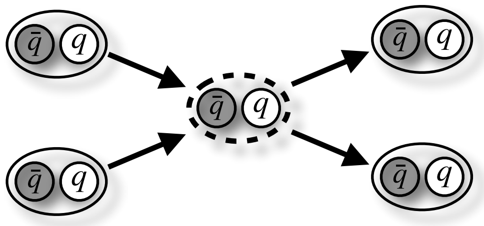
- Analysis of baryon resonances in **more realistic** setups
    - need **larger volume** and **NLO analysis in derivative expansion** with meson-baryon source operators

- Application to **exotic hadrons**
    - $\Lambda(1405)$, P_c pentaquarks, charm version of Θ^+ pentaquarks

Backups

Analysis of hadron resonances in lattice QCD

• meson-meson ➡ meson resonances



FV

Many studies being done

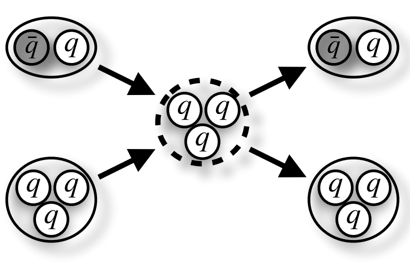
- ρ [M. Werner et al., 2019]
- σ, f_0, f_2 [R. Briceno et al., 2018]
- κ, K^* [G. Rendon et al., 2020]

HAL QCD

I=1 P-wave $\pi\pi \rightarrow \rho$

[Akahoshi et al. 2021]

• meson–baryon ➡ baryon resonances



FV

I=3/2 P-wave $N\pi \rightarrow \Delta$

[S. Paul et al., 2018]
[C. W. Andersen et al., 2021]

HAL QCD

None

Hadron scattering in lattice QCD

2-body hadron state with energy W

NBS wave function: $\Psi^W(\mathbf{r}) = \langle 0 | O_1(\mathbf{x}+\mathbf{r},0) O_2(\mathbf{x},0) | 2H, W \rangle$

hadron operators

asymptotic behavior

$$\Psi^{W,l}(r) \underset{r>R}{\propto} \frac{\sin(kr - \frac{l}{2}\pi + \delta^l(k))}{kr} e^{i\delta^l(k)} \qquad (W = \sqrt{k^2+m_1^2} + \sqrt{k^2+m_2^2})$$

phase shift for 2-body hadron scattering

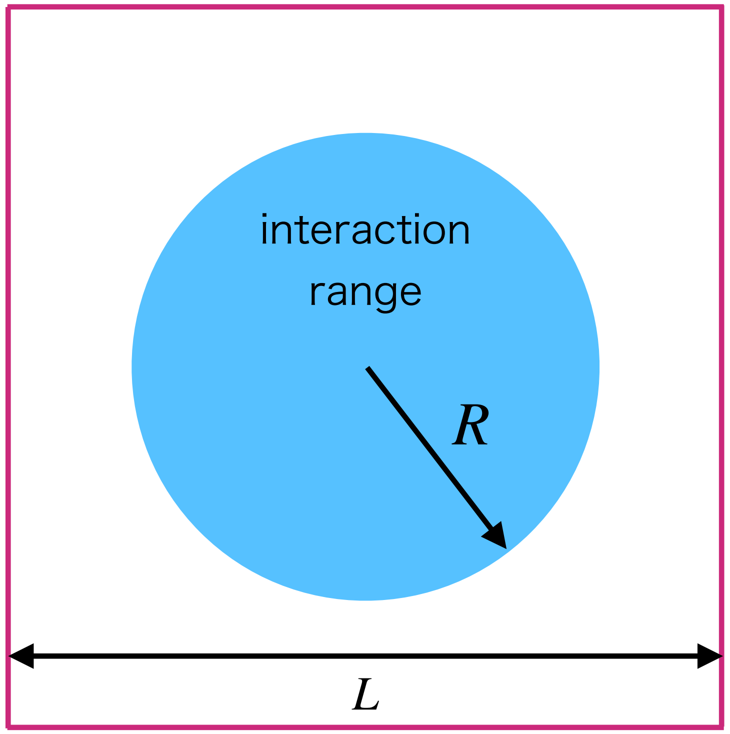
- Finite volume method [Lüscher, 1991]

：use periodic boundary condition of NBS wave functions in finite volume to extract phase shift

e.g.) 1+1 dim

$$\Psi^W(x=-L/2) = \Psi^W(x=L/2)$$

$$\Longleftrightarrow \quad e^{i(-\frac{kL}{2}-\delta(k))} = e^{i(\frac{kL}{2}+\delta(k))}$$



- HAL QCD method [Ishii, Aoki, Hatsuda, 2007]

Δ source opeartors (3-quark type)

$$\bar{\Delta}^{++}_{+3/2}(t_0) = -\sum_{\mathbf{y}} \epsilon_{abc}(\bar{u}_b(\mathbf{y},t_0)\Gamma_+\bar{u}^T_c(\mathbf{y},t_0))\bar{u}_{a,0}(\mathbf{y},t_0)$$

$$\bar{\Delta}^{++}_{+1/2}(t_0) = -\frac{1}{\sqrt{3}}\sum_{\mathbf{y}} \epsilon_{abc}[\sqrt{2}(\bar{u}_b(\mathbf{y},t_0)\Gamma_z\bar{u}^T_c(\mathbf{y},t_0))\bar{u}_{a,0}(\mathbf{y},t_0) + (\bar{u}_b(\mathbf{y},t_0)\Gamma_+\bar{u}^T_c(\mathbf{y},t_0))\bar{u}_{a,1}(\mathbf{y},t_0)]$$

$$\bar{\Delta}^{++}_{-1/2}(t_0) = \frac{1}{\sqrt{3}}\sum_{\mathbf{y}} \epsilon_{abc}[\sqrt{2}(\bar{u}_b(\mathbf{y},t_0)\Gamma_z\bar{u}^T_c(\mathbf{y},t_0))\bar{u}_{a,1}(\mathbf{y},t_0) + (\bar{u}_b(\mathbf{y},t_0)\Gamma_-\bar{u}^T_c(\mathbf{y},t_0))\bar{u}_{a,0}(\mathbf{y},t_0)]$$

$$\bar{\Delta}^{++}_{-3/2}(t_0) = \sum_{\mathbf{y}} \epsilon_{abc}(\bar{u}_b(\mathbf{y},t_0)\Gamma_-\bar{u}^T_c(\mathbf{y},t_0))\bar{u}_{a,1}(\mathbf{y},t_0)$$

$$\left(\Gamma_\pm = \frac{1}{2}C(\gamma_2 \pm i\gamma_1),\ \Gamma_z = \frac{-i}{\sqrt{2}}C\gamma_3\right)$$

Ω source opeartors (3-quark type)

$$\bar{\Omega}^{++}_{+3/2}(t_0) = -\sum_{\mathbf{y}} \epsilon_{abc}(\bar{s}_b(\mathbf{y},t_0)\Gamma_+\bar{s}^T_c(\mathbf{y},t_0))\bar{s}_{a,0}(\mathbf{y},t_0)$$

$$\bar{\Omega}^{++}_{+1/2}(t_0) = -\frac{1}{\sqrt{3}}\sum_{\mathbf{y}} \epsilon_{abc}[\sqrt{2}(\bar{s}_b(\mathbf{y},t_0)\Gamma_z\bar{s}^T_c(\mathbf{y},t_0))\bar{s}_{a,0}(\mathbf{y},t_0) + (\bar{s}_b(\mathbf{y},t_0)\Gamma_+\bar{s}^T_c(\mathbf{y},t_0))\bar{s}_{a,1}(\mathbf{y},t_0)]$$

$$\bar{\Omega}^{++}_{-1/2}(t_0) = \frac{1}{\sqrt{3}}\sum_{\mathbf{y}} \epsilon_{abc}[\sqrt{2}(\bar{s}_b(\mathbf{y},t_0)\Gamma_z\bar{s}^T_c(\mathbf{y},t_0))\bar{s}_{a,1}(\mathbf{y},t_0) + (\bar{s}_b(\mathbf{y},t_0)\Gamma_-\bar{s}^T_c(\mathbf{y},t_0))\bar{s}_{a,0}(\mathbf{y},t_0)]$$

$$\bar{\Omega}^{++}_{-3/2}(t_0) = \sum_{\mathbf{y}} \epsilon_{abc}(\bar{s}_b(\mathbf{y},t_0)\Gamma_-\bar{s}^T_c(\mathbf{y},t_0))\bar{s}_{a,1}(\mathbf{y},t_0)$$

Stochastic estimation

$\underset{a}{\eta(x)_\alpha}$ $\cdots$ noise vector that satisfies

$$\begin{cases} \langle\langle \underset{a}{\eta(x)_\alpha}\,\underset{b}{\eta^*(y)_\beta} \rangle\rangle = \delta_{xy}\delta_{ab}\delta_{\alpha\beta} \\ \underset{a}{\eta(x)_\alpha}\,\underset{a}{\eta^*(x)_\alpha} = 1 \ (\textbf{for all } x, a, \alpha) \end{cases}$$

Propagator D^{-1} can be written as

$$\underset{a}{q(x)_\alpha} \longleftarrow \underset{b}{\bar{q}(y)_\beta} = D^{-1}(x,y)^{ab}_{\alpha\beta} = \sum_{c,\gamma,z} D^{-1}(x,z)^{ac}_{\alpha\gamma}\underline{\delta_{zy}\delta_{cb}\delta_{\gamma\beta}}$$

$$= \sum_{c,\gamma,z} D^{-1}(x,z)^{ac}_{\alpha\gamma}\underline{\langle\langle \underset{c}{\eta(z)_\gamma}\,\underset{b}{\eta^*(y)_\beta} \rangle\rangle}$$

$$= \langle\langle (\underbrace{D^{-1}\eta}_{\equiv\psi})\underset{a}{(x)_\alpha}\,\underset{b}{\eta^*(y)_\beta} \rangle\rangle = \langle\langle (\underset{a}{\psi(x)_\alpha}\,\underset{b}{\eta^*(y)_\beta} \rangle\rangle$$

Stochastic estimation

$$\Leftrightarrow D^{-1}(x,y)^{ab}_{\alpha\beta} = \lim_{N_r\to\infty} \frac{1}{N_r} \sum_{r=1}^{N_r} \psi_{[r]}(x)_{\substack{\alpha \\ a}} \eta^*_{[r]}(y)_{\substack{\beta \\ b}}$$

$$\left(\psi \cdots \textbf{solution} \ \sum_{b,\beta,y} D(x,y)^{ab}_{\alpha\beta} \psi(y)_{\substack{\beta \\ b}} = \eta(x)_{\substack{\alpha \\ a}}\right)$$

Therefore, D^{-1} can be estimated by

$$\boxed{D^{-1}(x,y)^{ab}_{\alpha\beta} \approx \frac{1}{N_r} \sum_{r=1}^{N_r} \psi_{[r]}(x)_{\substack{\alpha \\ a}} \eta^*_{[r]}(y)_{\substack{\beta \\ b}}}$$

noisy estimation: very noisy ← $\eta(x)_{\substack{\alpha \\ a}}$ itself has $O(1)$ error

→ this noise can be reduced by using “dilution”

Stochastic estimation with dilution

ex) time dilution

decompose the noise vector

$$\eta(x)_{\substack{\alpha \\ a}} = \sum_{j=0}^{N_t-1} \eta^{(j)}(x)_{\substack{\alpha \\ a}} \quad \text{where} \quad \eta^{(j)}(x)_{\substack{\alpha \\ a}} = \begin{cases} \eta(x)_{\substack{\alpha \\ a}} & (\textbf{for } j = t) \\ 0 & (\textbf{for } j \neq t) \end{cases}$$

$$\begin{bmatrix} \eta(t=0) \\ \eta(t=1) \\ \eta(t=2) \\ \vdots \\ \vdots \end{bmatrix} = \underbrace{\begin{bmatrix} \eta(t=0) \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}}_{=\eta^{(0)}(t)} + \underbrace{\begin{bmatrix} 0 \\ \eta(t=1) \\ 0 \\ 0 \\ \vdots \end{bmatrix}}_{=\eta^{(1)}(t)} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \eta(t=2) \\ 0 \\ \vdots \end{bmatrix}}_{=\eta^{(2)}(t)} + \cdots$$

Stochastic estimation with dilution

ex) time dilution

$$D^{-1}(x,y)^{ab}_{\alpha\beta} = \sum_{c,\gamma,z} D^{-1}(x,z)^{ac}_{\alpha\gamma} \langle\langle \underset{c}{\eta(z)_\gamma} \underset{b}{\eta^*(y)_\beta} \rangle\rangle$$

$$= \sum_{c,\gamma,z} D^{-1}(x,z)^{ac}_{\alpha\gamma} \sum_{j,k=0}^{N_t-1} \underline{\langle\langle \underset{c}{\eta^{(j)}(z)_\gamma} \underset{b}{\eta^{(k)*}(y)_\beta} \rangle\rangle}$$

$j \neq k$ terms are noisy parts, not signals

$$\to \sum_{c,\gamma,z} D^{-1}(x,z)^{ac}_{\alpha\gamma} \sum_{j=0}^{N_t-1} \underline{\langle\langle \underset{c}{\eta^{(j)}(z)_\gamma} \underset{b}{\eta^{(j)*}(y)_\beta} \rangle\rangle}$$

Stochastic estimation with dilution

ex) time dilution

$$D^{-1}(x,y)^{ab}_{\alpha\beta} = \sum_{j=0}^{N_t-1} \langle\langle \underset{a}{(\psi^{(j)}(x)_\alpha} \underset{b}{\eta^{(j)*}(y)_\beta)} \rangle\rangle$$

$$\Big(\sum_{b,\beta,y} D(x,y)^{ab}_{\alpha\beta} \underset{b}{\psi^{(i)}(y)_\beta} = \underset{a}{\eta^{(i)}(x)_\alpha} \Big)$$

Therefore,

$$\boxed{D^{-1}(x,y)^{ab}_{\alpha\beta} \approx \frac{1}{N_r} \sum_{r=1}^{N_r} \sum_{j} \underset{a}{\psi^{(j)}_{[r]}(x)_\alpha} \underset{b}{\eta^{(j)*}_{[r]}(y)_\beta}}$$

Covariant approximation averaging (CAA)

$O[U] \cdots$ observable that is covariant under symmetry G

$$\Leftrightarrow O[U^g] = O^g[U] \text{ for all } g \in G$$

(ex) $G \cdots$ translation $x \to x + a$)

We define

$$O_G[U] = \frac{1}{N_G} \sum_{g \in G} O[U^g] = \frac{1}{N_G} \sum_{g \in G} O^g[U]$$

($N_G \cdots$ number of the element of G)

This variable satisfies

$$\langle O[U] \rangle = \langle O_G[U] \rangle \quad (\because \langle O[U^g] \rangle = \langle O[U] \rangle)$$

Covariant approximation averaging (CAA)

$O^{(appx)}[U] \cdots$ approximation of G which reduces computational cost

and we introduce

$$O_G^{(appx)}[U] = \frac{1}{N_G}\sum_{g\in G} O^{(appx)}[U^g] = \frac{1}{N_G}\sum_{g\in G} O^{(appx)g}[U]$$

Improved estimator is defined by

$$\boxed{O^{(imp)}[U] = O[U] - O^{(appx)}[U] + O_G^{(appx)}[U]}$$

and this satisfies

$$\langle O^{(imp)}[U]\rangle = \langle O[U]\rangle - \langle O^{(appx)}[U]\rangle + \underline{\langle O_G^{(appx)}[U]\rangle}$$

$$= \langle O^{(appx)}[U]\rangle$$

$$= \langle O[U]\rangle$$

CAA + Truncated solver method (TSM)

(same as all-mode averaging without low mode)

$$O^{(appx)} = O[S^{(appx)}[U]]$$

$$O_G^{(appx)} = \frac{1}{N_G} \sum_{g \in G} O[S^{(appx)g}[U]]$$

where

$$(S^{(appx)}b)_i = \sum_{i=1}^{N_{CG}} (H^i) c_i$$

relaxed stopping criterion in the CG method
(Truncated solver method)

CAA + TSM in this situation

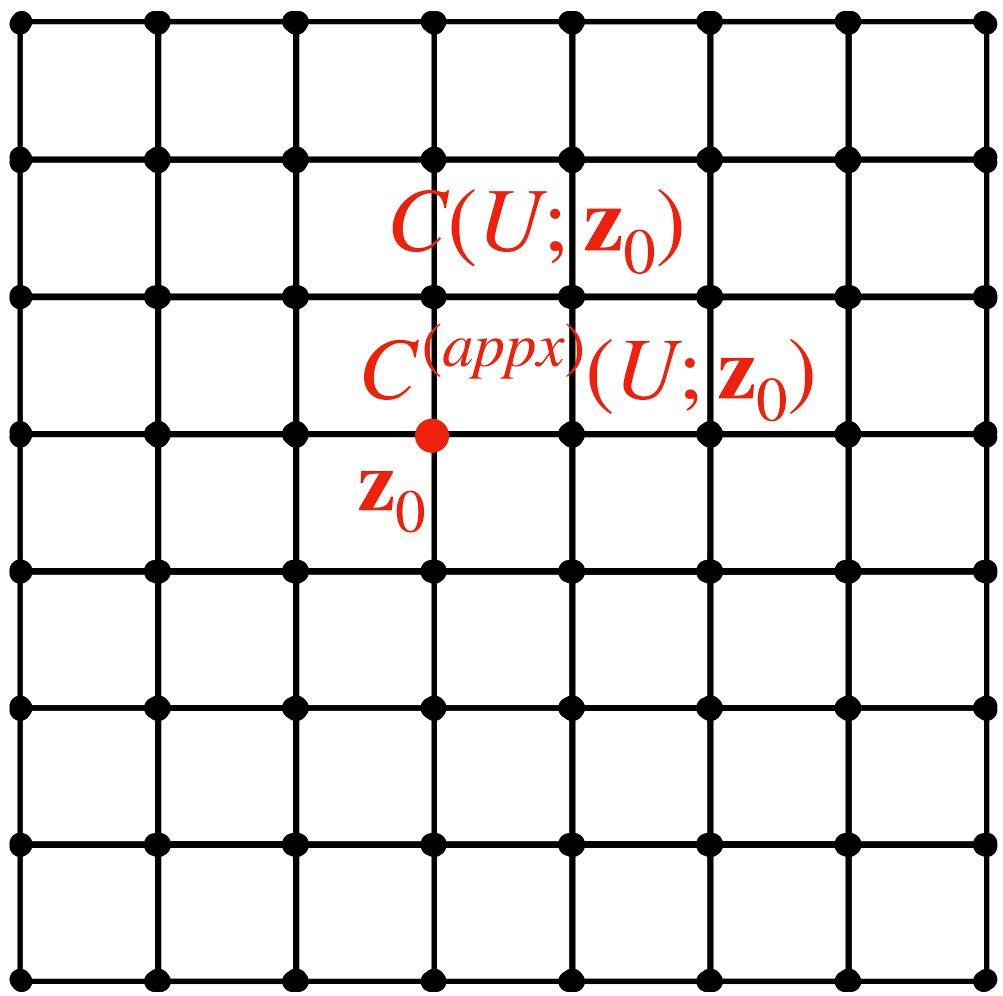
$C(U;\mathbf{z}_0)$: correlation function at gauge conf. U with the hadron source operator at $\mathbf{z}_0$

$C^{(appx)}(U;\mathbf{z}_i)$: <u>approximated correlation function</u> at gauge conf. U with the hadron source operator at $\mathbf{z}_i$

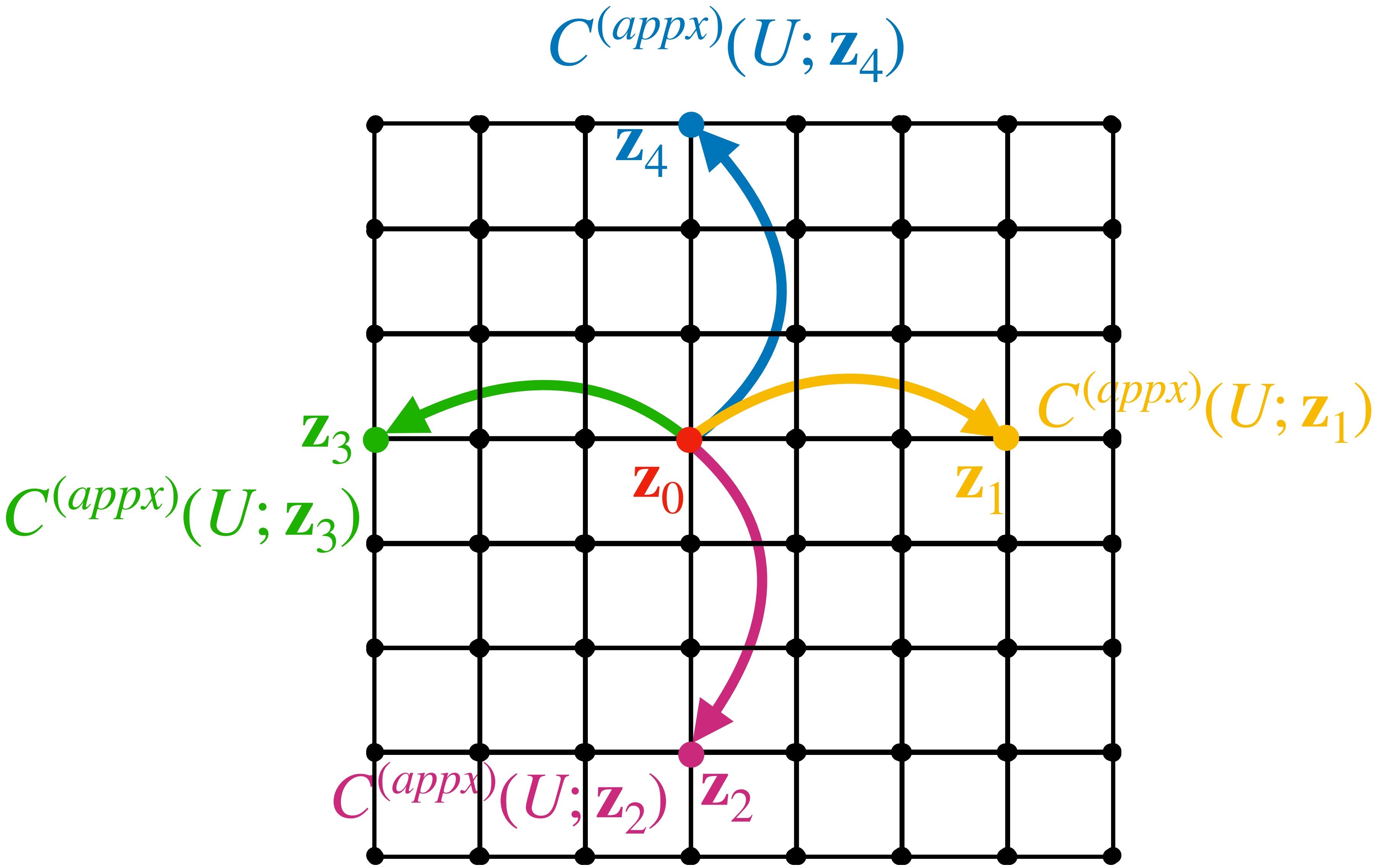
by relaxing stopping condition $||D\psi - s||/||s|| < \epsilon$ in BiCG solver

CAA + TSM in this situation

1. For each gauge conf., we calculate $C(U;\mathbf{z}_0)$ and $C^{(appx)}(U;\mathbf{z}_0)$ for some $\mathbf{z}_0$.



2. Tranlate $\mathbf{z}_0$ and calculate $C^{(appx)}(U;\mathbf{z}_i)$ at each source point.



CAA + TSM in this situation

3. The improve estimator is constructed from $C(U;\mathbf{z}_0)$

and $\{C^{(appx)}(U;\mathbf{z}_i)\}_{i=0,1\cdots N_s}$

$$\boxed{C^{(imp)}(U) = C(U;\mathbf{z}_0) - C^{(appx)}(U;\mathbf{z}_0) + \frac{1}{N_s}\sum_{i=1}^{N_s} C^{(appx)}(U;\mathbf{z}_i)}$$

this satisfies

$$\langle C^{(imp)}(U)\rangle = \langle C(U;\mathbf{z}_0)\rangle - \langle C^{(appx)}(U;\mathbf{z}_0)\rangle + \underbrace{\frac{1}{N_s}\sum_{i=1}^{N_s}\langle C^{(appx)}(U;\mathbf{z}_i)\rangle}_{= \langle C^{(appx)}(U;\mathbf{z}_0)\rangle}$$

$$= \langle C(U;\mathbf{z}_0)\rangle$$

Test for the non-locality contributions from smeared sink

- smearing function at the sink:

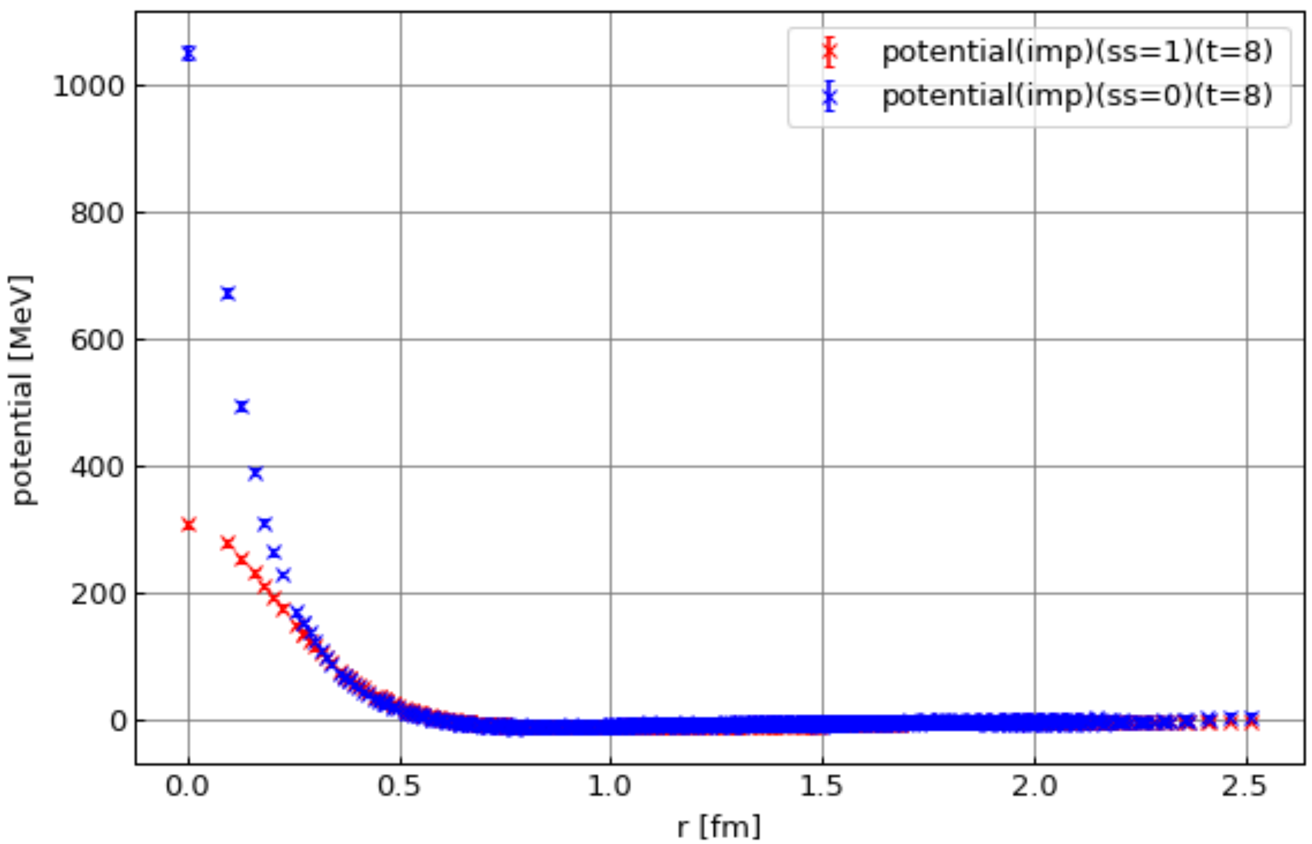
$$f_{A,B}(\mathbf{x}) = \begin{cases} Ae^{-B|\mathbf{x}|} & (|\mathbf{x}| < \frac{L-1}{2}) \\ 1 & (|\mathbf{x}| = 0) \\ 0 & (|\mathbf{x}| \geq \frac{L-1}{2}) \end{cases}$$

w/ $(A, B) = (1.0, 1/0.7)$

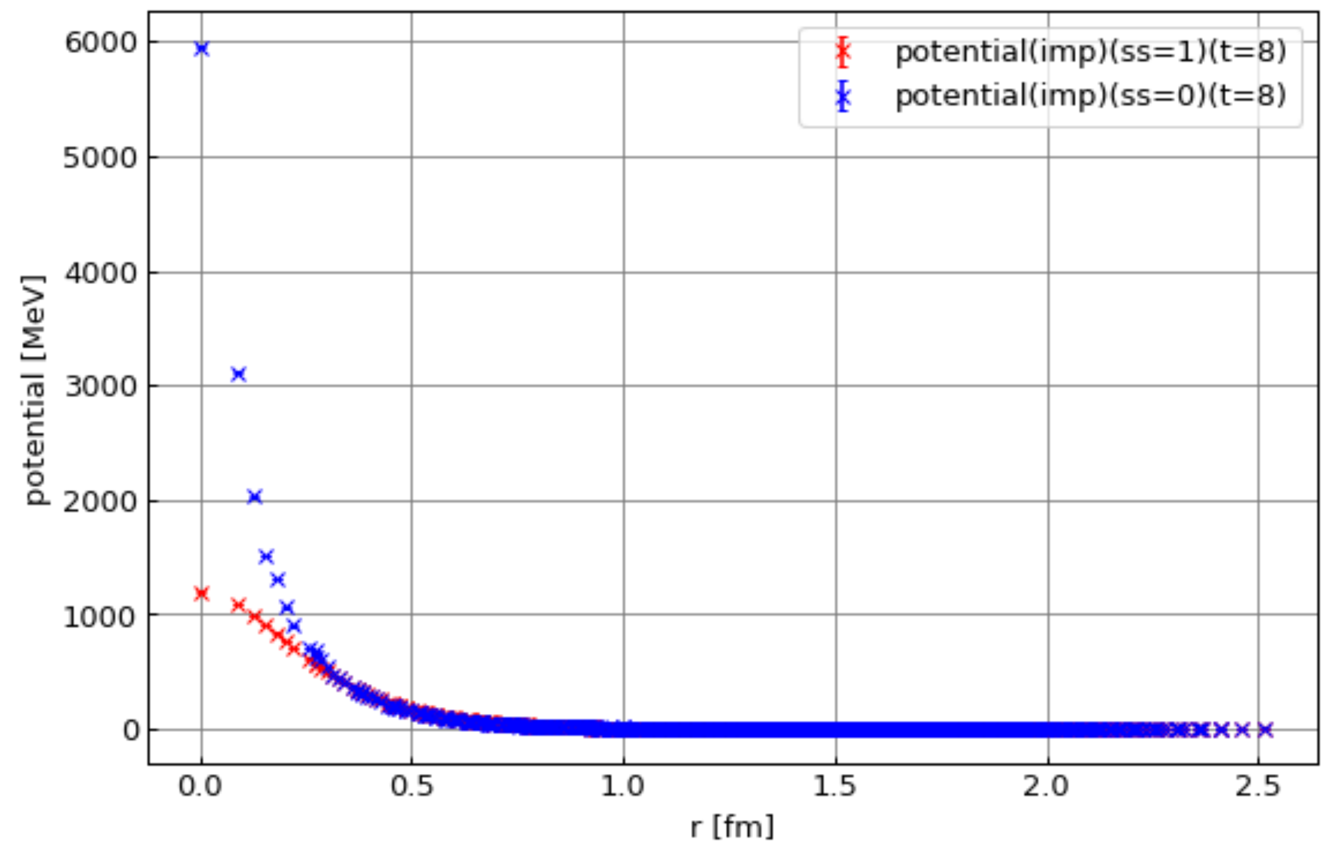
blue ⋯ point sink
red ⋯ smeared sink

- potential of S-wave NK w/ $m_s = m_l$

I=0



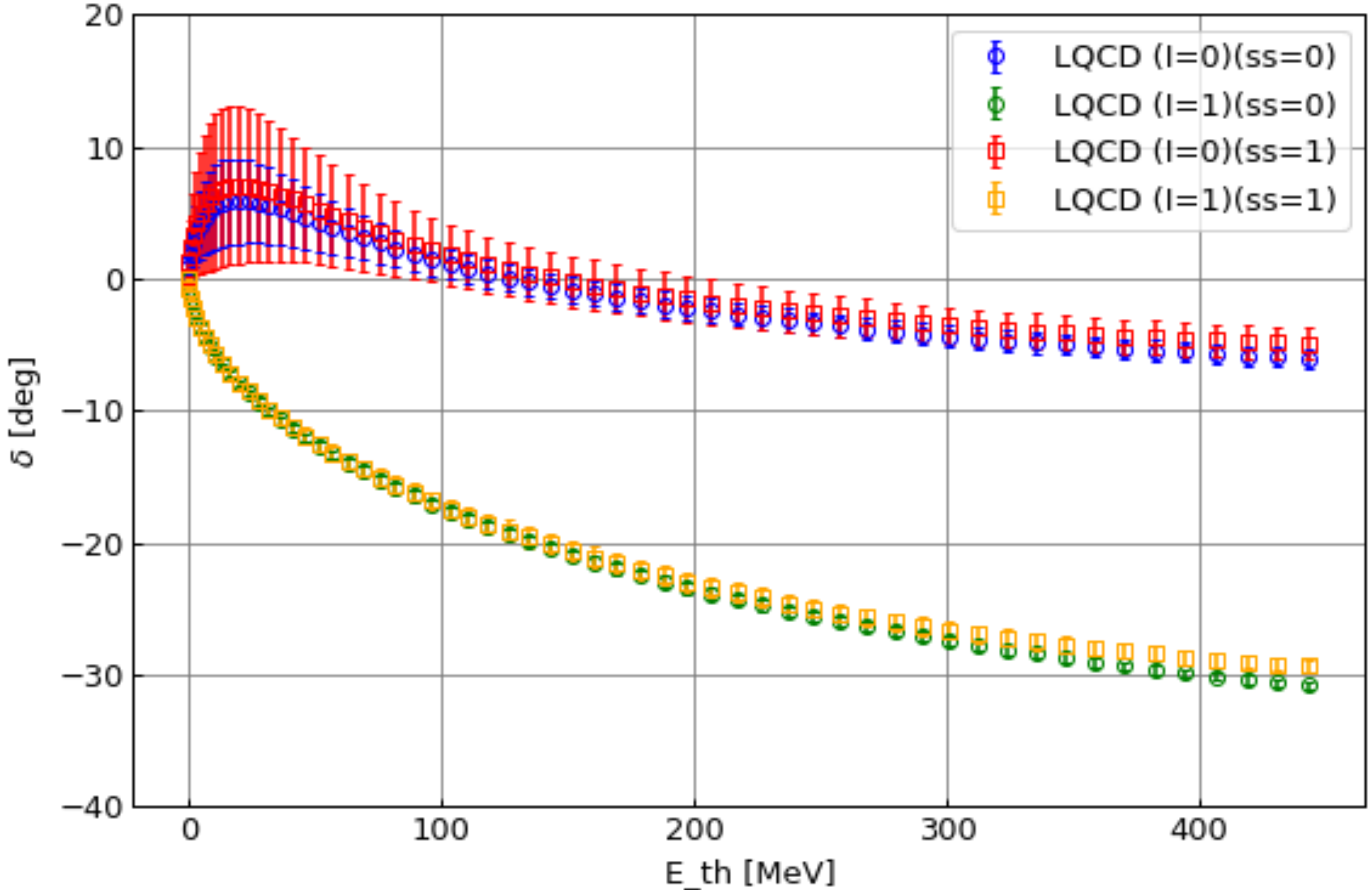
I=1 (similar to $I = 3/2\ N\pi$)



Test for the non-locality contributions

- phase shift of S-wave NK w/ $m_s = m_l$ (t=8)

blue: point sink (I=0)
red: smeared sink (I=0)
green: point sink (I=1)
orange: smeared sink (I=1)



we can ignore the non-locality contributions

in $0 < E_{th} \lesssim 100$ MeV

Details of setups

- 3-point functions are projected onto H_g representation
- smaering function: $f_{A,B}(\mathbf{x})$ ($(A,B) = (1.0,0.38)$ for source,

$(A,B) = (1.0,1/0.7)$ for sink)

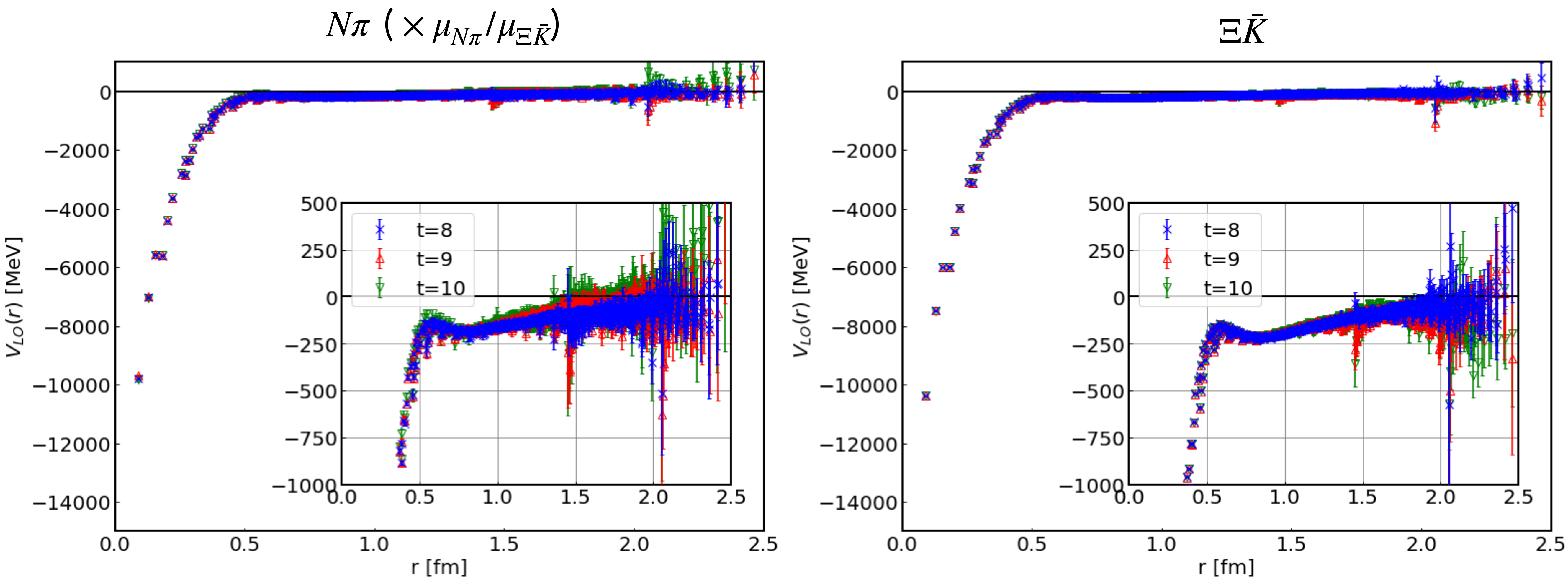
$$f_{A,B}(\mathbf{x}) = \begin{cases} Ae^{-B|\mathbf{x}|} & (|\mathbf{x}| < \frac{L-1}{2}) \\ 1 & (|\mathbf{x}| = 0) \\ 0 & (|\mathbf{x}| \geq \frac{L-1}{2}) \end{cases}$$

$$\eta^{(s_{dil})}(\mathbf{x}) = \begin{cases} \eta(\mathbf{x}) & (x+y+z \equiv s_{dil} \pmod 2) \\ 0 & (x+y+z \equiv s_{dil}+1 \pmod 2) \end{cases}, \; s_{dil} = 0,1,$$

$$\downarrow$$

- dilution for stochastic estimation: time, color, spinor, s2
- CAA+TSM: $\epsilon = 10^{-4}$ for relaxed condition
- we neglect $O(\Delta W_n^2)$ term for $N\pi$ and $O(\Delta W_n^4)$ term for $\Xi\bar{K}$

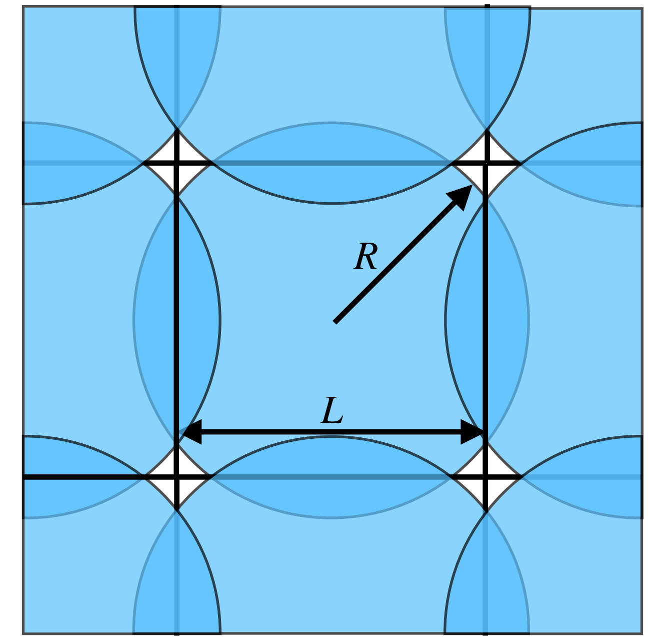
Direct comparison to $N\pi$ and $\Xi\bar{K}$ potentials



- $\Xi\bar{K}$ potential is slightly deeper

➡ $|(k^{(N\pi)}_{bound})^2| < |(k^{(\Xi\bar{K})}_{bound})^2|$ and depper bound for $\Xi\bar{K}$?

Setup for the fittings



- Fitting function:
  - 3 Gaussians

$$V^{3G}(\mathbf{r}) = a_0 e^{-(r/a_1)^2} + a_2 e^{-(r/a_3)^2} + a_4 e^{-(r/a_5)^2}$$

  - 3 Gaussians with finite volume effect [Akahoshi et al., 2020]

$$V_p^{3G}(\mathbf{r}) = V^{3G}(\mathbf{r}) + \sum_{\mathbf{n} \in \{(0,0,\pm1),(0,\pm1,0),(\pm1,0,0)\}} V^{3G}(\mathbf{r} + L\mathbf{n})$$

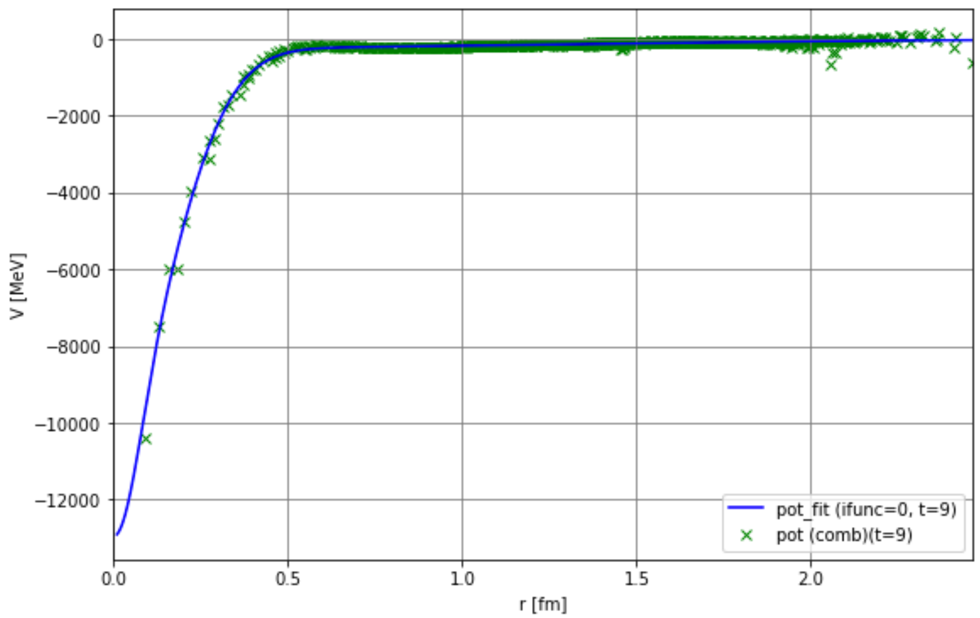
- Fit in the following 4 cases to see the lattice artifact in the short range

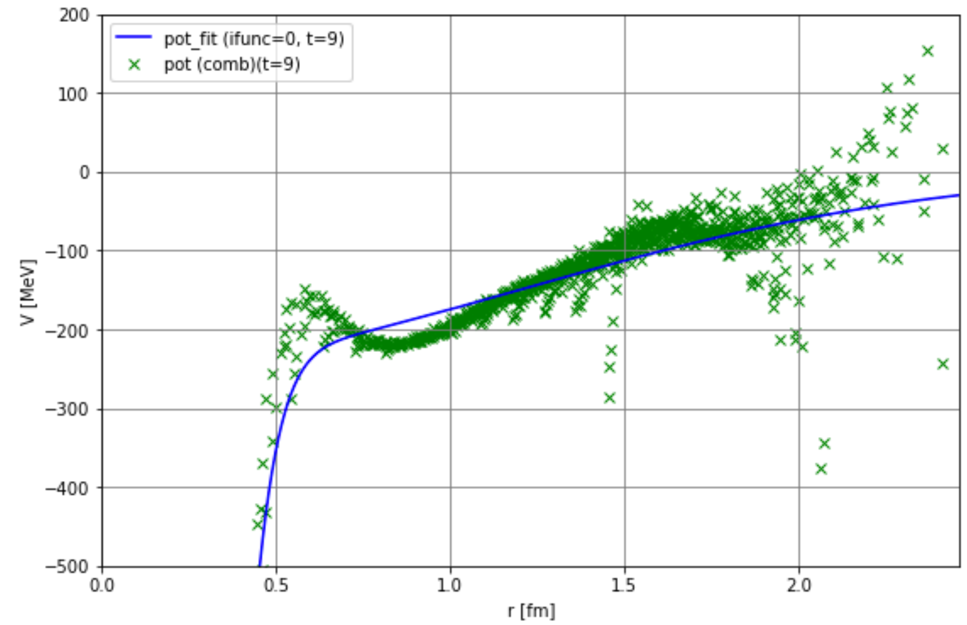
potentials w/ and w/o data at $r = a$ $\times$ potentials whose laplacian term is calculated w/ 2nd and 4th precision

- Estimate systematic error by including the 4 cases, fitting function, and different timeslices

Problem in the naive fittings

- It was found that some fitting results have the following behavior





- This is due to **the deep potential in a short distance** with **small statistical error**

- Indeed, this does not happen in the case w/o the shortest data and w/ 2nd-prec. laplacian

the shallowest case

- In order to avoid this behavior, I used **Bayesian analysis**

Bayesian analysis

[Lepage et al., 2002]

- a fitting where we add the bias term to χ^2/dof
- fit by using function

$$F = \chi^2 + \lambda\phi$$

where

$$\phi = \sum_n \frac{(a_n - \tilde{a}_n)^2}{\tilde{\sigma}_n^2}$$

- $\{a_n\}$: parameters we want to fit
- $\{\tilde{a}_n\}$: bias parameters, $\{\tilde{\sigma}_n\}$: relative weights
- λ: tunable parameter

- λ is tuned in the region where $\{a_n\}$ depend weakly on λ and take it as small value as possible

Setup for Bayesian analysis

(for convenience, each case is labeled as follows)

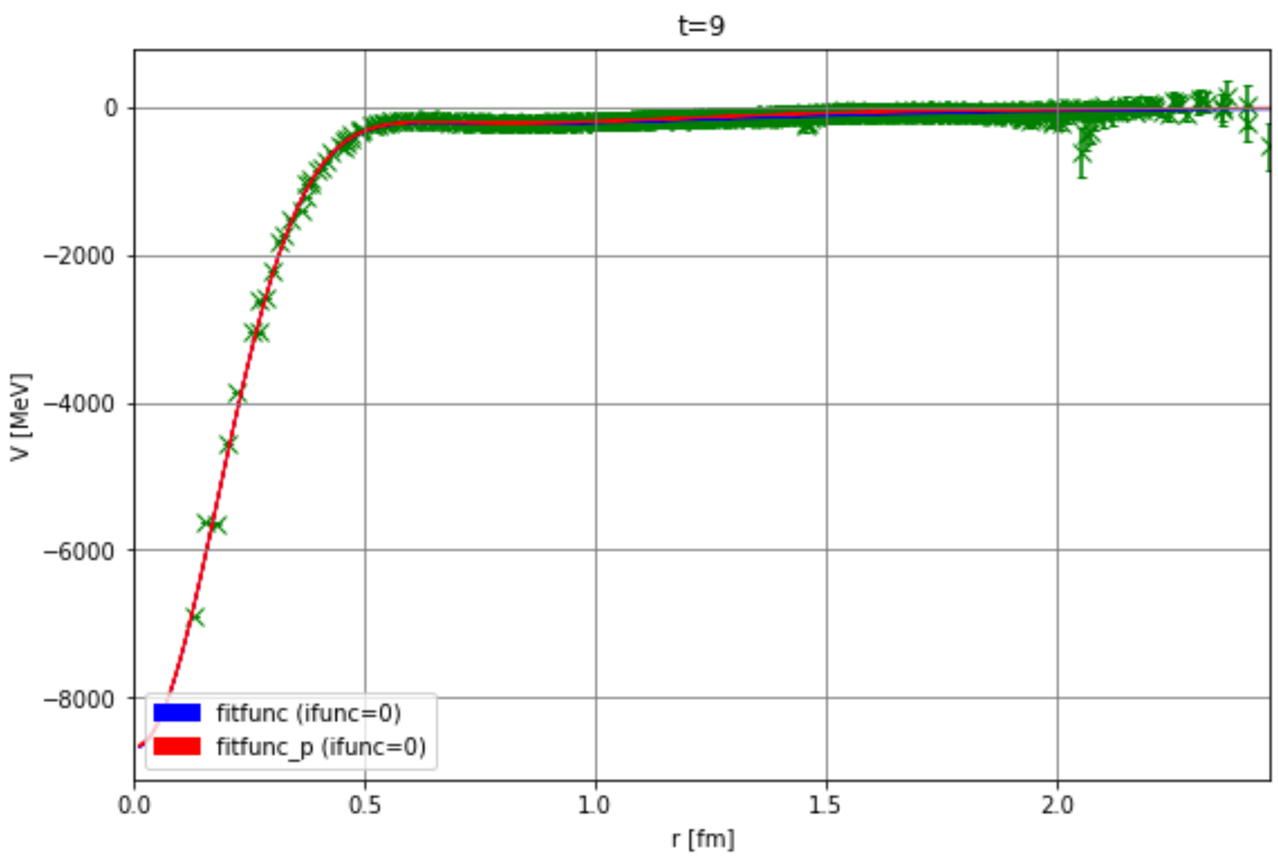
data at $r = a$ \ laplacian	2nd prec.	4th prec.
removed	①	②
not removed	③	④

- fit in case ① without Bayesian analysis and set $\{\tilde{a}_n\}$ and $\{\tilde{\sigma}_n\}$ as mean values and error of the results, respectively
- use Bayesian analysis in case ②, ③ and ④ with $\{\tilde{a}_n\}$ and $\{\tilde{\sigma}_n\}$ determined above
- use fitting potential in case ① for the central values and statistical error of the observables while in other cases for the systematic error

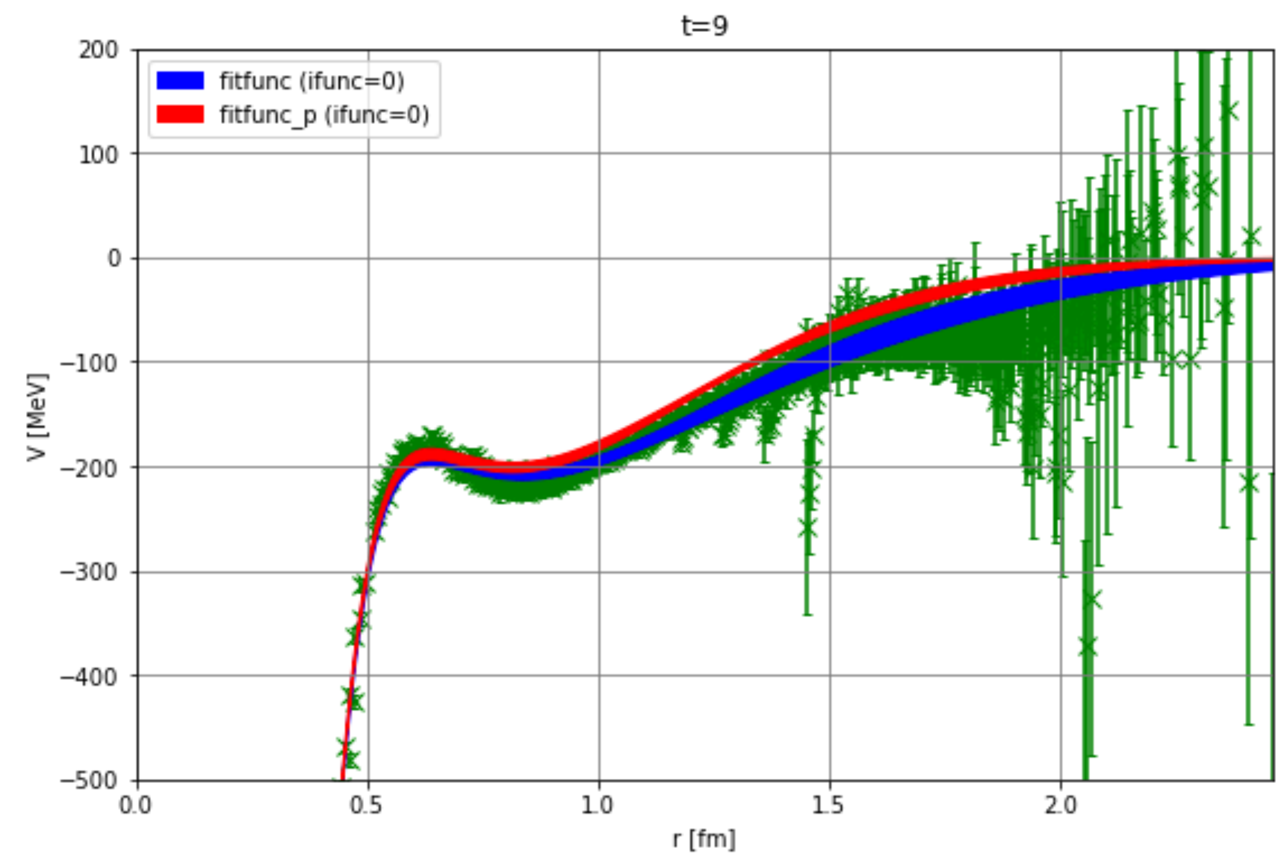
Fitting results in case ①

laplacian / data at $r = a$	2nd prec.	4th prec.
removed	①	②
not removed	③	④

XiKbar



XiKbar (zoomed)

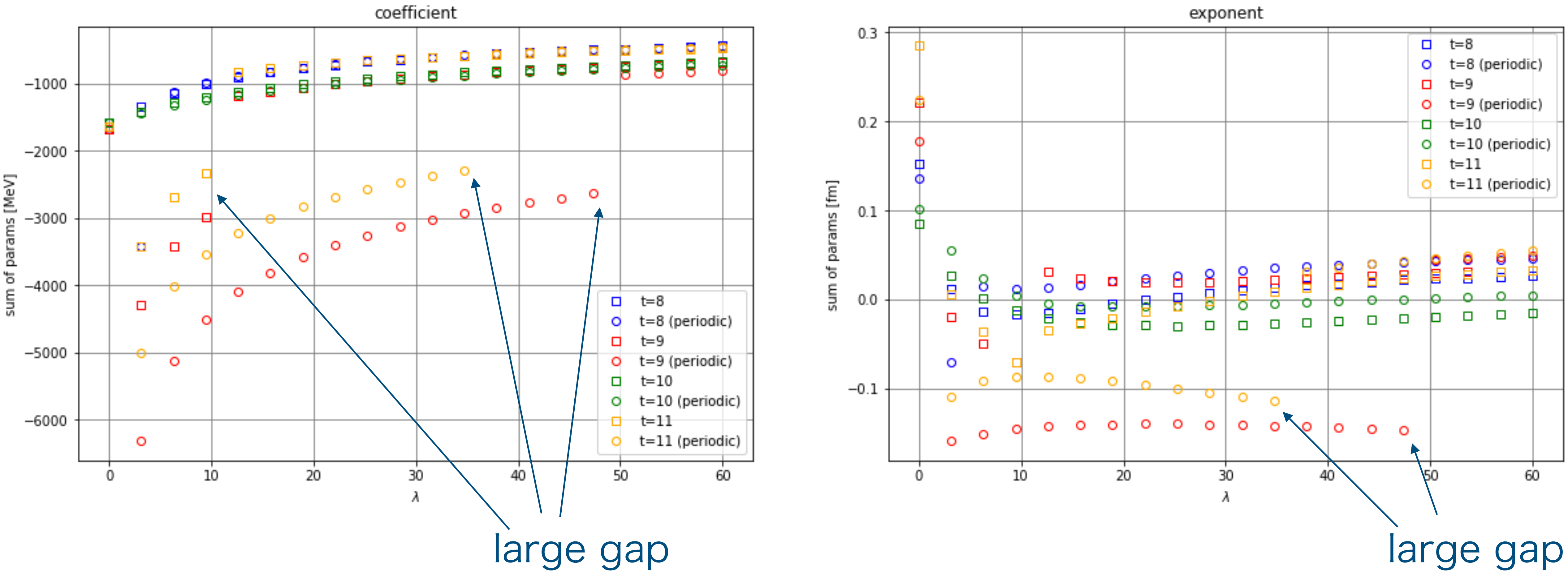


- it looks that the fitting works well in this case
- the range of the attraction becomes smaller when we take into account p.b.c.

Bayesian analysis in case ④

laplacian / data at $r = a$	2nd prec.	4th prec.
removed	①	②
not removed	③	④

- sum of the parameters for XiKbar



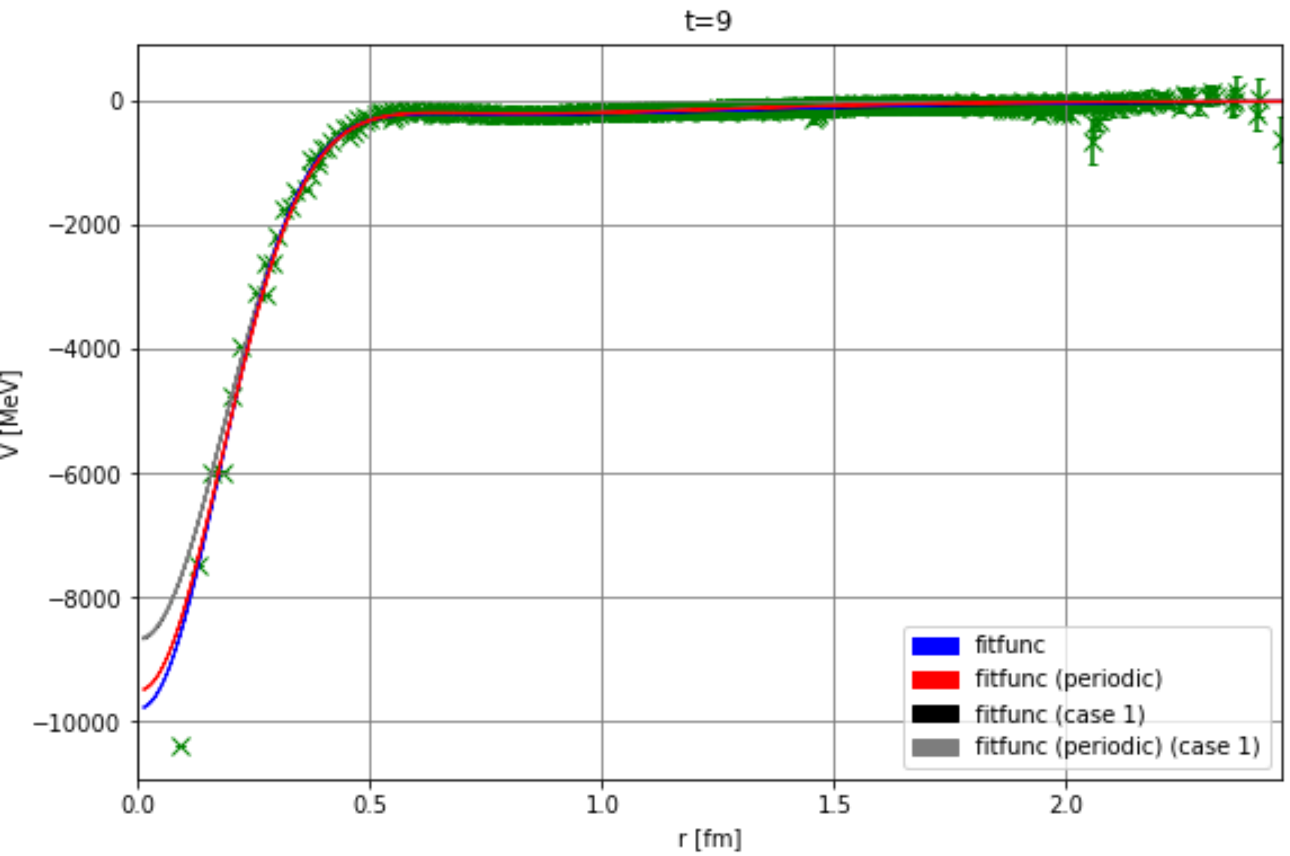
- λ is set to the minimum value in the region after the large gap in each case

- if there is no large gap, λ is set to zero (w/o Bayesian)

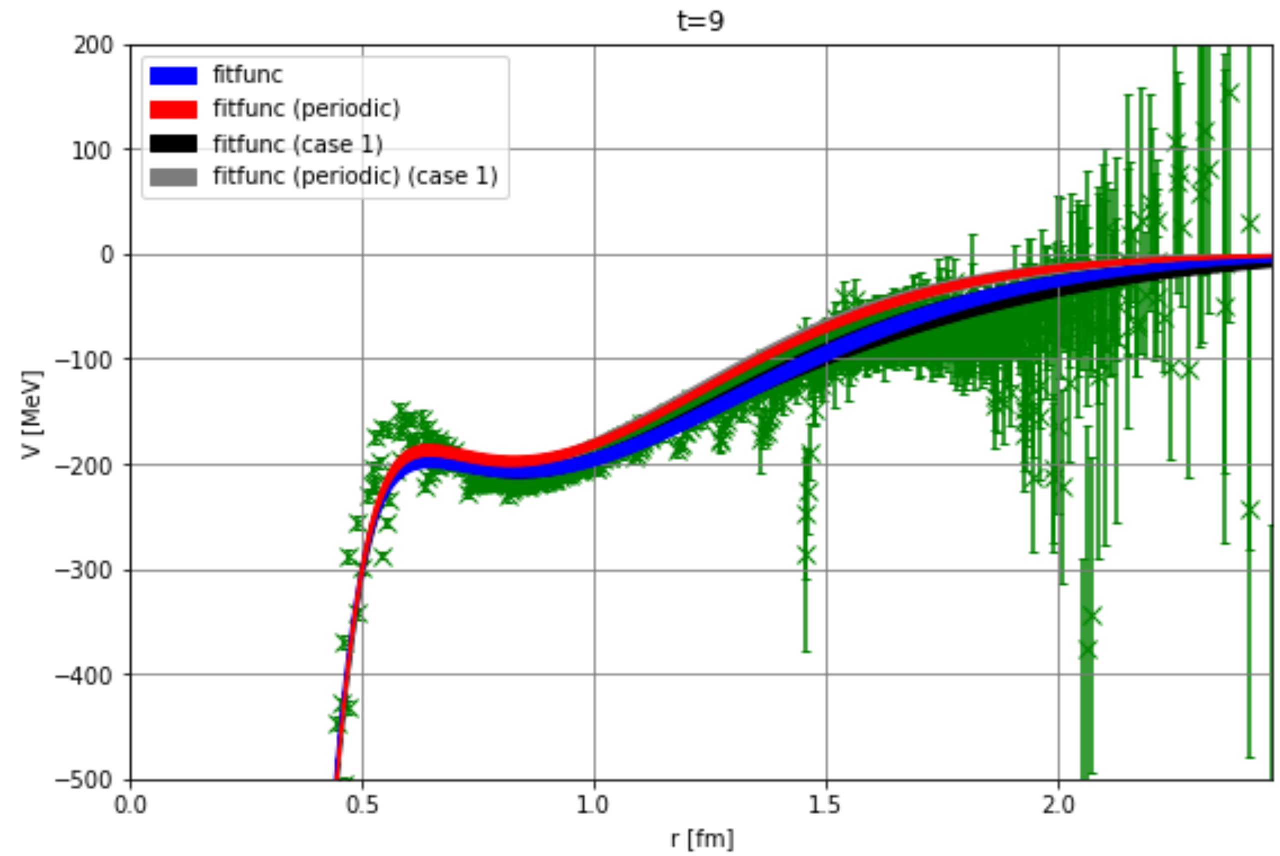
Fitting results in case ④

laplacian / data at $r = a$	2nd prec.	4th prec.
removed	①	②
not removed	③	④

XiKbar



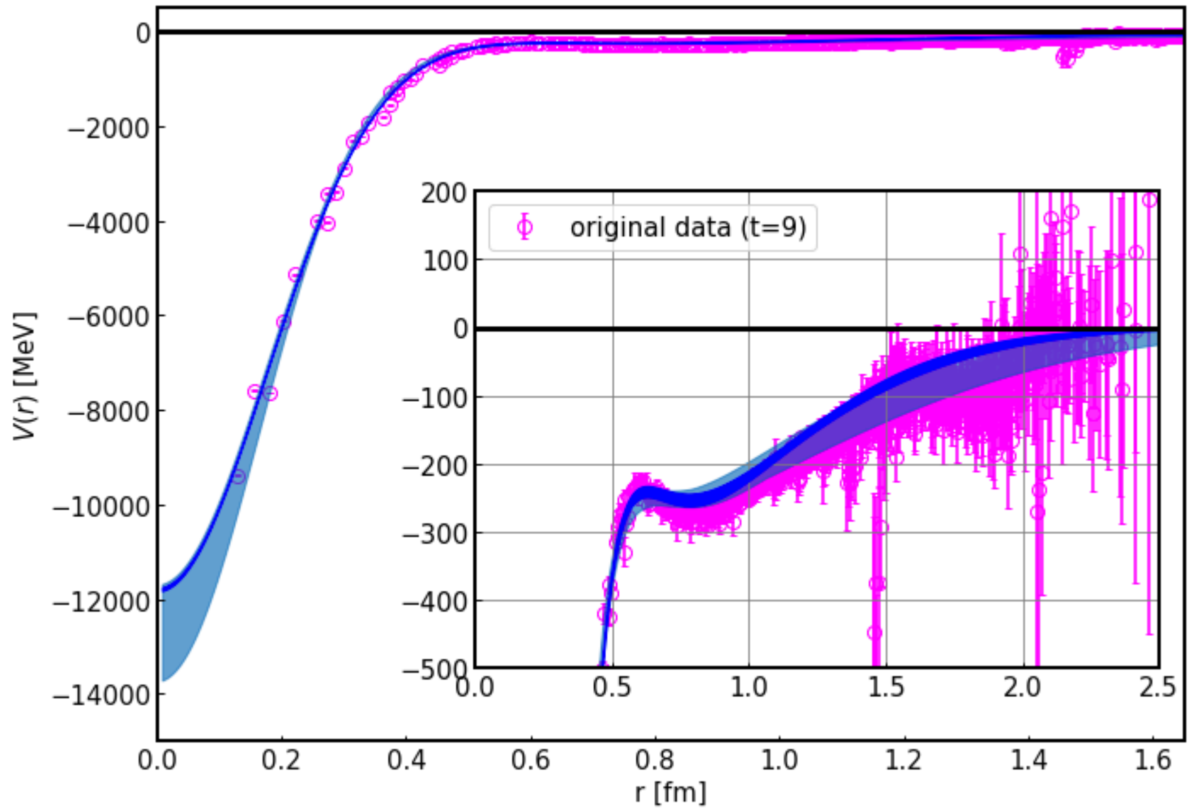
XiKbar (zoomed)

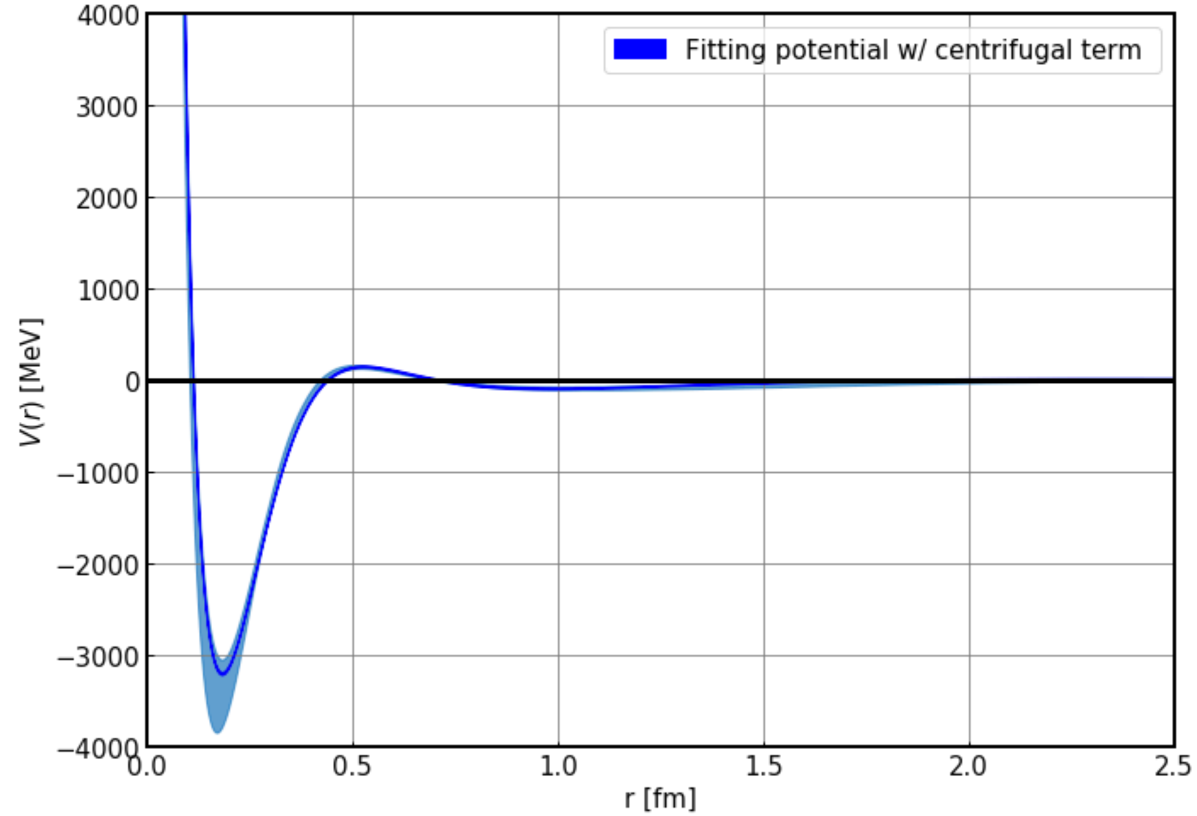


- in the long distance, the results are the same as those in case ①
- in the short distance, the bound becomes deeper

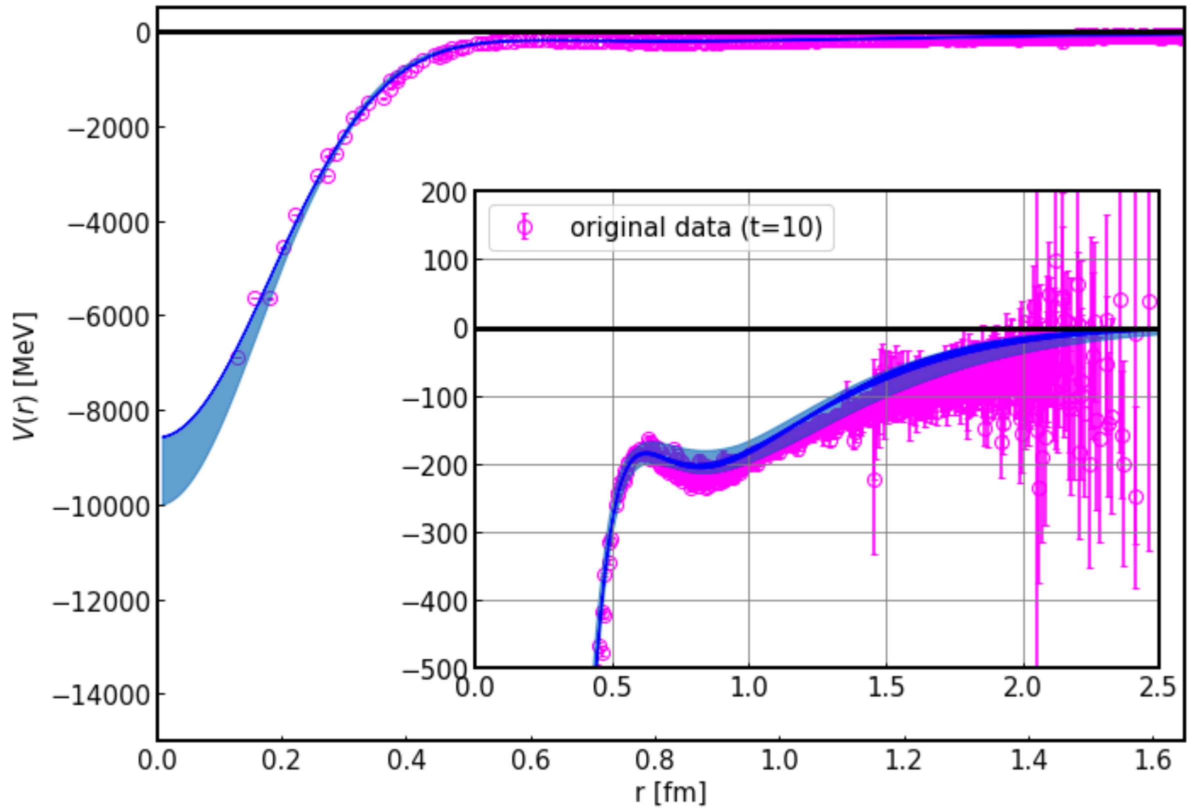
Fitting results with systematic error

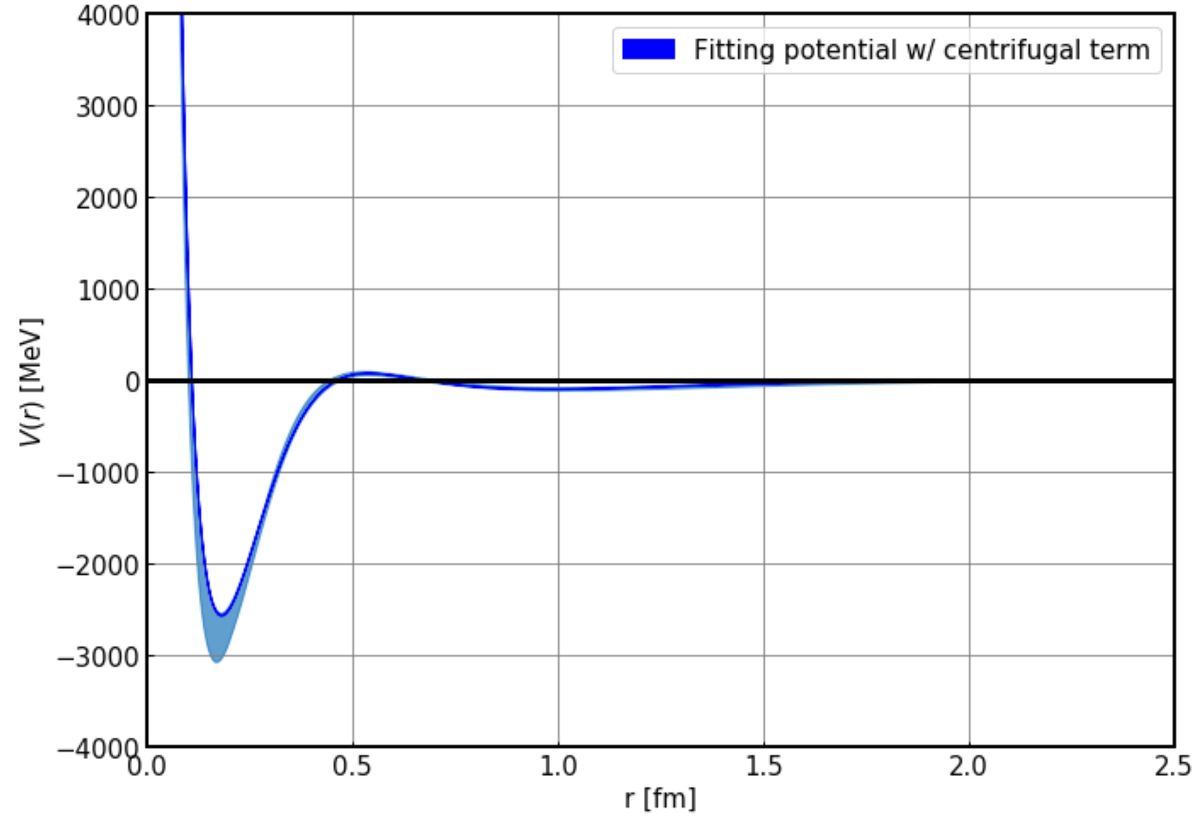
$N\pi$





$\Xi\bar{K}$

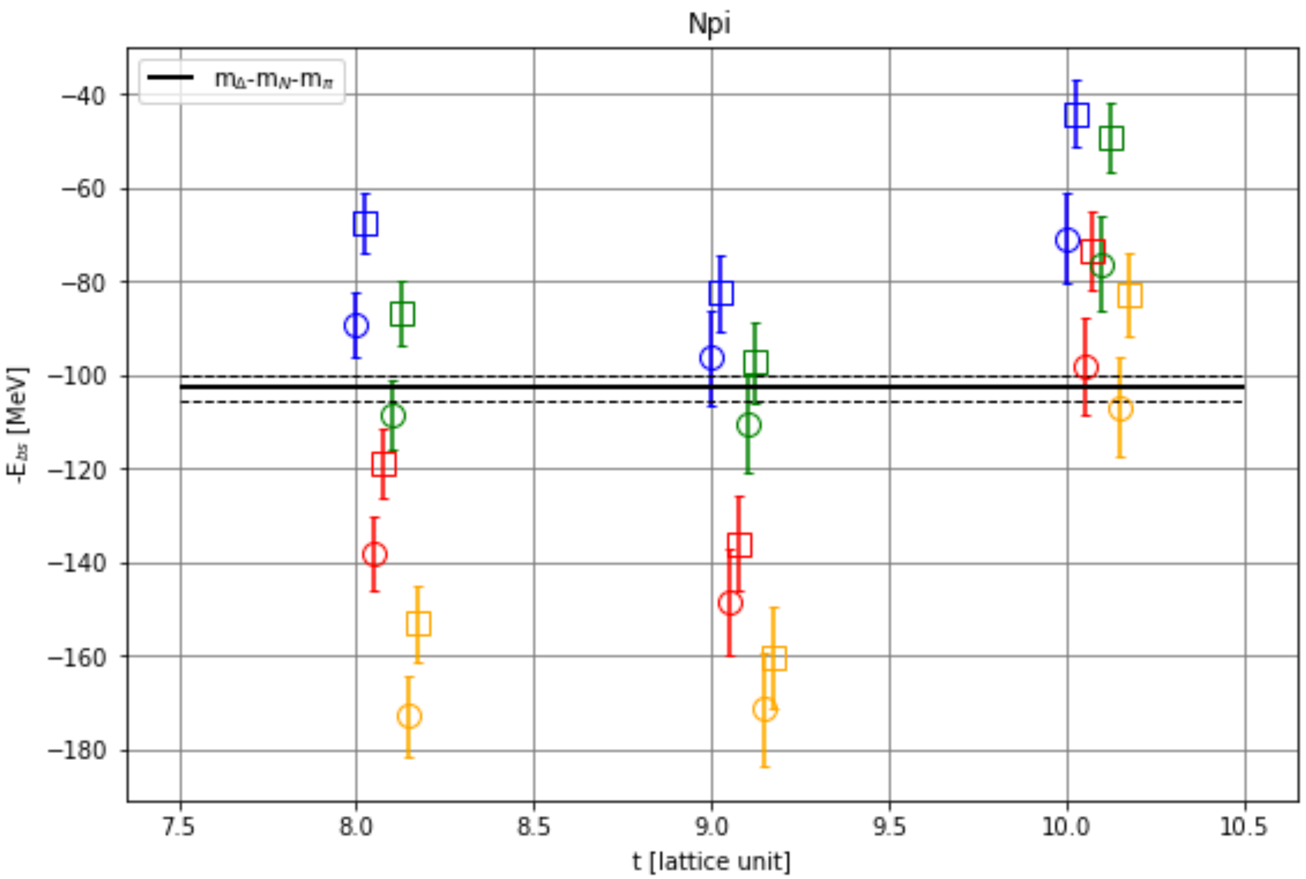


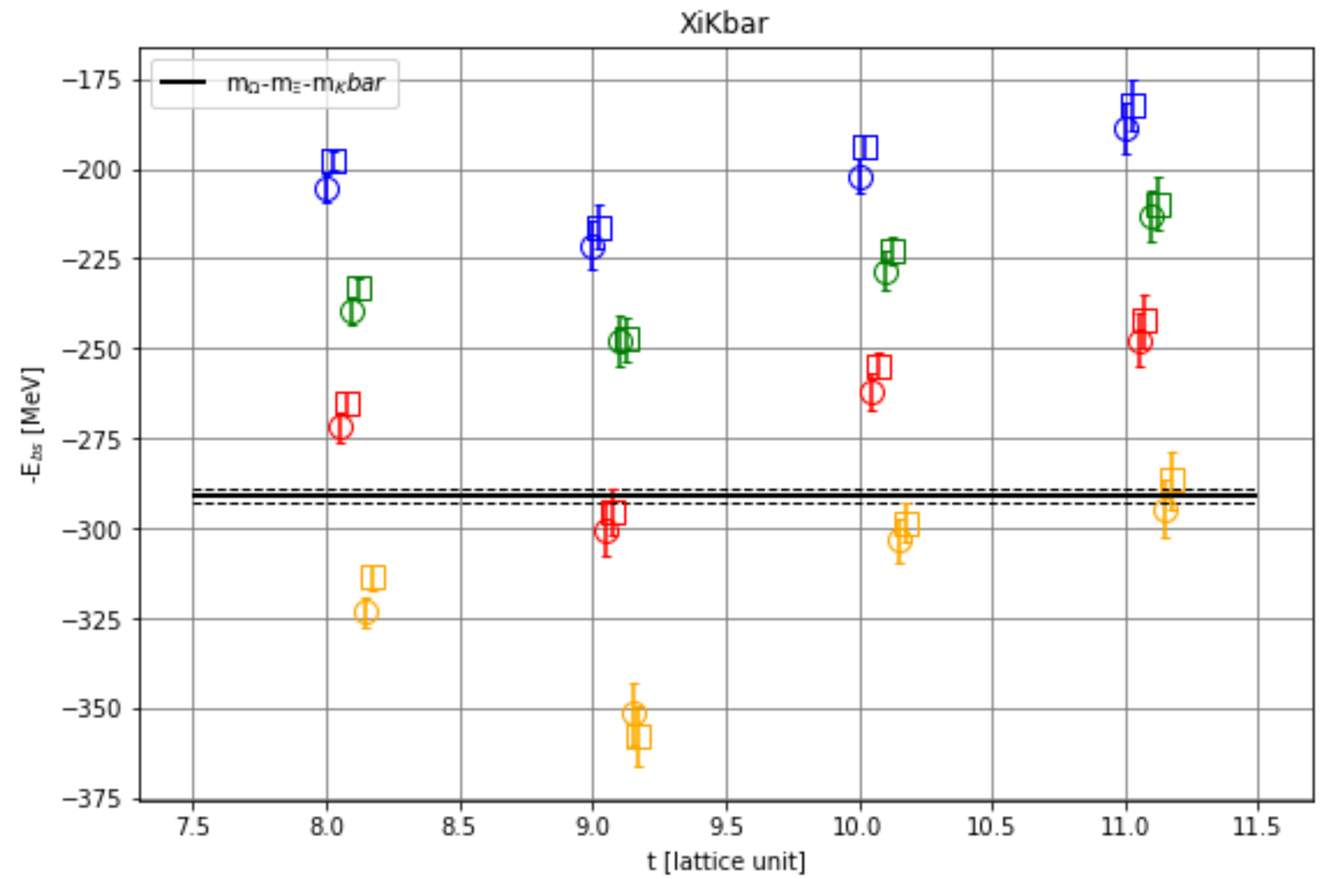


Binding energy in all cases

data at $r=a$ \ laplacian	2nd prec.	4th prec.
removed	①	②
not removed	③	④

(o: usual, □: periodic)

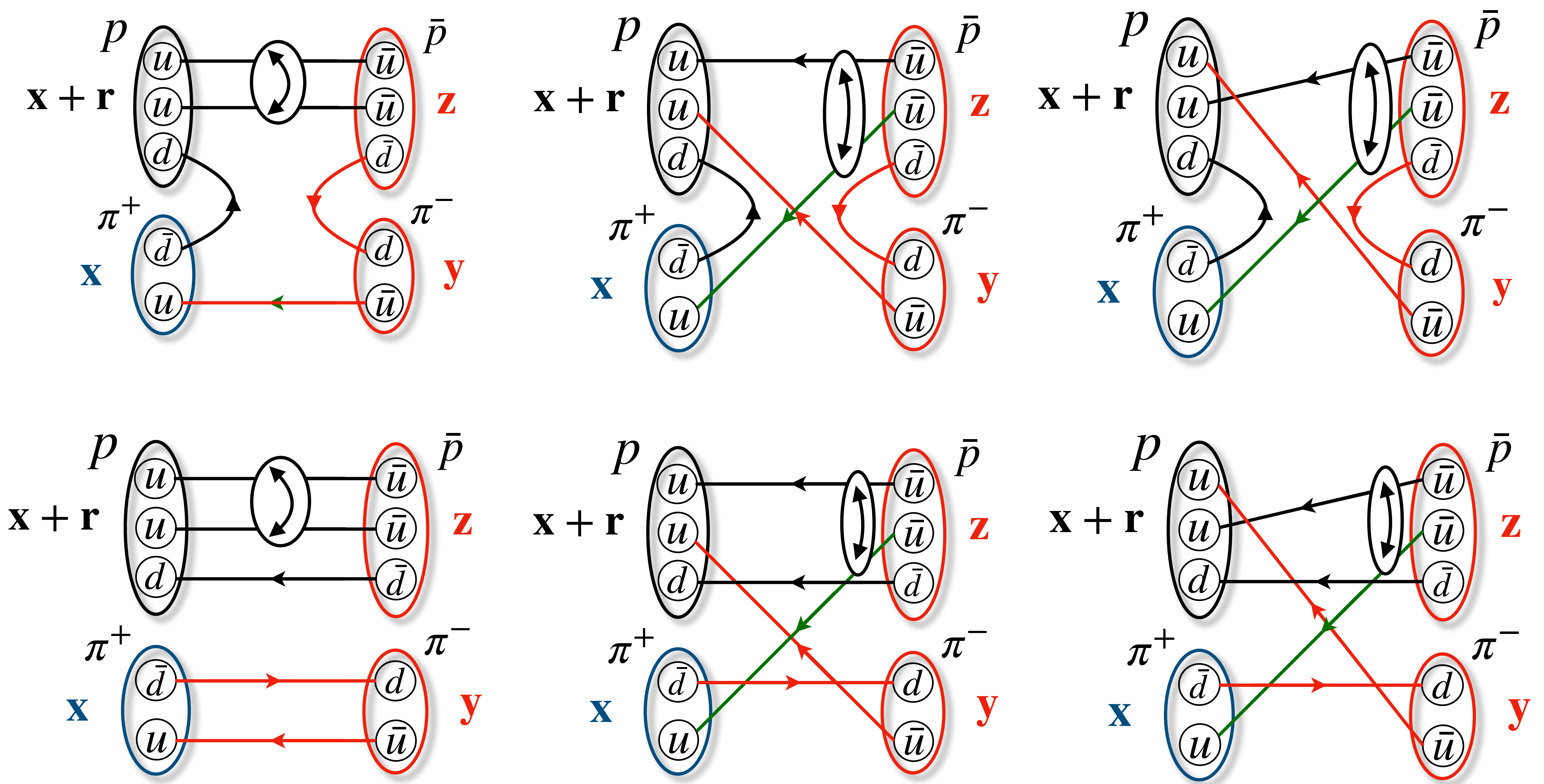




$$E_B^{N\pi} = 82.6(8.2)_{stat} \begin{pmatrix} +90.3 \\ -38.4 \end{pmatrix}_{sys} \text{ MeV}$$

$$E_B^{\Xi\bar{K}} = 193.9(3.2)_{stat} \begin{pmatrix} +163.9 \\ -11.4 \end{pmatrix}_{sys} \text{ MeV}$$

Quark contraction for the 4-point function with $N\pi$ source



point-to-all + stochastic+ one-end trick