Multi-level computation of the hadronic vacuum polarization contribution to $(g_\mu - 2)$

Leonardo Giusti

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Based on:

Dalla Brida, LG, Harris, Pepe, PLB 816 (2021) 136191 [arXiv:2007.02973]

Lattice 2021 - MIT - July 27th 2021

The bottleneck: signal/noise ratio for HVP (HLbL,...)

- The HVP contribution to $a_\mu = (g-2)_\mu/2$ reads

$$a_\mu^{\rm HVP} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dx_0 K(x_0, m_\mu) G(x_0)$$

where

$$G(x_0) = \int d^3x \langle J_k^{\rm em}(x) J_k^{\rm em}(0)\rangle$$

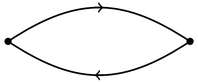
with $K(x_0, m_\mu)$ being a known function

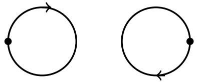
- For the light-connected contribution (by far the largest)

$$\frac{\sigma^2_{G^{\rm conn}_{u,d}}(x_0)}{[G^{\rm conn}_{u,d}(x_0)]^2} \propto \frac{1}{n_{cnfg}}\, e^{2(M_\rho - M_\pi)|x_0|}$$

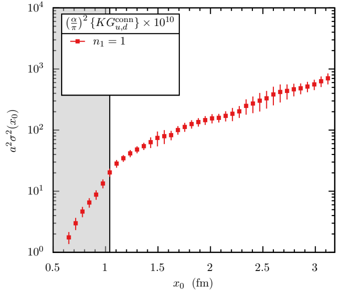
where M_ρ is the lightest state in that channel.

Signal lost after 1.5-2.5 fm (depending on $m_{u,d}$) due to exp. increase of statistical error





$n_{cnfg} = n_0 = 25\,,\;\; n_{tot} = n_0 \cdot n_1$



$a = 0.065$ fm , $M_\pi = 270$ MeV

$(V/a^4) = 96 \times 48^3$

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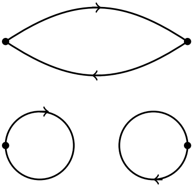

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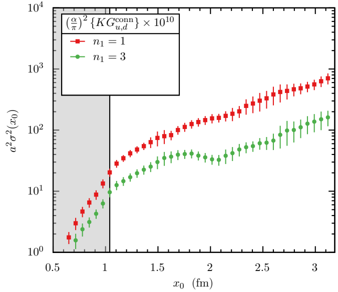

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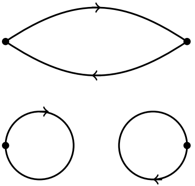
- Sharp rise of σ^2 with x_0 when computed by a standard 1-level integration (red points) is automatically flattened out by the 2-level integration (blue-points)

- Accurate computations can be obtained at large distances: no need for any modeling of the long-distance behaviour of $G^{\rm conn}_{u,d}$



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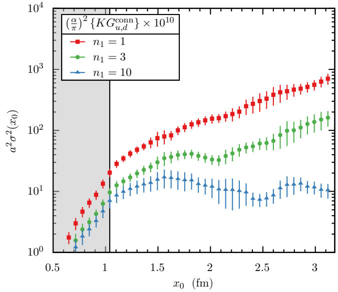

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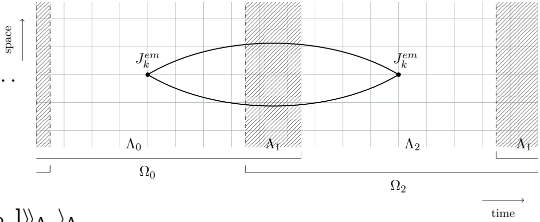
Why/How it works: multi-level integration

[Parisi, Petronzio, Rapuano 83; Lüscher, Weisz 01; . . . ; Meyer 02; LG, Della Morte 08 10, . . .]

- If the action and the observable can be factorized

$$S[U] = S_0[U_{\Omega_0}] + S_2[U_{\Omega_2}] + \ldots$$
$$O[U] = O_0[U_{\Omega_0}] \times O_2[U_{\Omega_2}]$$



then

$$\langle O[U] \rangle = \langle \langle\!\langle O_0[U_{\Omega_0}]\rangle\!\rangle_{\Lambda_0} \times \langle\!\langle O_2[U_{\Omega_2}]\rangle\!\rangle_{\Lambda_2} \rangle_{\Lambda_1}$$

where

$$\langle\!\langle O_0[U_{\Omega_0}]\rangle\!\rangle_{\Lambda_0} = \frac{1}{Z_{\Lambda_0}} \int \mathcal{D}U_{\Lambda_0}\, e^{-S_0[U_{\Omega_0}]}\, O_0[U_{\Omega_0}]$$

- Two-level integration:

  - n_0 configurations U_{Λ_1}
  - n_1 configurations U_{Λ_0} and U_{Λ_2} for each U_{Λ_1}

- If $\langle\!\langle \cdot \rangle\!\rangle_{\Lambda_i}$ can be computed efficiently with a statistical error comparable to its central value, then the prefactor in the signal/noise ratio changes as (until S/N problem solved)

$$n_0 \to n_0 n_1^2$$

at the cost of generating approximatively $n_0 n_1$ level-0 configurations

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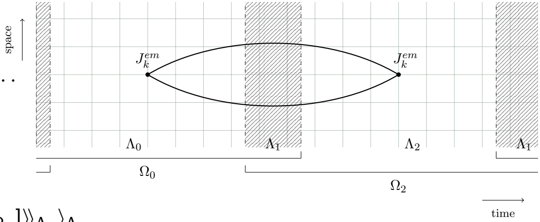

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- With more active blocks, at the cost of approximatively $n_0 n_1$ level-0 configurations,

$$n_0 \to n_0 n_1^{n_{\rm block}}$$

and the gain increases exponentially with the distance since $n_{\rm block} \propto |y_0 - x_0|$. For the same relative accuracy of the correlator, the computational effort would then increase approximatively linearly with the distance

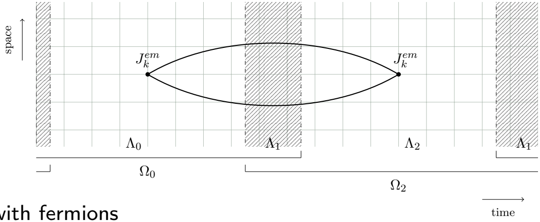
Multi-level integration with fermions

[Cè, LG, Schaefer 16; Dalla Brida, LG, Harris, Pepe 20]

- Thanks to

  ∗ Overlapping Domain Decomp.

  ∗ Multi-Boson representation



multi-level integration also possible with fermions

- The effective action (determinant of the Dirac operator) can be decomposed as

$$\det\, D = \frac{\det(1-\omega)}{\det\, D_{\Lambda_1}\, \det D_{\Omega_0}^{-1}\, \det D_{\Omega_2}^{-1}}$$

and for 2 flavours, for instance, can be represented as

$$\{\det D^\dagger D\}^2 = \int \mathcal{D}\phi \ldots \exp\{-S_0[U_{\Omega_0},\ldots] - S_1[U_{\Lambda_1},\ldots] - S_2[U_{\Omega_2},\ldots]\}$$

- Factorization thanks to different representations of various quark-path contributions:

  ∗ Pseudo-fermions for paths with no loops around Λ_1
  ∗ Multi-Bosons for paths with 1–N loops (N is the number of Multi-Bosons)
  ∗ Reweighting factor for paths with more than N loops

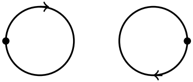
Split-even estimator of disconnected contribution

[LG, Harris, Nada Schaefer 19]

- Advantage of multi-level sets in when variances are due to fluctuations of gauge field
- The disconnected Wick contraction reads

$$t(x) = \mathrm{Tr}\left[\gamma_k\{D^{-1}_{m_u}(x,x) - D^{-1}_{m_s}(x,x)\}\right]$$

$$= (m_s - m_u)\,\mathrm{Tr}\left[\gamma_k D^{-1}_{m_u} D^{-1}_{m_s}(x,x)\right]$$

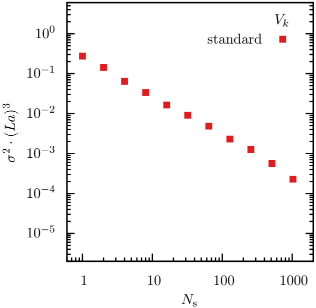


- Standard stochastic estimator $[\langle\eta(x)\eta^\dagger(y)\rangle = \delta_{xy}]$

$$\theta(x) = \frac{(m_s - m_u)}{N_s}\sum_{i=1}^{N_s} \mathrm{Im}\left[\eta_i^\dagger(x)\gamma_k\{D^{-1}_{m_u}D^{-1}_{m_s}\eta_i\}(x)\right]$$

is expensive. It requires $O(10^4)$ random fields η for its σ^2 to be dominated by gauge fluctuations

Why random noise much larger than gauge one?
Computable and understandable in QFT



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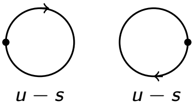
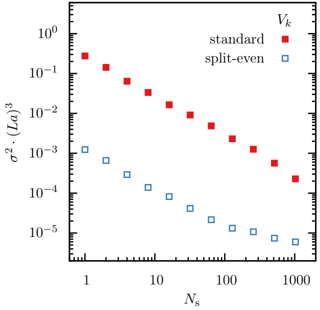


- Split-even stochastic estimator [$\langle \eta(x)\eta^\dagger(y)\rangle = \delta_{xy}$]

$$\tau(x) = \frac{(m_s - m_u)}{N_s}\sum_{i=1}^{N_s} \mathrm{Im}\left[\{\eta_i^\dagger D_{m_u}^{-1}\}(x)\,\gamma_k\,\{D_{m_s}^{-1}\eta_i\}(x)\right]$$

requires $O(10^2)$ random fields η to hit gauge noise. Gain: 2 orders of magnitude. Definition suggested by the QFT analysis of the variance.

Used in the past for pseudoscalar density in TMQCD (one-end trick) **[ETM Coll. 08, 12]**



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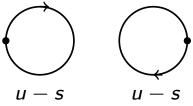

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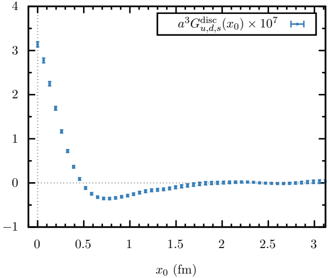
combined with multi-level integration is a solution for a precise computation of the disconnected contribution

It is already being applied in production phase for HVP by CLS (Mainz)



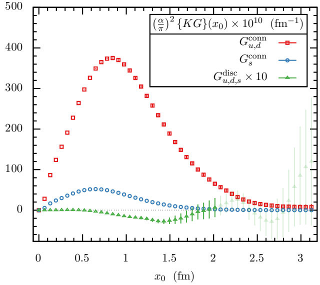
First multi-level computation of HVP

- Wilson glue with $O(a)$-improved Wilson quarks

$$\beta = 5.3\,, \quad (T/a)\times(L/a)^3 = 96\times 48^3$$

$$a = 0.065\ \mathrm{fm}\,, \quad M_\pi = 270\ \mathrm{MeV}$$

$$n_0 = 25\,, \qquad n_1 = 10$$



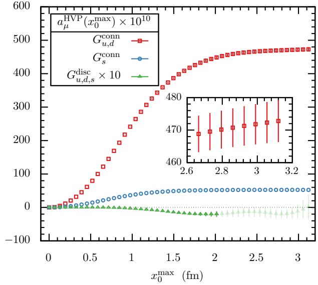
- Domain Decomposition adopted:

$\Lambda_0\ :\ x_0/a \in [0,39]\,,\quad \Lambda_1\ :\ x_0/a \in [40,47]\cup[88,95]$

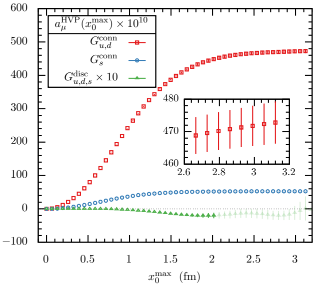
$\Lambda_2\ :\ x_0/a \in [48,87]$

- With 2-level integration achieved 1% precision with just $n_0\cdot n_1 = 250$ configurations

- With lighter quarks, the gain due to the 2-level integration is even more dramatic since $(M_\rho - M_\pi)$ increases significantly



Conclusions & Outlook



- Permille precision and accuracy on HVP is the challenge for lattice QCD

- Our strategy: new integration and estimators (better "machine" and "experiment")

- Multi-level integration reduces the variance exponentially:

  - with the time-distance of the currents

  - when pion mass gets lighter (physical point)

- Next step: R&D $\Longrightarrow$ production. Significant human and numerical resources needed

- Analogous variance-reduction pattern expected to work out also for lattice calibration, electromagnetic corrections, HLbL, ...