High precision scale setting on the lattice

Lukas Varnhorst for the BMW collaboration

26.07.2021

BMW collaboration

Talk based on:

S. Borsanyi, Z. Fodor, J. N. Guenther, C. Hoelbling, S. D. Katz, L. Lellouch, T. Lippert, K. Miura, L. Parato and K. K. Szabo, F. Stokes, B. C. Toth, Cs. Torok, L. Varnhorst,

"Leading-order hadronic vacuum polarization contribution to the muon magnetic moment from lattice QCD,"

Nature **593** (2021) no.7857, 51-55

[arXiv:2002.12347 [hep-lat]].

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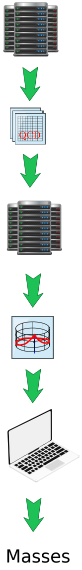
Strategy of Ω mass extraction

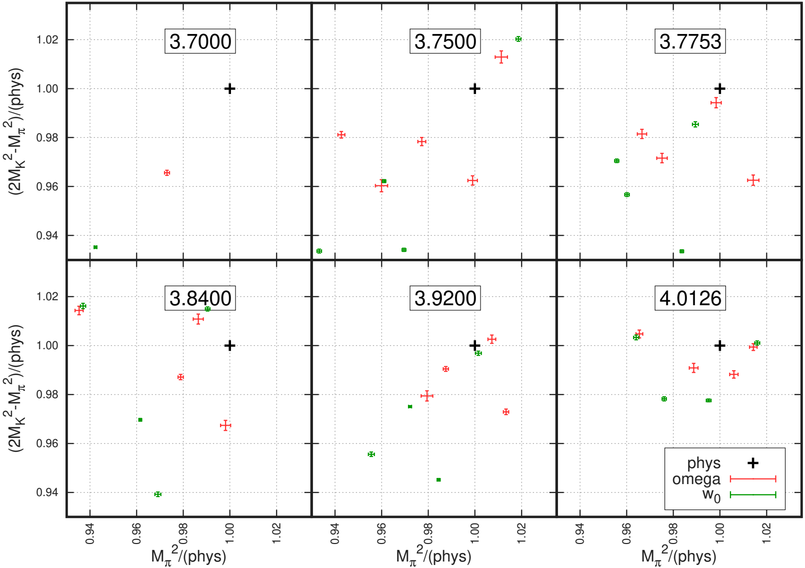


Determine the lattice scale with the Ω baryon mass. In this talk, the determination of w_0 from M_Ω is discussed.

We used an action with the following properties:

- Symmanzik improved gauge action
- $N_f = 2 + 1 + 1$
- Four steps of stout smearing of the gauge fields in the Dirac operator
- Staggered fermions

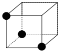
Bracketing of the physical point

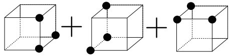


Baryon operators

Interpolating operator for baryons: Products of three quark fields. But: Dirac structure is encoded in spatial indices.

Classification of staggered baryon operators in [1]. For the Ω baryon, there are two candidates:

VI 

XI 

E.g. for the `VI` operator

$$O_{\mathrm{VI}} = \frac{1}{6}\epsilon^{abc}(D_1\chi^a(x))(D_2\chi^b(x))(D_3\chi^c(x))$$

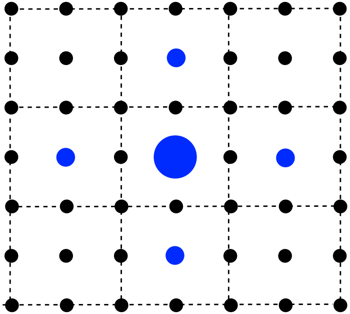
$$D_i = \frac{1}{2}(\chi(x-\hat{i}) + \chi(x+\hat{i}))$$

In [2] an additional flavor DOF is used to construct a operator `Ba`

$$O_{\mathrm{Ba}} = [2\delta_{\alpha 1}\delta_{\beta 2}\delta_{\gamma 3} - \delta_{\alpha 3}\delta_{\beta 1}\delta_{\gamma 2} - \delta_{\alpha 2}\delta_{\beta 3}\delta_{\gamma 1} + (\cdots \beta \leftrightarrow \gamma \cdots)] \times$$
$$\epsilon^{abc}\left((D_1\chi^a_\alpha(x))(D_{12}\chi^b_\beta(x))(D_{13}\chi^c_\gamma(x)) - (D_2\chi^a_\alpha(x))(D_{21}\chi^b_\beta(x))(D_{23}\chi^c_\gamma(x)) + (D_3\chi^a_\alpha(x))(D_{31}\chi^b_\beta(x))(D_{23}\chi^c_\gamma(x))\right)$$

[1] M. F. L. Golterman and J. Smit, "Lattice Baryons With Staggered Fermions," Nucl. Phys. B **255** (1985), 328-340
[2] J. A. Bailey, "Staggered baryon operators with flavor SU(3) quantum numbers," Phys. Rev. D **75** (2007), 114505 [arXiv:hep-lat/0611023 [hep-lat]]

Quark smearing



- Excited state contributions can be reduced by applying quark field smearing
- Smearing connects only next-to-neighbors to keep the taste structure intact.
- Gauge fields are 3d stout smeared fields.

$$Wv(x) = (1-\sigma)v(x) + \frac{\sigma}{6}\sum_{\mu=1,2,3}\Big(U_\mu^{\mathrm{3d}}(x)U_\mu^{\mathrm{3d}}(x+\hat{\mu})v(x+2\hat{\mu}) + U_\mu^{\mathrm{3d}\dagger}(x-\hat{\mu})U_\mu^{\mathrm{3d}\dagger}(x-2\hat{\mu})v(x-2\hat{\mu})\Big)$$

But: Suppression of excited states not enough for a single state fit.

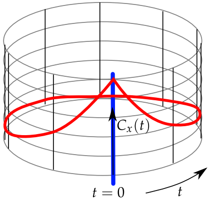
The Ω correlation function

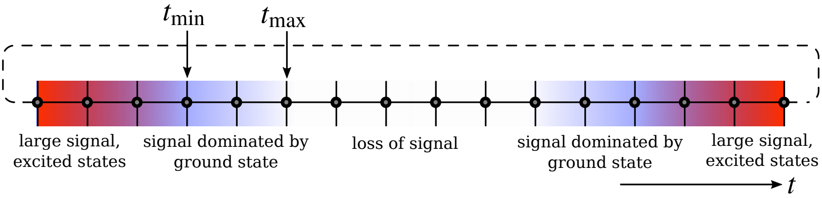
The correlation function of a staggered baryon has the form

$$H(t;A,M) = A_1 h_+(t;M_0) + A_2 h_-(t;M_2) + A_3 h_+(t;M_3) + A_4 h_-(t;M_4) + \cdots$$

with

$$h_+(t,M) = e^{-Mt} + (-1)^{t-1} e^{-M(t-T)}$$
$$h_-(t,M) = -h_+(T-t,M)$$





Extraction of the Ω mass:

4 state fit extraction

Fit propagator to

$$H(t;A,M) = A_1 h_+(t;M_0) + A_1 h_-(t;M_1) + A_2 h_+(t;M_2) + A_3 h_-(t;M_3)$$

with $h_+(t,M) = e^{-Mt} + (-1)^{t-1} e^{-M(t-T)}$ and $h_-(t,M) = -h_+(T-t,M)$.

Use priors for the excited state masses:

prior mean	rel. prior width
2012 MeV	0.10
2250 MeV	0.10
2400 MeV	0.15

GEVP based extraction

Construct matrix [1] from folded propagator H_t:

$$\mathcal{H}(t) = \begin{pmatrix} H_{t+0} & H_{t+1} & H_{t+2} & H_{t+3} \\ H_{t+1} & H_{t+2} & H_{t+3} & H_{t+4} \\ H_{t+2} & H_{t+3} & H_{t+4} & H_{t+5} \\ H_{t+3} & H_{t+4} & H_{t+5} & H_{t+6} \end{pmatrix}$$

and solve $\mathcal{H}(t_0)v(t_0,t_1) = \lambda(t_0,t_1)\mathcal{H}(t_1)v(t_0,t_1)$.

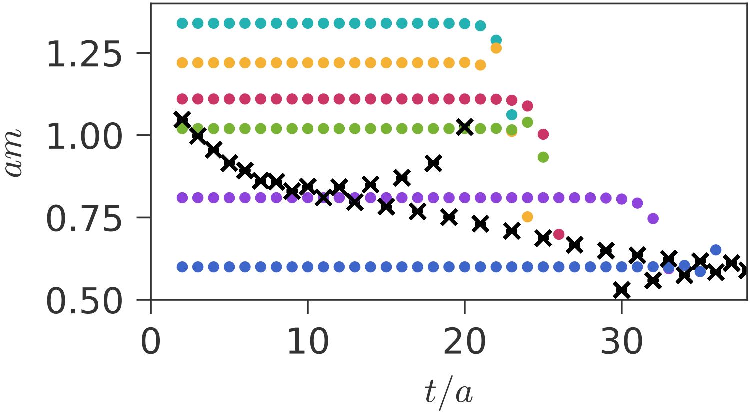
Then, extract mass from $C(t) = v^\dagger(t_0,t_1)\mathcal{H}(t)v(t_0,t_1)$ between t_{start} and t_{stop}.

No assumption on the masses of the excited states.

[1] C. Aubin and K. Orginos, "A new approach for Delta form factors," AIP Conf. Proc. **1374** (2011) no.1, 621-624 doi:10.1063/1.3647217 [arXiv:1010.0202 [hep-lat]].

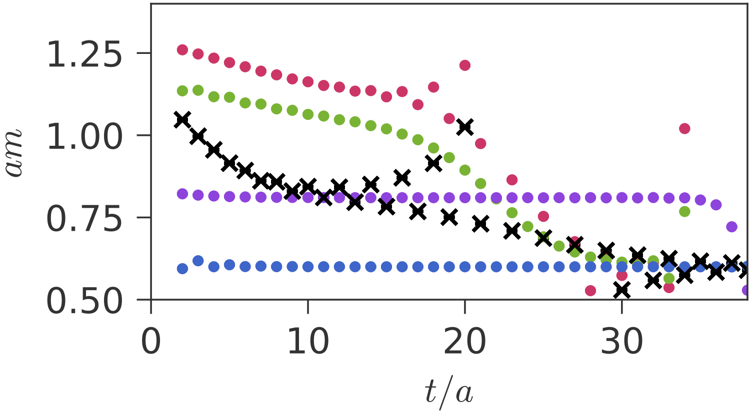
GEVP based extraction

Can improve overlap with ground state significantly → Mock analysis:



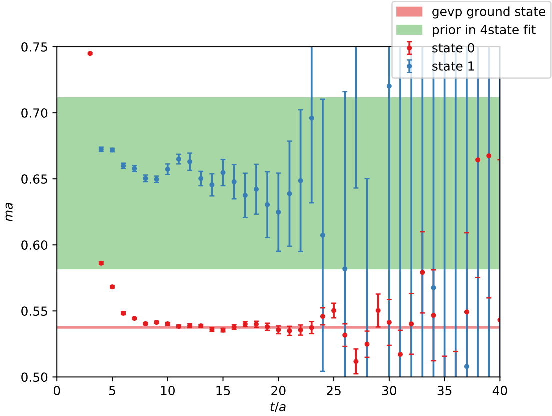
GEVP based extraction

Can improve overlap with ground state significantly → Mock analysis:



GEVP based extraction

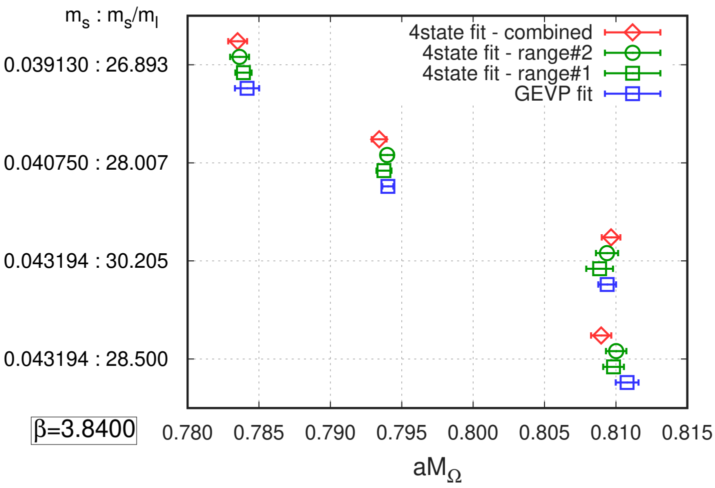
On real data ground state overlap can be significantly improved:



GEVP excited state agrees very well with prior in the 4 state fit.

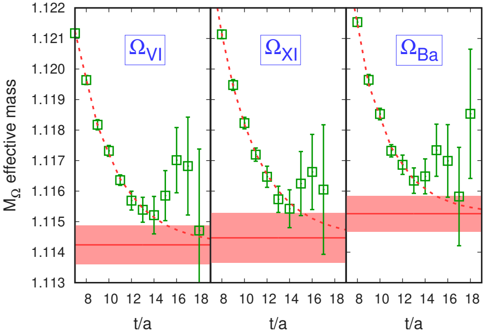
Comparison between methods

Good agreement between mass extraction with both methods:



Comparison of operators

We have investigate the 3 operators `VI`, `XI` and `Ba` on one ensemble with increased statistics:



`VI` and `XI` couple to two tastes, whereas `Ba` is constructed to couple only to one taste.

Difference safely within 0.1% error. → Use `VI` for the main analysis.

Isospin breaking for hadron masses

$$\langle \mathcal{C} \rangle_{\text{QCD+QED}} \approx \langle \mathcal{C} \rangle\vert_0 + \frac{\delta m}{m_l} \cdot \langle \mathcal{C} \rangle'_m + \frac{e_v^2}{2} \cdot \langle \mathcal{C} \rangle''_{20} + e_v e_s \cdot \langle \mathcal{C} \rangle''_{11} + \frac{e_s^2}{2} \cdot \langle \mathcal{C} \rangle''_{02}$$

Consider effective mass $\mathcal{M}[H]$ (H: correlation function) as observable:

$$M_0 = \mathcal{M}[\langle H_0 \rangle_0]$$

$$M''_{02} = \frac{\delta \mathcal{M}[H]}{\delta H}\bigg\vert_{\langle H_0 \rangle_0} \langle H \rangle''_{02} = \frac{\delta \mathcal{M}[H]}{\delta H}\bigg\vert_{\langle H_0 \rangle_0} \left\langle (H_0 - \langle H_0 \rangle_0) \frac{\text{dets}''_2}{\text{dets}_0} \right\rangle$$

$$M''_{20} \approx \frac{1}{e_v^2} (\mathcal{M}[\frac{1}{2} \langle H_+ + H_- \rangle_0] - \mathcal{M}[\langle H_0 \rangle_0])$$

$$M''_{11} \approx \frac{\delta \mathcal{M}[H]}{\delta H}\bigg\vert_{\langle H_+ + H_- \rangle_0} \left\langle \frac{H_+ - H_-}{2e_v} \frac{\text{dets}'_1}{\text{dets}_0} \right\rangle_0$$

$$M'_m \approx \frac{m_l}{\delta m} (\mathcal{M}[\langle H_{\delta m} \rangle_0] - \mathcal{M}[\langle H_0 \rangle_0])$$

Note that derivatives of $\mathcal{M}[H]$ can be calculated analytically.

For more details on isospin breaking: Talk by Letizia Parato Jul 27, 2021, 6:15 AM UET

Global fits

Using `Type I` fit for $Y = w_0 M_\Omega$:

$$Y = A + BX_l + CX_s + DX_{\delta m} + Ee_v^2 + Fe_v e_s + Ge_s^2$$

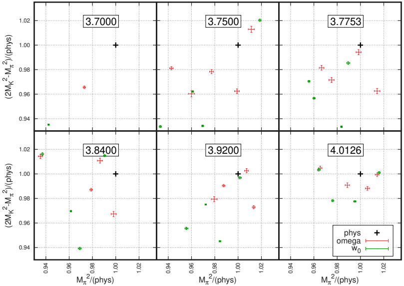
with

$$X_l = M_{\pi^0}^2 / M_\Omega^2 - \left[M_{\pi^0}^2 / M_\Omega^2 \right]_*$$

$$X_s = M_{K_\chi}^2 / M_\Omega^2 - \left[M_{K_\chi}^2 / M_\Omega^2 \right]_*$$

$$X_{\delta m} = \frac{(M_{K^0} - M_{K^+})^2}{M_\Omega^2}$$

$$M_{K_\chi}^2 = \frac{1}{2}(M_{K^0}^2 + M_{K^+}^2 - M_{\pi^+}^2)$$



The coefficients $A, \ldots, G$ themselves depend on the lattice spacing and X_l and X_s.

Global fits

Using `Type I` fit for $Y = w_0 M_\Omega$:

$$Y = A + BX_l + CX_s + DX_{\delta m} + Ee_v^2 + Fe_v e_s + Ge_s^2$$

The coefficients $A, \ldots, G$ themselves depend on the lattice spacing and X_l and X_s.

$$\begin{aligned}
A &= A_0 + A_2 \left[a^2 \alpha_s^n(1/a)\right] + A_4 \left[a^2 \alpha_s^n(1/a)\right]^2 + A_6 \left[a^2 \alpha_s^n(1/a)\right]^3 \\
B &= B_0 + B_2 a^2 \\
C &= C_0 + C_2 a^2 \\
D &= D_0 + D_2 a^2 + D_4 a^4 + D_l X_l + D_s X_s \\
E &= E_0 + E_2 a^2 + E_4 a^4 + E_l X_l + E_s X_s \\
F &= F_0 + F_2 a^2 \\
G &= G_0 + G_2 a^2
\end{aligned}$$

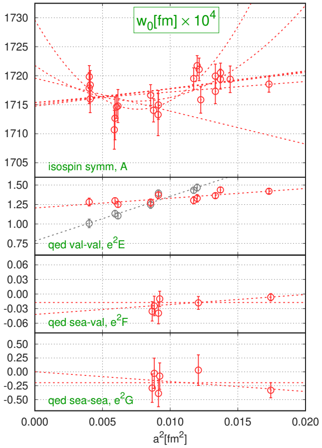
Global fits

Using Type I fit for $Y = w_0 M_\Omega$:

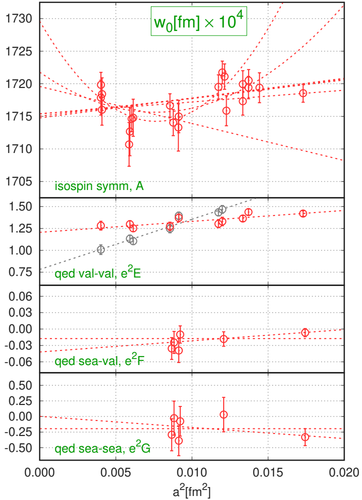
$$Y = A + BX_l + CX_s + DX_{\delta m} + Ee_v^2 + Fe_v e_s + Ge_s^2$$

Write as coupled system of equations

$$[Y]_0 = [A + BX_l + CX_s]_0$$
$$[Y]'_m = [DX_{\delta m}]'_m$$
$$[Y]''_{20} = [A + BX_l + CX_s + DX_{\delta m}]''_{20} + [E]_0$$
$$[Y]''_{11} = [A + BX_l + CX_s + DX_{\delta m}]''_{11} + [F]_0$$
$$[Y]''_{02} = [A + BX_l + CX_s + DX_{\delta m}]''_{02} + [G]_0$$



w_0 from Ω mass:



Systematic error:

- AIC weight for different fit functions or lattice spacing cuts.
- Flat weight for rest.

$$w_0 = 0.17236(29)(63)[70]\,\mathrm{fm}$$

Thank you!

backup slides

Isospin splitting of w_0

In general we have

$$\langle W_{\tau=w_0^2(e)} \rangle = 0.3 \quad \text{with} \quad W_\tau = \frac{\mathrm{d}(\tau^2 E)}{\mathrm{d}\log\tau}$$

We then expanded

$$\langle W_\tau \rangle = \langle W_\tau \rangle_0 + e_s^2 \left\langle (W_\tau - \langle W_\tau \rangle_0) \frac{\text{dets}_2''}{\text{dets}_0} \right\rangle_0$$

$$w_0(e_s) = w_0 + e_s^2 \delta w_0$$

$$W_{\tau=w_0^2(e_s)} = W_{\tau=w_0^2} + e_s^2 \cdot 2 w_0 \delta w_0 \cdot \left. \frac{\mathrm{d}W}{\mathrm{d}\tau} \right|_{\tau=w_0^2}$$

So that

$$\delta w_0 = -\left[\frac{1}{2\sqrt{\tau}} \left\langle \frac{\mathrm{d}W}{\mathrm{d}\tau} \right\rangle_0^{-1} \left\langle (W_\tau - \langle W_\tau \rangle_0 \frac{\text{dets}_2''}{\text{dets}_0} \right\rangle_0 \right]_{\tau=w_0^2}$$

QCD+QED

$$\langle\mathcal{C}\rangle_{\text{QCD+QED}} \approx \langle\mathcal{C}\rangle\!|_0 + \frac{\delta m}{m_l}\cdot\langle\mathcal{C}\rangle'_m + \frac{e_v^2}{2}\cdot\langle\mathcal{C}\rangle''_{20} + e_v e_s\cdot\langle\mathcal{C}\rangle''_{11} + \frac{e_s^2}{2}\cdot\langle\mathcal{C}\rangle''_{02}$$

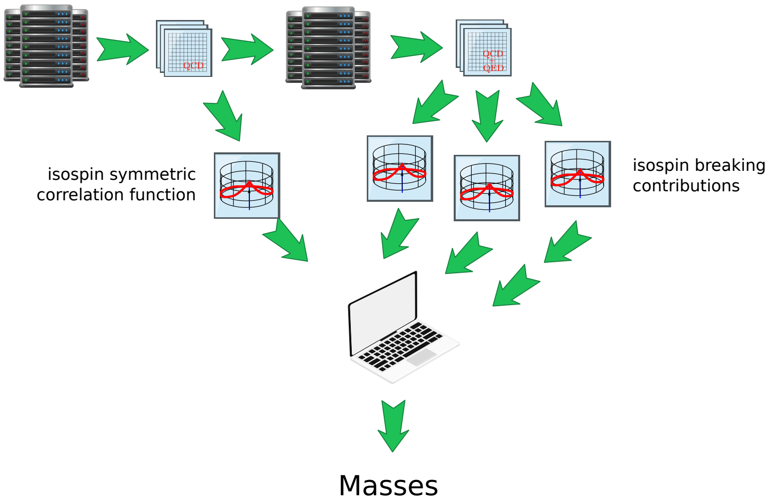
$$\begin{aligned}
\langle\mathcal{C}\rangle\!|_0 &= \langle\mathcal{C}_0(U)\rangle_U \\
\langle\mathcal{C}\rangle'_m &= \langle\mathcal{C}'_m(U)\rangle_U \\
\langle\mathcal{C}\rangle''_{20} &= \left\langle\langle\mathcal{C}''_2(U,A)\rangle_{A,\text{q.}}\right\rangle_U \\
\langle\mathcal{C}\rangle''_{11} &= \left\langle\left\langle\mathcal{C}'_1(U,A)\cdot\frac{d_1(U,A)}{d_0(U)}\right\rangle_{A,\text{q.}}\right\rangle_U \\
\langle\mathcal{C}\rangle''_{02} &= \left\langle\left(\mathcal{C}_0(U)-\langle\mathcal{C}_0(U)\rangle_U\right)\cdot\left\langle\frac{d_2(U,A)}{d_0(U)}\right\rangle_{A,\text{q.}}\right\rangle_U
\end{aligned}$$

Strategy:

- Take isospin symmetric $SU(3)$ configurations: U
- Measure $\mathcal{C}_0(U)$ and $\mathcal{C}'_m(U)$
- For each U, generate quenched $U(1)$ fields: A
- Measure $\mathcal{C}'_1(U,A)$, $\mathcal{C}''_2(U,A)$, $\frac{d_1(U,A)}{d_0(U)}$ and $\frac{d_2(U,A)}{d_0(U)}$

For more details: Talk by Letitzia Parato Jul 27, 2021, 6:15 AM UET

QCD+QED


QCD
QCD
+
QED
isospin symmetric
correlation function
isospin breaking
contributions
Masses