Controlling unwanted exponentials in lattice calculations of radiative leptonic decays

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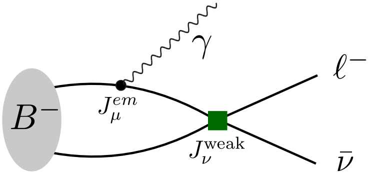
$B^- \to \ell^- \bar{\nu}\gamma$

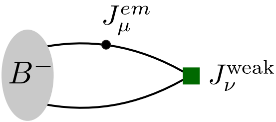


- Hard photon removes helicity suppression $(m_\ell/m_B)^2$
- For large $E_\gamma^{(0)}$, simplest decay that probes the inverse moment of the B meson light-cone distribution amplitude
$$\frac{1}{\lambda_B} = \int_0^\infty \mathrm{d}\omega\ \frac{\Phi_{B+}(\omega)}{\omega}$$
- λ_B important input in QCD factorization approach to exclusive B decays, currently not well known
[See e.g., Beneke, Braun, Ji, Wei, arXiv:1804.04962/JHEP 2018;

  Beneke, Buchalla, Neubert, Sachrajda, arXiv:hep-ph/9905312/PRL 1999]
- Belle: $\mathcal{B}(B^+ \to \ell^+\nu\gamma) < 3.0 \times 10^{-6}$ $(E_\gamma^{(0)} > 1$ GeV)

  [arXiv:1810.12976/PRD 2018]

Hadronic Tensor and Form Factors



$$J_\mu^{em} = \sum_q e_q \bar{q}\gamma_\mu q, \qquad J_\nu^{weak} = \bar{u}\gamma_\nu(1-\gamma_5)b$$

$$T_{\mu\nu} = -i \int \mathrm{d}^4x \; e^{ip_\gamma\cdot x} \langle 0|\, \mathbf{T}\big(J_\mu^{em}(x) J_\nu^{weak}(0)\big) \,\big|B^-(\vec{\mathbf{p}}_B)\big\rangle$$

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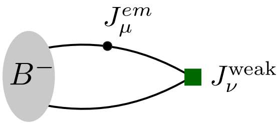

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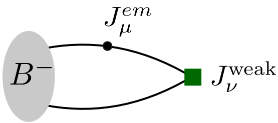
$$= \epsilon_{\mu\nu\tau\rho}p_\gamma^\tau v^\rho F_V + i\big[-g_{\mu\nu}(v\cdot p_\gamma) + v_\mu(p_\gamma)_\nu\big]F_A - i\frac{v_\mu v_\nu}{(v\cdot p_\gamma)}m_B f_B$$

$$+ (p_\gamma)_\mu\text{-terms}$$

$$F_{A,SD} = F_A + f_B/E_\gamma^{(0)}, \qquad E_\gamma^{(0)} = p_B\cdot p_\gamma/m_B$$

Hadronic Tensor and Form Factors



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$$+\,(p_\gamma)_\mu\text{-terms}$$

$$F_{A,SD} = F_A + f_B/E_\gamma^{(0)}, \qquad E_\gamma^{(0)} = p_B\cdot p_\gamma/m_B$$

Goal: Calculate $F_V, F_{A,SD}$ as a function of $E_\gamma^{(0)}$

Euclidean correlation function

(* all times are now Euclidean)

$$C_{3,\mu\nu}(t_{em}, t_H) = \int d^3x \int d^3y \; e^{-i\vec{\mathbf{p}}_\gamma\cdot\vec{\mathbf{x}}} e^{i\vec{\mathbf{p}}_H\cdot\vec{\mathbf{y}}} \langle J_\mu^{\mathrm{em}}(t_{em}, \vec{\mathbf{x}}) J_\nu^{\mathrm{weak}}(0) \phi_H^\dagger(t_H, \vec{\mathbf{y}}) \rangle$$

$$\phi_H^\dagger \sim \bar{Q}\gamma_5 u$$

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$$I_{\mu\nu}^{<}(T, t_H) = \int_{-T}^{0} dt_{em} e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

$$I_{\mu\nu}^{>}(T, t_H) = \int_{0}^{T} dt_{em} e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

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$$I_{\mu\nu}(T, t_H) = I^{<}(T, t_H) + I^{>}(T, t_H)$$

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$$I_{\mu\nu}^<(T, t_H) = \int_{-T}^{0} dt_{em} e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

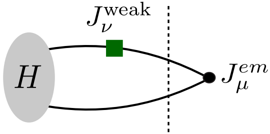
$$I_{\mu\nu}^>(T, t_H) = \int_{0}^{T} dt_{em} e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

$$I_{\mu\nu}(T, t_H) = I^<(T, t_H) + I^>(T, t_H)$$

Show relation between $I_{\mu\nu}(T, t_H)$ and $T_{\mu\nu}$
$\rightarrow$ compare spectral decompositions of both time orderings of $I_{\mu\nu}$ and $T_{\mu\nu}$

Euclidean spectral decomposition of $I_{\mu\nu}^{>}$

Time ordering: $t_{em} > 0$



$$T_{\mu\nu}^{>} = -\sum_n \frac{\langle 0|\, J_\mu^{em}(0)\,|n(\vec{\mathbf{p}}_\gamma)\rangle \,\langle n(\vec{\mathbf{p}}_\gamma)|\, J_\nu^{weak}(0)\,|H(\vec{\mathbf{p}}_H)\rangle}{2E_{n,\vec{\mathbf{p}}_\gamma}(E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma})}$$

Euclidean spectral decomposition of $I^{>}_{\mu\nu}$

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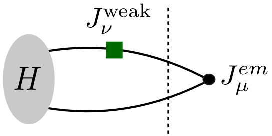

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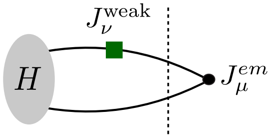

$$T^{>}_{\mu\nu} = -\sum_n \frac{\langle 0| J^{em}_\mu(0) |n(\vec{\mathbf{p}}_\gamma)\rangle \langle n(\vec{\mathbf{p}}_\gamma)| J^{weak}_\nu(0) |H(\vec{\mathbf{p}}_H)\rangle}{2E_{n,\vec{\mathbf{p}}_\gamma}(E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma})}$$

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$$\times \sum_n \frac{\langle 0| J^{em}_\mu(0) |n(\vec{\mathbf{p}}_\gamma)\rangle \langle n(\vec{\mathbf{p}}_\gamma)| J^{weak}_\nu(0) |m(\vec{\mathbf{p}}_H)\rangle}{2E_{n,\vec{\mathbf{p}}_\gamma}(E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma})} \left[1 - e^{(E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma})T}\right]$$

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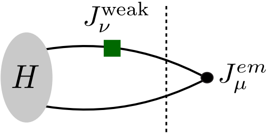

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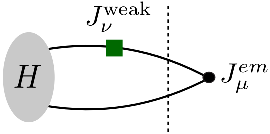
$t_H \to -\infty$ to achieve ground state saturation

Euclidean spectral decomposition of $I^{>}_{\mu\nu}$

Time ordering: $t_{em} > 0$



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$t_H \to -\infty$ to achieve ground state saturation

$T \to \infty$ to remove unwanted exponentials that come with intermediate states

Final relation

For $\mathbf{p}_\gamma \neq \mathbf{0}$,

$$T_{\mu\nu} = \lim_{T\to\infty} \lim_{t_H\to-\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\vec{\mathbf{p}}_H)|\, \phi_H^\dagger \,|0\rangle} \underbrace{\int_{-T}^{T} dt_{em}\ e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)}_{I_{\mu\nu}(T, t_H)}$$

Calculating $I_{\mu\nu}(T, t_H)$

$$T_{\mu\nu} = \lim_{T\to\infty} \lim_{t_H\to-\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\vec{\mathbf{p}}_H)|\, \phi_H^\dagger \,|0\rangle} \underbrace{\int_{-T}^{T} dt_{em} \; e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)}_{I_{\mu\nu}(T,t_H)}$$

Two methods to calculate $I_{\mu\nu}(T, t_H)$:

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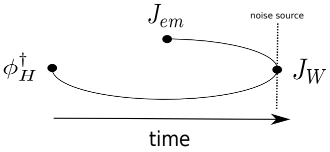
1: 3d (timeslice) sequential propagator through $\phi_H^\dagger \to$ calculate $C_{3,\mu\nu}(t_{em}, t_H)$ on lattice, fixed t_H get all t_{em} for free



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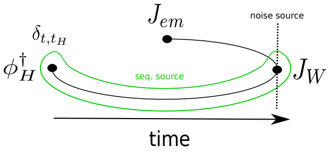

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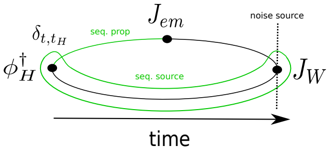

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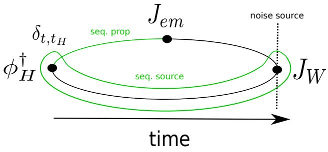
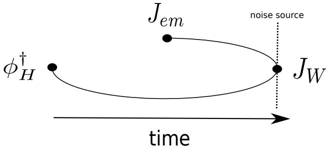


2: 4d sequential propagator through J_μ^{em} $\to$ calculate $I_{\mu\nu}(T, t_H)$ on lattice, fixed T get all t_H for free



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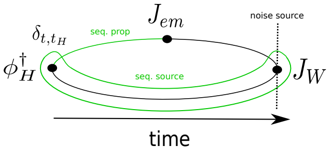

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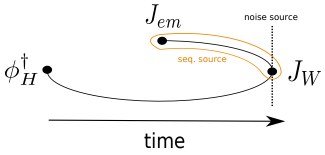

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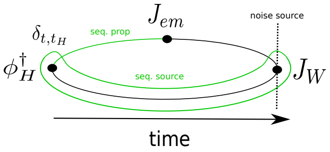

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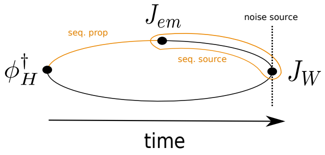

Past lattice studies

$$T_{\mu\nu} = \lim_{T\to\infty} \lim_{t_H\to-\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\vec{\mathbf{p}}_H)|\,\phi_H^\dagger\,|0\rangle} \underbrace{\int_{-T}^{T} dt_{em}\; e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)}_{I_{\mu\nu}(T,t_H)}$$

- [1] we presented results at Lattice 2019 using 3d method
  - fitting to a constant looking for plateaus in T and t_H
  - all calculations were done in the rest frame of the meson
- [2] use 4d method to perform realistic physical calculation
  - set $T = N_T/2$ and fit to constant in t_H where data has plateaued

[1] [Kane, Lehner, Meinel, Soni, arXiv:1907.00279]

[2] [Desiderio, Frezzotti, Garofalo, Giusti, Hansen, Lubicz, Martinelli, Sachrajda, Sanfilippo, Simula, Tantalo, PRD 2021, arXiv:2006.05358]

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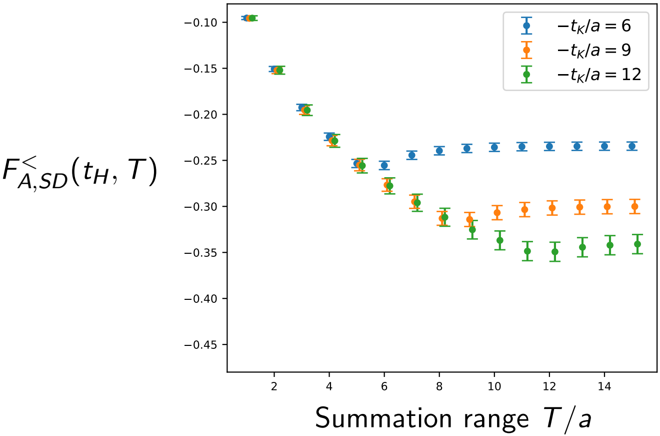
- We have new data in moving frame of meson where fitting to a constant is not possible

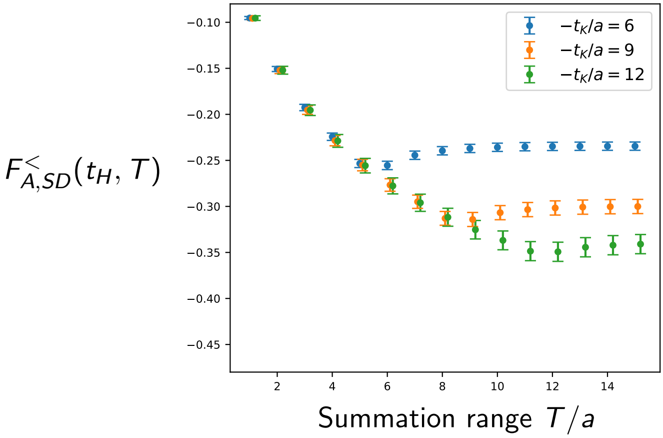
$K^- \to \gamma \ell^- \bar{\nu}_\ell$: $p_K = \frac{2\pi}{L}(0,0,1)$, $p_\gamma = \frac{2\pi}{L}(0,0,1)$

3d method $t_{em} < 0$ time ordering



$K^- \to \gamma \ell^- \bar{\nu}_\ell$: $p_K = \frac{2\pi}{L}(0,0,1)$, $p_\gamma = \frac{2\pi}{L}(0,0,1)$

3d method $t_{em} < 0$ time ordering



Must perform more complicated fits to take $\lim_{T\to\infty}$ and $\lim_{t_H} \to -\infty$

Current work

Developed fitting methods to take $\lim_{T\to\infty}$ and $\lim_{t_H} \to -\infty$ by:

- fitting only 3d sequential propagator data
- fitting only 4d sequential propagator data
- performing global fits to both 3d and 4d method data

Compare each of the fitting methods

Simulation parameters

So far we have considered:

$$K^- \to \gamma \ell^- \bar{\nu}, \qquad D^+ \to \gamma \ell^+ \nu, \qquad D_s^+ \to \gamma \ell^+ \nu$$

- RBC/UKQCD: $24^3 \times 64$, $m_\pi \approx 340$ MeV, $a \approx 0.11$ fm (K^-, D^+, D_s^+)
- RBC/UKQCD: $32^3 \times 64$, $m_\pi \approx 340$ MeV, $a \approx 0.11$fm (K^-)
- $\mathbb{Z}_2$ random wall sources at weak current location
- Up/down/strange valence quarks: same domain-wall action as sea quarks
- Neglect disconnected diagrams
- Charm valence quarks: Möbius domain-wall with "stout" smearing
- "Mostly nonperturbative" renormalization
- All-mode averaging with 16 sloppy and 1 exact samples per config

see [Kane, Lehner, Meinel, Soni, arXiv:1907.00279] for more details

$K^- \to \gamma \ell^- \bar{\nu}$ runs

RBC/UKQCD Ensembles: $32^3 \times 64$, $m_\pi \approx 340$ (MeV), $a \approx 0.11$(fm)

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3d method:

$-t_K/a$	$\frac{L}{2\pi}\mathbf{p}_K$	$\frac{L}{2\pi}\mathbf{p}_\gamma$	# configs
$\{6, 9, 12\}$	(0,0,1)	(0,0,1)	20
	(0,0,2)		

$K^- \to \gamma \ell^- \bar{\nu}$ runs

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4d method:

T/a	$\frac{L}{2\pi}\mathbf{p}_K$	$\frac{L}{2\pi}\mathbf{p}_\gamma$	# configs
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	(0,0,2)		

Fit form: 3d method

Include terms to fit
(1) unwanted exponential from first intermediate state
(2) first excited state

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Fit form factors F_V and $F_{A,SD}$ directly instead of $I_{\mu\nu}$

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Include terms to fit
(1) unwanted exponential from first intermediate state
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Fit form factors F_V and $F_{A,SD}$ directly instead of $I_{\mu\nu}$

Time ordering $t_{em} < 0$:

$$F^{<}(t_H, T) = F^{<} + B_F^{<}(1 + B_{F,\mathrm{exc}}^{<}\, e^{\Delta E(T+t_H)})\, e^{-(E_\gamma - E_H + E^{<})T} + C_F^{<}\, e^{\Delta E\, t_H}$$

■ fit parameters

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■ fit parameters

Only have three values of t_H, fitting multiple exponentials not possible

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Only have three values of t_H, fitting multiple exponentials not possible
$\rightarrow$ Determine ΔE from the pseudoscalar two-point correlation function
$\rightarrow$ use result as Gaussian prior in form factor fits

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■ fit parameters

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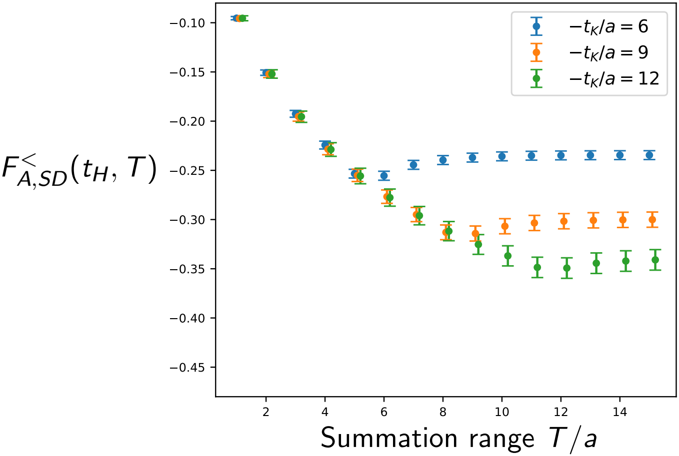
Look at preliminary data and fit results for $K^- \rightarrow \gamma \ell^- \bar{\nu}_\ell$

$K^- \to \gamma \ell^- \bar{\nu}_\ell$: 3d method $p_K = \frac{2\pi}{L}(0,0,1)$ $p_\gamma = \frac{2\pi}{L}(0,0,1)$

Time ordering $t_{em} < 0$:

$$F^<_{A,SD}(t_H, T) = F^<_{A,SD} + B^<_{F_{A,SD}}(1 + B^<_{F_{A,SD},\mathrm{exc}} e^{\Delta E(T+t_H)}) e^{-(E_\gamma - E_K + E^<_A)T} + C^<_{F_{A,SD}} e^{\Delta E t_H}$$



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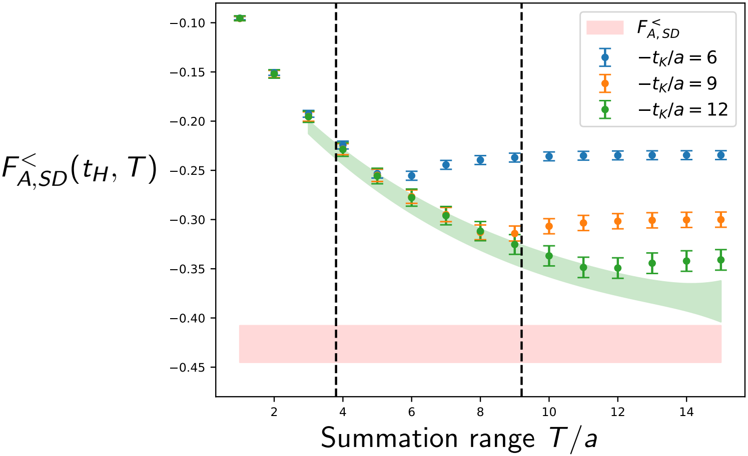

Fit form: 4d method

Use fit ranges where data has plateaued in t_H, i.e. $t_H \to -\infty$

Include terms to fit
(1) unwanted exponential from first intermediate state

Fit form: 4d method

Use fit ranges where data has plateaued in t_H, i.e. $t_H \to -\infty$

Include terms to fit
(1) unwanted exponential from first intermediate state

Sum of both time orderings $I_{\mu\nu}(T, t_H) = I^{<}_{\mu\nu}(T, t_H) + I^{>}_{\mu\nu}(T, t_H)$

Fit form: 4d method

Use fit ranges where data has plateaued in t_H, i.e. $t_H \to -\infty$

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(1) unwanted exponential from first intermediate state

Sum of both time orderings $I_{\mu\nu}(T, t_H) = I^{<}_{\mu\nu}(T, t_H) + I^{>}_{\mu\nu}(T, t_H)$

$$F(t_H, T) = F + B_F^{<}\, e^{-(E_\gamma - E_H + E^{<})T} + B_F^{>}\, e^{(E_\gamma - E^{>})T}$$

■ fit parameters

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Use fit ranges where data has plateaued in t_H, i.e. $t_H \to -\infty$

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$$F(t_H, T) = F + B_F^{<} \underbrace{e^{-(E_\gamma - E_H + E^{<})T}}_{t_{em}<0} + B_F^{>} \underbrace{e^{(E_\gamma - E^{>})T}}_{t_{em}>0}$$

■ fit parameters

Fit form: 4d method

Use fit ranges where data has plateaued in t_H, i.e. $t_H \to -\infty$

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■ fit parameters

Only have three values of T, fitting multiple exponentials not possible
$\to$ Use broad Gaussian prior on $E^{>}$ exclude unphysical values

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■ fit parameters

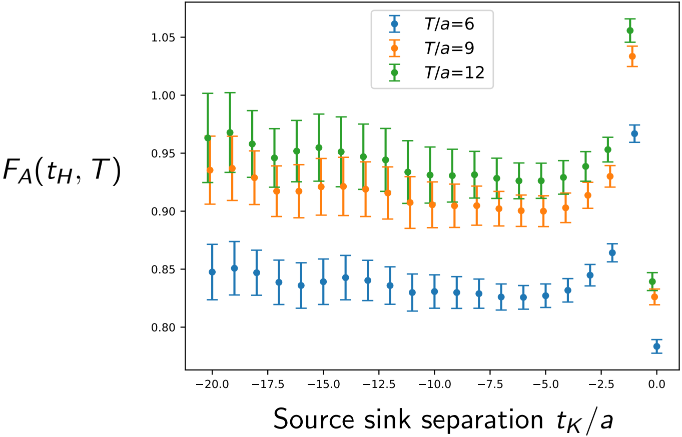
Only have three values of T, fitting multiple exponentials not possible
$\to$ Use broad Gaussian prior on $E^>$ exclude unphysical values

Look at preliminary data and fit results for $K^- \to \gamma \ell^- \bar{\nu}_\ell$

$K^- \to \gamma \ell^- \bar{\nu}_\ell$: 4d method $p_K = \frac{2\pi}{L}(0,0,1)$ $p_\gamma = \frac{2\pi}{L}(0,0,1)$

Sum of both time orderings $t_{em} < 0 + t_{em} > 0$:

$$F_A(t_H, T) = F_A + B^{<}_{F_A} e^{-(E_\gamma - E_K + E_A^{<})T} + B^{>}_{F_A} e^{(E_\gamma - E_A^{>})T}$$

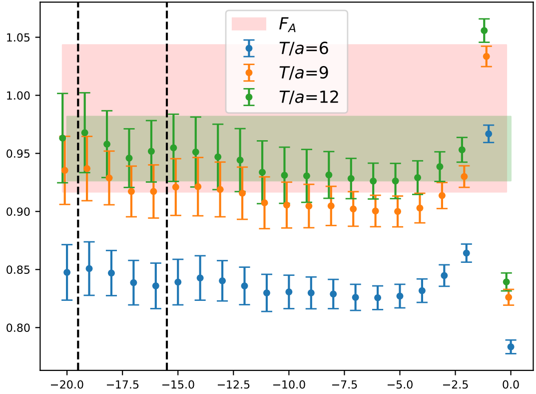


$K^- \to \gamma \ell^- \bar{\nu}_\ell$: 4d method $p_K = \frac{2\pi}{L}(0,0,1)$ $p_\gamma = \frac{2\pi}{L}(0,0,1)$

Sum of both time orderings $t_{em} < 0 + t_{em} > 0$:

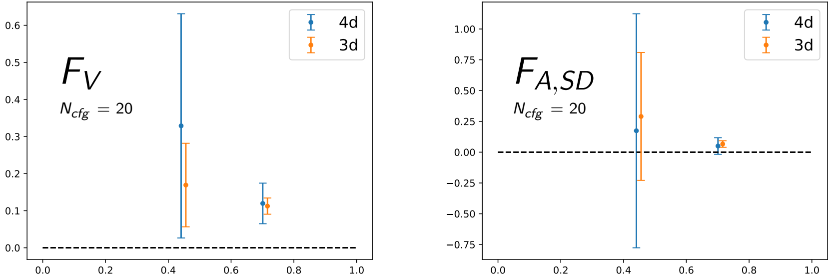
$$F_A(t_H, T) = F_A + B_{F_A}^{<} e^{-(E_\gamma - E_K + E_A^{<})T} + B_{F_A}^{>} e^{(E_\gamma - E_A^{>})T}$$

$F_A(t_H, T)$



Source sink separation t_K/a

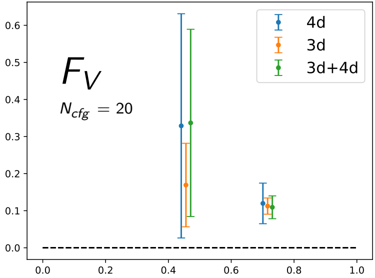
$K^- \to \gamma \ell^- \bar{\nu}_\ell$: Independent 3d and 4d analysis results



$$x_\gamma = 2E_\gamma^{(0)}/m_K \qquad x_\gamma = 2E_\gamma^{(0)}/m_K$$

$$0 < x_\gamma < 1$$

$K^- \to \gamma\ell^-\bar{\nu}_\ell$: Combined 3d and 4d analysis results



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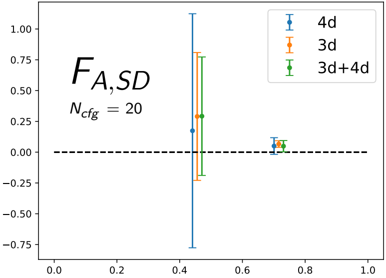
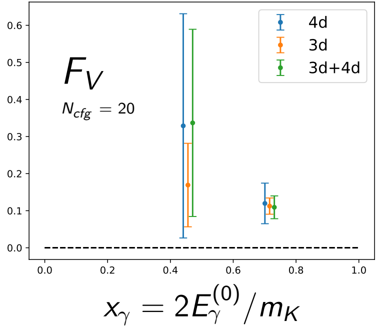
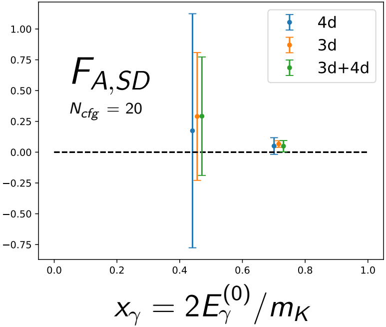


$$x_\gamma = 2E_\gamma^{(0)}/m_K$$

$$0 < x_\gamma < 1$$

$K^- \to \gamma \ell^- \bar{\nu}_\ell$: 3d and 4d analysis results





- Combined analysis $\to$ remove prior on the parameter $E^>$
- Still need two-point function to constrain excited state energy gap ΔE
- Combined fits have error similar or larger to 3d fits alone
- Similar findings for the $D_s^+ \to \gamma \ell^+ \nu_\ell$ decay process

Future plans

$$T_{\mu\nu} = \lim_{T\to\infty}\lim_{t_H\to-\infty}\frac{-2E_H e^{-E_H t_H}}{\langle H(\vec{\mathbf{p}}_H)|\,\phi_H^\dagger\,|0\rangle}\underbrace{\int_{-T}^{T} dt_{em}\;e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em},t_H)}_{I_{\mu\nu}(T,t_H)}$$

- Calculate $I^{<}_{\mu\nu}(T,t_H)$ and $I^{>}_{\mu\nu}(T,t)$ separately using the 4d method
- Twisted boundary conditions to get to small photon energies $E_\gamma^{(0)}$
- Realistic physical calculation for K and $D_{(s)}$, $B_{(s)}$ mesons

Backup slides

Summary

$$T_{\mu\nu} = \lim_{T\to\infty} \lim_{t_B\to-\infty} \frac{-2E_B e^{-E_B t_B}}{\langle B(\vec{\mathbf{p}}_B)|\,\phi_B^\dagger\,|0\rangle} \underbrace{\int_{-T}^{T} dt_{em}\; e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_B)}_{I_{\mu\nu}(T,t_H)}$$

Not always possible to fit to a constant in T and t_H

3d sequential propagator: fixed t_B get all t_{em} for free

- good control over $\lim_{T\to\infty}$ for both $t_{em} > 0$ and $t_{em} < 0$
- need two-point function to constrain excited state energy to take $\lim_{t_B\to-\infty}$

4d sequential propagator: fixed T get all t_B for free

- good control over $\lim_{t_B\to-\infty}$
- cannot resolve separate time orderings for t_{em}, need prior on energy to take $\lim_{T\to\infty}$

Global fit result:

- do not need prior on 4d fit parameter
- error similar or greater in size to only 3d data fits

$D^+ \to \gamma\ell^+\nu$ and $D_s^+ \to \gamma\ell^+\nu$ runs

RBC/UKQCD Ensemble: $24^3 \times 64$, $m_\pi \approx 340$ MeV, $a \approx 0.11$ fm

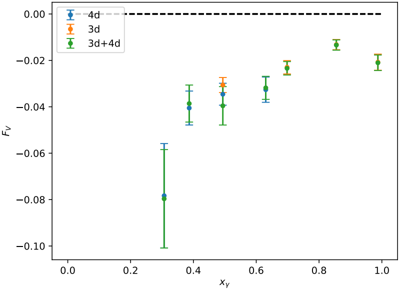
3d method:

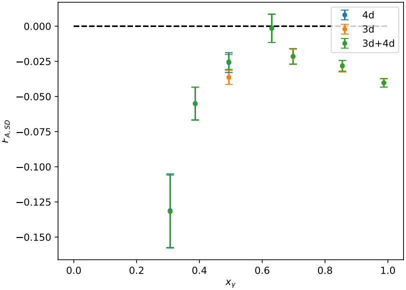
Lattice	$-t_{D_s}/a$	$\mathbf{p}_{D_{(s)}}/(\frac{2\pi}{L})$	$\mathbf{p}_\gamma^2/(\frac{2\pi}{L})^2$	# configs
$24^3 \times 64$	$\{6, 9, 12\}$	(0,0,0)	$\{1, 2, 3, 4\}$	25

4d method:

Lattice	T/a	$\mathbf{p}_{D_{(s)}}/(\frac{2\pi}{L})$	$\mathbf{p}_\gamma/(\frac{2\pi}{L})$	# configs
$24^3 \times 64$	$\{6, 9, 12\}$	(0,0,0)	(0,0,1)	25
		(0,0,1)		
		(0,0,2)		
		(0,0,-1)		

$D_s^+ \to \gamma\ell^+\nu_\ell$: Combined 3d and 4d analysis results





Minkowski spectral decomposition of $T_{\mu\nu}$

Time ordering $t_{em} < 0$:

$$\widehat{1} = |0\rangle \langle 0| + \sum_n \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_n(\vec{\mathbf{p}})} |n(\vec{\mathbf{p}})\rangle \langle n(\vec{\mathbf{p}})|$$

$$\begin{aligned} T^{<}_{\mu\nu} &= -i \int_{-\infty(1-i\epsilon)}^{0} dt_{em} \int \mathrm{d}^3x \; e^{ip_\gamma \cdot x} \langle 0| J^{weak}_\nu(0) \widehat{1} J^{em}_\mu(x) \left| B^-(\vec{\mathbf{p}}_B) \right\rangle \\ &= -\sum_n \frac{1}{2E_{n,\vec{\mathbf{p}}_B - \vec{\mathbf{p}}_\gamma}} \frac{1}{E_\gamma + E_{n,\vec{\mathbf{p}}_B - \vec{\mathbf{p}}_\gamma} - E_{B,\vec{\mathbf{p}}_B} - i\epsilon} \\ &\qquad \times \langle 0| J^{weak}_\nu(0) \left| n(\vec{\mathbf{p}}_B - \vec{\mathbf{p}}_\gamma) \right\rangle \langle n(\vec{\mathbf{p}}_B - \vec{\mathbf{p}}_\gamma)| J^{em}_\mu(0) \left| B(\vec{\mathbf{p}}_B) \right\rangle \end{aligned}$$

(In infinite volume, the sum over n includes an integral over the continuous spectrum of multi-particle states.)

Euclidean spectral decomposition of $I_{\mu\nu}$

Time ordering $t_{em} < 0$: (for large negative t_B)

$$I^{<}_{\mu\nu}(t_B, T) = \int_{-T}^{0} dt_{em}\ e^{E_\gamma t} C_{3,\mu\nu}(t_{em}, t_B)$$

(* all times are now Euclidean)

$$= \langle B(\vec{\mathbf{p}}_B)|\, \phi_B^\dagger(0)\, |0\rangle \frac{1}{2E_{B,\vec{\mathbf{p}}_B}} e^{E_B t_B}$$

$$\times \sum_n \frac{1}{2E_{n,\vec{\mathbf{p}}_B-\vec{\mathbf{p}}_\gamma}} \frac{\langle 0|\, J_\nu^{weak}(0)\, |n(\vec{\mathbf{p}}_B - \vec{\mathbf{p}}_\gamma)\rangle \langle n(\vec{\mathbf{p}}_B - \vec{\mathbf{p}}_\gamma)|\, J_\mu^{em}(0)\, |B(\vec{\mathbf{p}}_B)\rangle}{E_\gamma + E_{n,\vec{\mathbf{p}}_B-\vec{\mathbf{p}}_\gamma} - E_{B,\vec{\mathbf{p}}_B}}$$

$$\times \left[1 - e^{-(E_\gamma + E_{n,\vec{\mathbf{p}}_B-\vec{\mathbf{p}}_\gamma} - E_{B,\vec{\mathbf{p}}_B})T}\right]$$

Require $E_\gamma + E_{n,\vec{\mathbf{p}}_B-\vec{\mathbf{p}}_\gamma} - E_{B,\vec{\mathbf{p}}_B} > 0$ to get rid of unwanted exponential

States $|n(\vec{\mathbf{p}}_B - \vec{\mathbf{p}}_\gamma)\rangle$ has same flavor quantum numbers as B meson

$\rightarrow E_{n,\mathbf{p}_B-\mathbf{p}_\gamma} \geq E_{B,\mathbf{p}_B-\mathbf{p}_\gamma} = \sqrt{m_B^2 + (\mathbf{p}_B - \mathbf{p}_\gamma)^2}$

For $\mathbf{p}_\gamma \neq 0$, $|\mathbf{p}_\gamma| + \sqrt{m_n^2 + (\mathbf{p}_B - \mathbf{p}_\gamma)^2} > \sqrt{m_B^2 + \mathbf{p}_B}$ is automatically satisfied

Minkowski spectral decomposition of $T_{\mu\nu}$

Time ordering $t_{em} > 0$:

$$T^{>}_{\mu\nu} = -i \int_0^{\infty(1-i\epsilon)} dt_{em} \int d^3x \; e^{ip_\gamma \cdot x} \langle 0 | \, J^{em}_\mu(t_{em}, \vec{\mathbf{x}}) J^{weak}_\nu(0) \left| B^-(\vec{\mathbf{p}}_B) \right\rangle$$

$$= \sum_m \frac{1}{2E_{m,\vec{\mathbf{p}}_\gamma}} \frac{1}{E_\gamma - E_{m,\vec{\mathbf{p}}_\gamma} - i\epsilon} \times \langle 0 | \, J^{em}_\mu(0) \, | m(\vec{\mathbf{p}}_\gamma) \rangle \, \langle m(\vec{\mathbf{p}}_\gamma) | \, J^{weak}_\nu(0) \, | B(\vec{\mathbf{p}}_B) \rangle$$

(In infinite volume, the sum over n includes an integral over the continuous spectrum of multi-particle states.)

Euclidean spectral decomposition of $I_{\mu\nu}$

Time ordering: $t_{em} > 0$ (for large negative t_B)

$$
\begin{aligned}
I^{>}_{\mu\nu}(t_B, T) &= \int_0^T dt_{em}\; e^{E_\gamma t} C_{\mu\nu}(t_{em}, t_B) \qquad \text{(* all times are now Euclidean)}\\
&= -\langle B(\vec{\mathbf{p}}_B)|\, \phi^\dagger_B(0)\, |0\rangle \, \frac{1}{2E_{B,\vec{\mathbf{p}}_B}} e^{E_B t_B}\\
&\times \sum_n \frac{1}{2E_{n,\vec{\mathbf{p}}_\gamma}} \langle 0|\, J^{em}_\mu(0)\, |n(\vec{\mathbf{p}}_\gamma)\rangle \langle n(\vec{\mathbf{p}}_\gamma)|\, J^{weak}_\nu(0)\, |B(\vec{\mathbf{p}}_B)\rangle\\
&\times \frac{1}{E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma}} \left[1 - e^{(E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma})T}\right]
\end{aligned}
$$

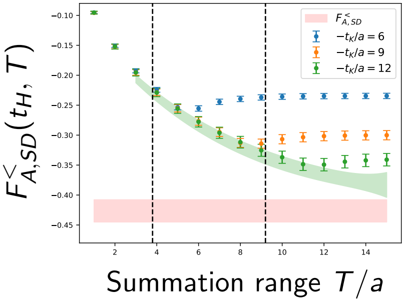
Require $E_\gamma - E_{n,\vec{\mathbf{p}}_\gamma} < 0$

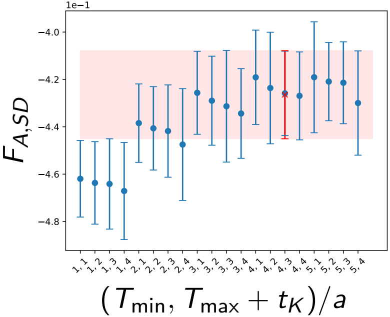
Because the states $\left|n(\mathbf{p}_\gamma)\right\rangle$ have mass, $\sqrt{m_n^2 + \mathbf{p}_\gamma^2} > |\mathbf{p}_\gamma|$ is automatically satisfied

$K^- \to \gamma \ell^- \bar{\nu}_\ell$: 3d method $p_K = \frac{2\pi}{L}(0,0,1)$ $p_\gamma = \frac{2\pi}{L}(0,0,1)$

Time ordering $t_{em} < 0$: Stability fits

$$F^<_{A,SD}(t_H, T) = F^<_{A,SD} + B^<_{F_{A,SD}}(1 + B^<_{F_{A,SD},\text{exc}} e^{\Delta E(T+t_H)}) e^{-(E_\gamma - E_K + E^<_A)T} + C^<_{F_{A,SD}} e^{\Delta E t_H}$$





Right: result of the $F^<_{A,SD}$ parameter for different fit ranges. The red data point/band shows the chosen fit range. Left: result of the fit for that chosen fit range on top of the data.