38th International Symposium on Lattice Field Theory

QED and strong isospin corrections in the hardonic vacuum polarization contribution to the anomalous magnetic moment of the muon [Nature **593** (2021) 51–55]

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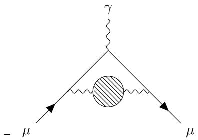
$$a_\mu = [g_\mu - 2]/2 = 0.00116592061(41)$$ [BNL+FNAL 2021]

- Fermilab g-2 experiment precision goal 0.14 ppm.
- The largest theoretical uncertainty (≈ 0.47 ppm) is associated with lowest-order hadronic vacuum polarization (LO-HVP) which have a relative error of 0.6%.



- To match the target experimental uncertainty, $a_\mu^{\text{LO-HVP}}$ has to be computed at 0.2% precision.
- To reach subpercent precision on the lattice:
  - $a|_{\%,\text{err}}$ propagates into $\sim 2a_\mu^{\text{LO-HVP}}|_{\%,\text{err}} \rightarrow$ few permil precision in the scale.
  - $\mathcal{O}\left(\frac{1}{N_c}\left(\frac{\Lambda}{m_c}\right)^2\right) \sim \mathcal{O}(10^{-2}) \rightarrow$ the charm quark has to be included.
  - $\mathcal{O}\left(\frac{m_d - m_u}{\Lambda}\right) \sim \mathcal{O}(10^{-2}) \rightarrow m_u$ and m_d can't be degenerate.
  - $\mathcal{O}(\alpha) \sim \mathcal{O}(10^{-2}) \rightarrow$ QED effects have to be included.

Expectation value of obs $O(e, \delta m)$ at first order in $\alpha = \frac{e^2}{4\pi}$ and $\delta m = m_d - m_u$:

$$\langle O \rangle = \frac{\int [dU][dA] e^{-S[U,A]}\, dets_0 \left(\mathbf{1} + e_s \frac{dets_1'}{dets_0} + e_s^2 \frac{dets_2''}{dets_0}\right)\left(O_0 + \frac{\delta m}{m_l} O_m' + e_v O_1' + e_v^2 O_2''\right)}{\int [dU] e^{-S_g[U]} \int [dA] e^{-S_\gamma[A]}\, dets_0 \left(\mathbf{1} + e_s \frac{dets_1'}{dets_0} + e_s^2 \frac{dets_2''}{dets_0}\right)}$$

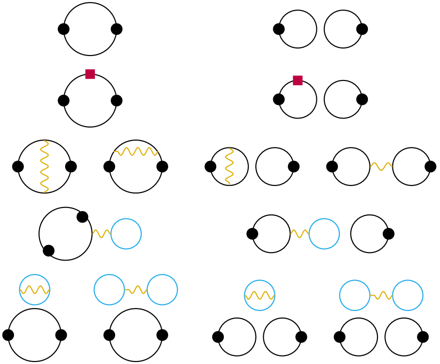
$$= Z_0^{-1} \int [dU] e^{-S[U]} dets_0 O_0 \qquad \langle O_0 \rangle_0$$

$$+ \frac{\delta m}{m_l} \langle O_m' \rangle_0 \qquad \equiv \langle O \rangle_m'$$

$$+ e_v^2 \langle O_{20}'' \rangle_0 \qquad \equiv \langle O \rangle_{20}''$$

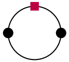
$$+ e_v e_s \left\langle O_1' \frac{dets_1'}{dets_0} \right\rangle_0 \qquad \equiv \langle O \rangle_{11}''$$

$$+ e_s^2 \left\langle [O_0 - \langle O_0 \rangle_0] \frac{dets_2''}{dets_0} \right\rangle_0 \qquad \equiv \langle O \rangle_{02}''$$



where $dets[U, A; \{m_f\}, \{q_f\}, e] = \prod_f det\, M_f[V_U e^{ieq_f A}, m_f]^{1/4}$ (M_f fermionic matrix),
$O_0 = O(0,0)$, $O_m' = m_l \partial_{\delta m} O|_{(0,0)}$, $O_1' = \partial_{e_v} O|_{(0,0)}$ and $O_2'' = \frac{1}{2}\partial_{e_v}^2 O|_{(0,0)}$.

Strong-isospin breaking $\langle O\rangle'_m$

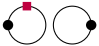


- SIB contributions to hadron masses computed as the finite difference

$$M'_m \approx \frac{m_l}{\delta m}\left(\mathcal{M}[\langle H_{\delta m}\rangle_0] - \mathcal{M}[\langle H_0\rangle_0]\right)$$

with $H_{\delta m}$ hadron propagator evaluated at $\delta m = 2m_l \frac{1-r}{1+r}|_{r=0.485}$ and $e_v = 0$.

- Connected current propagator $[C^{\mathrm{conn}}]'_m$ computed via insertion of operator corresponding to mass derivative.



- $[C^{\mathrm{disc}}]'_m$: computed as a finite difference using $I(m_l, e=0)$ and $I(0.9m_l, e=0)$, where $I(m_l, e)$ is the trace of the light quark propagator.

Valence-valence $\langle O\rangle''_{20}$

- QED inserted as a free stochastic $U(1)$ field.

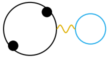
- e_v^2 contributions to hadron masses computed as the finite difference

$$M''_{20} = \frac{1}{e_v^2}\left(\mathcal{M}[\tfrac{1}{2}\langle H_+ + H_-\rangle_0] - \mathcal{M}[\langle H_0\rangle_0]\right)$$

- $[C^{\mathrm{conn}}]''_{20}$ computed as a finite difference using $C^{\mathrm{conn}}(m_s, 0)$ and $C^{\mathrm{conn}}(m_s, \pm\frac{1}{3}e_*)$ for strange, $C^{\mathrm{conn}}(m_\kappa, 0)$ and $C^{\mathrm{conn}}(m_\kappa, \pm\frac{1}{3}e_*)$ for light, with $\kappa = m_{\mathrm{val}}/m_{\mathrm{sea}} = 3, 5, 7$, then linear chiral extrapolation to $\kappa = 1$.

- $[C^{\mathrm{disc}}]''_{20}$ computed as a finite difference using $I(m_l, 0)$, $I(m_l, \pm\frac{1}{3}e_*)$, $I(m_s, 0)$ and $I(m_s, \pm\frac{1}{3}e_*)$.

Valence-sea $\langle O \rangle''_{11}$



- $\langle O \rangle''_{02}$ values averaged on A then U:

$$\langle O \rangle''_{11} = \left\langle \left\langle O'_1 \, \frac{dets'_1}{dets_0} \right\rangle_A \right\rangle_U$$

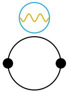
- First derivatives of observables estimated as finite differences: $O'_1 = \frac{1}{2e_V}(O_+ - O_-)$.
- Hadron masses are given in a mixed form:

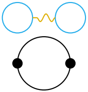
$$M''_{11} = \frac{\delta \mathcal{M}}{\delta H}\Big|_{\langle H_+ + H_- \rangle_0} \cdot \left\langle \frac{H_+ - H_-}{2e_V} \, \frac{dets'_1}{dets_0} \right\rangle$$

with hadron propagators $H_\pm$ evaluated at $e_* = \pm\sqrt{4\pi\alpha_*}$ and $\delta m = 0$.

- $[C^{\mathrm{disc}}]''_{11}$ requires $dets'_1$ and $I(m_f, e)'_1$, obtained as a finite difference.

Sea-sea $\langle O \rangle''_{02}$



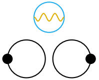


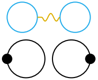
- $\langle O \rangle''_{02}$ averaged on A then U:

$$\langle O \rangle''_{02} = \left\langle [O_0 - \langle O_0 \rangle_U] \left\langle \frac{dets''_2}{dets_0} \right\rangle_A \right\rangle_U$$

- Hadron masses are given in a mixed form:

$$M''_{02} = \frac{\delta \mathcal{M}}{\delta H}\Big|_{\langle H_0 \rangle_0} \cdot \langle (H_0 - \langle H_0 \rangle) \frac{dets''_2}{dets_0} \rangle_0$$



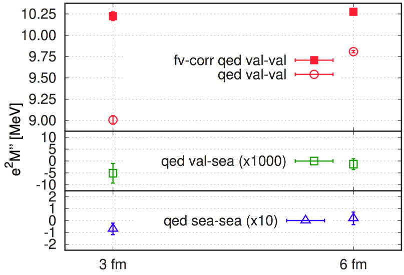


- $[C^{\mathrm{disc}}]''_{02}$ requires $dets''_2$, $I(m_{f=l,s}, 0)$.
- We use random sources, truncated solver method, low-mode averaging to compute $dets'_1$ and $dets''_2$.

- QED is defined in the QED$_L$ scheme.
- $O(L^{-1})$ effects in hadron masses due to QED_L scheme:

$$M(L) - M = -\frac{(Qe)^2 c}{8\pi}\left[\frac{1}{L} + \frac{2}{ML^2} + O(L^{-3})\right]$$

- Only M''_{20} is corrected for these effects.
- QED effects are evaluated on a dedicated set of ensembles with 4stout action:
  - valence effects on $L = 6fm$ boxes, dynamical (sea) effects on $L = 3fm$ boxes.
  - At $\beta = 3.700$ both volumes have been used → no difference in dynamical contributions evaluated on $L = 6fm$ and $L = 3fm$ boxes.
  - At $L = 3fm$ one order magnitude less computer time for same precision.



- Lattice QCD parameters: $\{a, m_d + m_u, m_d - m_u, m_s, e\}$ ($m_c = 11.85 m_s$).
- To fix these parameters we use the following physical observables:

$$\begin{aligned} &M_{\Omega^-} \to a \\ &M^2_{\pi_\chi} \equiv \tfrac{1}{2}(M^2_{uu} + M^2_{dd}) \approx M^2_{\pi_0} \\ &M^2_{K_\chi} \equiv \tfrac{1}{2}(M^2_{ds} + M^2_{us} - M^2_{ud}) = \tfrac{1}{2}(M^2_{K_0} + M^2_{K_+} - M^2_{\pi_+}) \\ &\Delta M^2_K \equiv M^2_{ds} - M^2_{us} = M^2_{K_0} - M^2_{K_+} \end{aligned} \tag{1}$$

- All interpolations of physical quantities to the physical point are performed in these variables.
  [We call "type-I fits" the fitting procedure with dependences based on (1)]
- An **isospin-breaking decomposition** requires a suitable set of quantities which allow one to separate a physical quantity into an isospin-symmetric contribution and corrections in δm and α:

$$\begin{aligned} &w_0 \to a \\ &M^2_{ss} \\ &\Delta M^2 \equiv M^2_{dd} - M^2_{uu} \end{aligned}$$

  plus e and $M^2_{\pi_\chi} \equiv \frac{1}{2}(M^2_{uu} + M^2_{dd})$ as in (1).
- Some of the observable above are not in the PDG, we computed them using type-I fits:

$$\begin{aligned} &[w_0]_* = 0.17236(29)(63)[70]\,\mathrm{fm} \\ &[M_{ss}]_* = 689.89(28)(40)[49]\,\mathrm{MeV}^2 \\ &[\Delta M^2]_* = 13170(320)(270)[420]\,\mathrm{MeV}^2 \end{aligned}$$

  where the errors are statistical, systematic and total.

- An isospin breaking decomposition requires matching QCD+QED to QCD_{iso}.
- Definition of our isospin decomposition:

$$\langle O\rangle_{\text{QED}} \equiv e^2 \cdot \frac{\partial\langle O\rangle}{\partial e^2}\Big|_{(M_{\pi_\chi} w_0, M_{ss} w_0, \frac{L}{w_0}, \Delta M\omega_0, e=0)}$$

$$\langle O\rangle_{\text{SIB}} \equiv (\Delta M w_0)^2 \frac{\partial\langle O\rangle}{\partial(\Delta M w_0)}\Big|_{(M_{\pi_\chi} w_0, M_{ss} w_0, \frac{L}{w_0}, \Delta M\omega_0=0, e=0)}$$

$$\langle O\rangle_{\text{ISO}} \equiv O(M_{\pi_\chi} w_0, M_{ss} w_0, \frac{L}{w_0}, \Delta M\omega_0 = 0, e = 0)$$

- $\langle O\rangle_{\text{QED}}$, $\langle O\rangle_{\text{SIB}}$ and $\langle O\rangle_{\text{ISO}}$ are obtained by “type-II fits”: $O = f(\{X\}, A, B, \dots)$ with $A = A(w_0) = A_0 + a^2 A_2 + \dots$ and $\{X\} = \{w_0, M_{\pi_\chi}, M_{ss}, \Delta M, e\}$

$$\begin{cases} [O]_0 = A + BX_l(M^2_{\pi_\chi}\omega_0^2) + CX_s(M^2_{ss}\omega_0^2) \\ [O]'_m = [D(w_0)\Delta M]'_m \\ [O]''_{20} = [A + BX_l + CX_s + DX_{\delta m}]''_{20} + [E]_0 \\ [O]''_{11} = [A + BX_l + CX_s + DX_{\delta m}]''_{11} + [F]_0 \\ [O]''_{11} = [A + BX_l + CX_s + DX_{\delta m}]''_{02} + [G]_0 \end{cases} \xrightarrow{a \to 0} \begin{cases} \langle O\rangle_{\text{ISO}} = A_0 \\ \langle O\rangle_{\text{SIB}} = D_0[\Delta M^2 w_0^2]_* \\ \langle O\rangle_{\text{QED-vv}} = e_*^2 E_0 \\ \langle O\rangle_{\text{QED-vvs}} = e_*^2 F_0 \\ \langle O\rangle_{\text{QED-ss}} = e_*^2 G_0 \end{cases}$$

$$X_l = M^2_{\pi_\chi} w_0^2 - \left[M^2_{\pi_\chi} w_0^2\right]_*$$

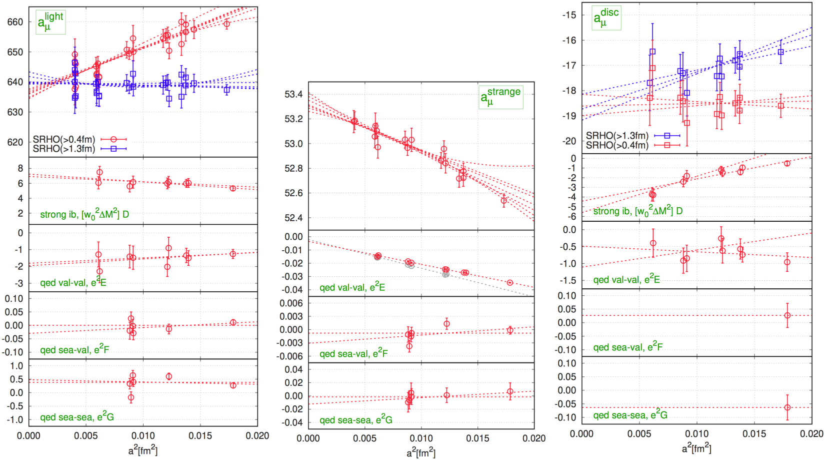
$$X_s = M^2_{ss} w_0^2 - \left[M^2_{ss} w_0^2\right]_*$$

$$X_{\delta m} = \Delta M^2 w_0^2$$

[Further details on type-I fits in Lukas Varnhorst’s talk - Monday 14h]

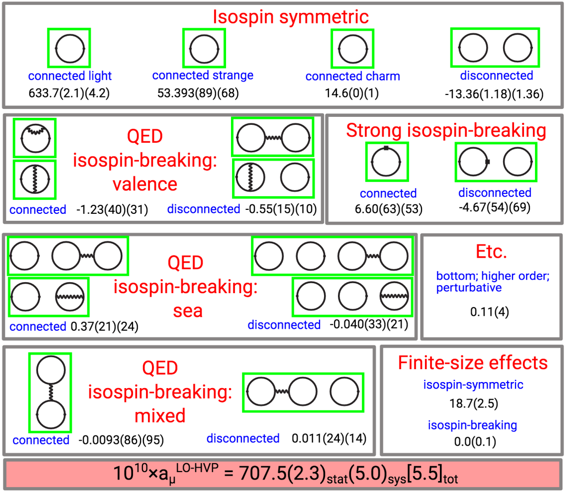
- $\langle O\rangle_{\text{ISO}} + \langle O\rangle_{\text{SIB}} + \langle O\rangle_{\text{QED}}$ is scheme independent and equal to $\langle O\rangle$ obtained via type-I fits.

Results for a_μ^{light}, a_μ^{strange} and a_μ^{disc}



Figures 25, 26 and 27 in BMW'20: continuum extrapolation to $a_\mu^{\text{light}}(L_{\text{ref}}, T_{\text{ref}})$, $a_\mu^{\text{strange}}(L_{\text{ref}}, T_{\text{ref}})$ and $a_\mu^{\text{disc}}(L_{\text{ref}}, T_{\text{ref}})$. First window is total result form type-I fit, other windows are IB decompositions from type-II fits.

All the results for $a_\mu^{\text{LO-HVP}}$



Figures 1 in BMW'20

- Strong-isospin-breaking and QED corrections to the LO-HVP contribution to a_μ are necessary in order to reach the precision needed for comparison to future experiments.
- We have included IB corrections by taking derivatives with respect to isospin-breaking parameters and by measuring these derivatives on the isospin-symmetric configurations.
- We have decomposed a_μ in isospin-symmetric part and isospin-breaking corrections by means of a suitable set of variables: $\{w_0, M_{ss}, \frac{1}{2}(M_{uu}^2 + M_{dd}^2), M_{dd}^2 - M_{uu}^2, e\}$.
- The IB decomposition is useful to crosscheck results amongst different collaborations.
  - Connected strong IB corrections can be crosschecked with
    RBC/UKQCD [Phys.Rev.Lett. 121 (2018) 2, 022003],
    ETM [Phys. Rev. D 99, 114502 (2019)],
    FHM [Phys.Rev.Lett. 120 (2018) 15, 152001],
    LM [Phys.Rev.D 101 (2020) 074515].
  - Connected QED valence-valence contributions can be crosschecked with
    RBC/UKQCD [Phys.Rev.Lett. 121 (2018) 2, 022003],
    ETM [Phys. Rev. D 99, 114502 (2019)].

Thanks for the attention