Finite size effects in the leading hadronic vacuum polarisation contribution to $(g-2)_\mu$

[Nature 593 (2021) 7857, 51-55]

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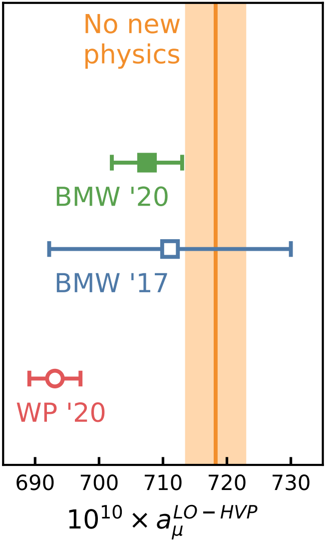
on behalf of the Budapest-Marseille-Wuppertal collaboration

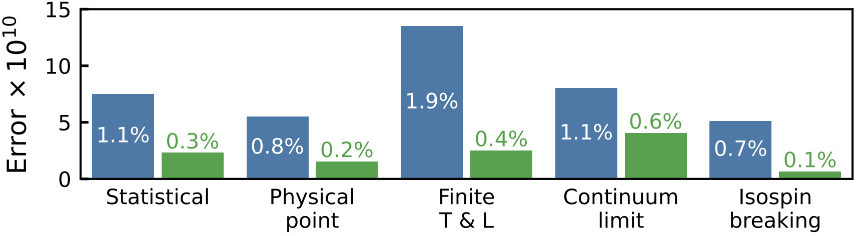
Borsanyi, Fodor, Guenther, Hoelbling, Katz, Lellouch, Lippert,
Miura, Parato, Szabo, Stokes, Toth, Torok, Varnhorst

Three years of progress

- Recently published sub-percent determination of HVP contribution to $g_\mu - 2$ [BMWc '20]
- First lattice calculation with errors comparable to data-driven determinations
- 3.4× increase in precision over our earlier work [BMWc '17]
- Many improvements needed to attain this precision, thanks to the work of many groups around the world

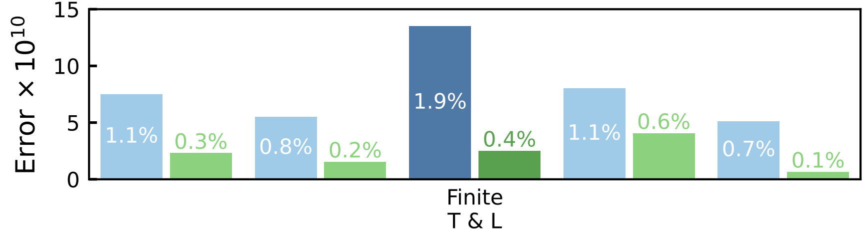


Key improvements



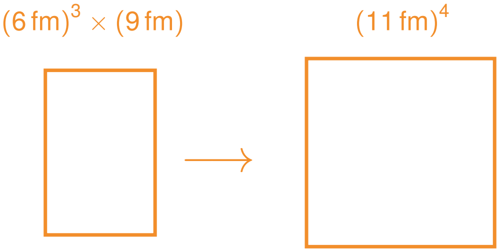
- Finite-size errors were dominant in our previous calculation
- In this talk I will explain how we reduced this error by 4.8×
- Other errors become dominant, requiring further improvements
  - Continuum limit [K. Szabo, Mon 13:15 EDT]
  - Physical point [L. Varnhorst, Mon 14:00 EDT]
  - Isospin breaking [L. Parato, Tue 06:15 EDT]

Finite size effects



- Even in our large volumes ($L \gtrsim 6.1\,\text{fm}$, $T \geq 8.7\,\text{fm}$), exponentially suppressed FV effects are significant
- One-loop SU(2) χPT[Aubin et al 16] suggests $\sim 2\%$ effect
- Perform dedicated FV study with even larger volumes: $(\sim 11\,\text{fm})^4$
- χPT& other models validated by comparing to lattice data
- Use two-loop χPT[Aubin et al 20] for tiny, residual correction

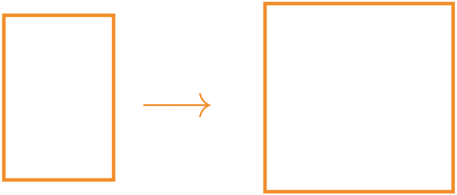
Dedicated finite-volume study



- Perform continuum extrapolation at reference volume
- Apply finite-size corrections in continuum
- Taste breaking distorts finite-size effects
- Large volumes only practical with coarse lattices ($a = 0.112\,\mathrm{fm}$)
- Perform dedicated simulations with reduced taste breaking
  - DBW2 action [Takaishi et al '96] and 4HEX smearing [Capitani et al '06] to suppress UV fluctuations
  - Tune pion masses with HMS mass instead of Goldstone pion

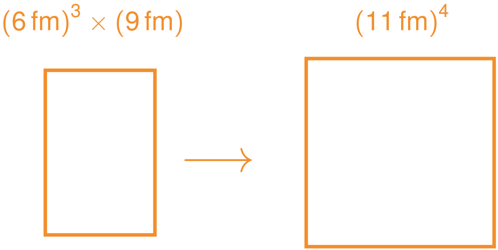
Dedicated finite-volume study

$(6\,\mathrm{fm})^3 \times (9\,\mathrm{fm})$ $(11\,\mathrm{fm})^4$



4HEX	18.1 ± 2.4	$\times 10^{-10}$

Model cross-check



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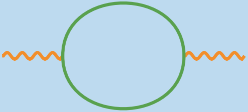
NNLO χPT		$\times 10^{-10}$
MLLGS		$\times 10^{-10}$

Chiral perturbation theory

NLO: free two-pion contribution



Chiral perturbation theory

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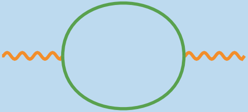
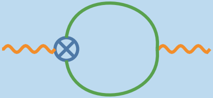
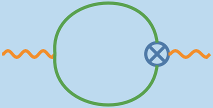


NNLO: L_9 contribution





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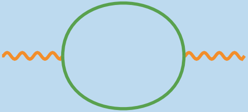

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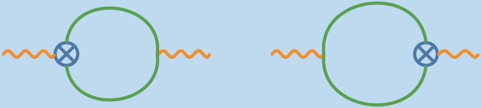
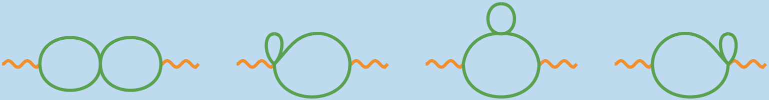
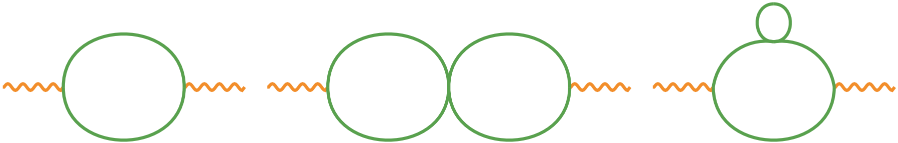


NNLO: pion interactions



Finite size effects



- Integrate over loop momenta

$$\frac{1}{3}\int\frac{d^3p}{(2\pi)^3}\frac{e^{-2E_pt}}{E_p^2}p^2\left[1+\frac{16}{F^2}L_9E_p^2-\frac{2}{F^2}\int\frac{d^3r}{(2\pi)^3}\frac{1}{2E_r}\right.$$
$$\left.+\frac{1}{3}\int\frac{d^3r}{(2\pi)^3}\frac{1}{E_r}\frac{r^2}{E_r^2-E_p^2}\right]$$

- Finite volume/time effects:

$$\int\frac{d^3p}{(2\pi)^3}\to\frac{1}{L^3}\sum_p,\qquad\int\frac{dp_4}{2\pi}\to\frac{1}{T}\sum_{p_4}$$

Meyer-Lellouch-Lüscher-Gounaris-Sakurai model

- Phenomenological model of interacting two-pion states
- Obtain correlator as Laplace transform of spectral function [Bernecker & Meyer '11]

$$\rho(E(k))|_{\pi\pi} = \frac{1}{6\pi^2}\left(\frac{k}{E(k)}\right)^3 |F_\pi(k)|^2$$

- Take $F_\pi(k)$ from the Gounaris-Sakurai (GS) parameterisation [Gounaris & Sakurai '68]
- In finite volume, becomes a sum over discrete two-pion states

$$\sum_{n>0} |\vec{A}_n|^2 e^{-E_n|t|}$$

- Energy levels determined from GS phase shift by Lüscher's formula [Lüscher '91]
- Amplitudes determined from phase shift and form factor through a Lellouch-Lüscher equation [Lellouch & Lüscher '01]

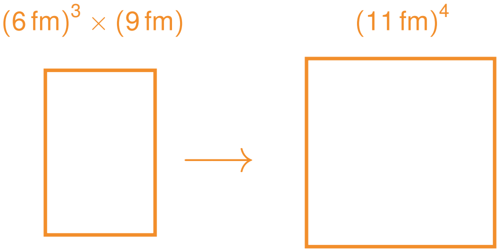
Model comparison

- Two more models for finite L (but not T)
  - Generic field-theory approach [Hansen & Patella '19, '20] (HP) relates the finite-size effect to $F_\pi(k)$
  - Rho-pion-gamma model [Chakraborty et al '17] (RHO) incorporates the $\rho(770)$ resonance directly into a χPT-like framework

- Compare finite L corrections for reference volume in infinite-T limit
- All four models agree within $\sim 2.5 \times 10^{-10}$

NNLO χPT	16.7	$\times 10^{-10}$
MLLGS	18.8	$\times 10^{-10}$
HP	17.7	$\times 10^{-10}$
RHO	16.2	$\times 10^{-10}$

Model cross-check



4HEX	18.1 ± 2.4	$\times 10^{-10}$
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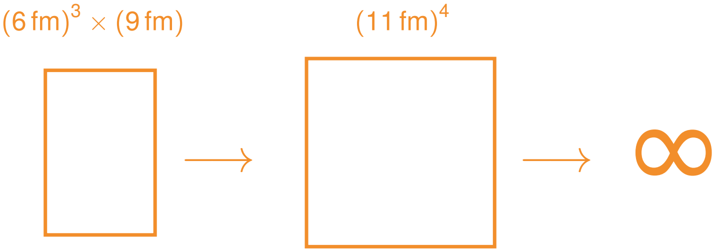
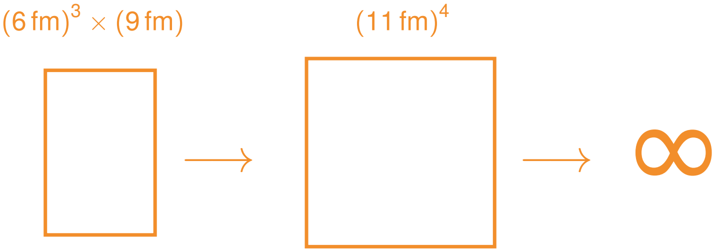
Residual correction



4HEX	18.1 ± 2.4		$\times 10^{-10}$
NNLO χPT	15.7	0.6 ± 0.3	$\times 10^{-10}$
MLLGS	17.8		$\times 10^{-10}$

Residual correction



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- Estimated $I = 0$ effect: 0.0(0.6)
- Estimated isospin-breaking correction: 0.0(0.1)

Conclusion

- Final correction of 18.7(2.5)
- Obtained by combination of:
  - Dedicated FV study
  - NNLO χPT
  - Other models
- Reduction in finite-size error contribution from 1.9% to 0.4%
- Overall reduction in total error from 2.7% to 0.8%
- Discrepancy with data-driven determination surprising
- Cross-checks with other lattice groups important