Lattice calculation of $K \to \ell \nu_\ell \ell'^+ \ell'^-$ decay width

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Outline

1 Introduction

2 Lattice calculation techniques

3 Numerical results ($m_\pi = 352 MeV$)

4 Conclusion

based on **arXiv:2103.11331**

1. Introduction: The return of Kaon physics

"The return of Kaon physics."

A. J. Buras, 2018 (arxiv: 1805.11096)

- Kaon physics: high precision test of the Standard Model

1. Suitable for identifying new physics：Flavor Changing Neutral Current (FCNC)， CP violation
2. Experiments and theoretical predictions have achieved high accuracy.

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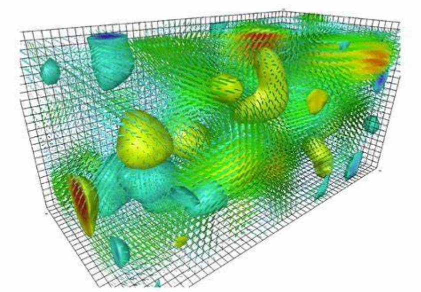
1. Suitable for identifying new physics：Flavor Changing Neutral Current (FCNC)， CP violation
2. Experiments and theoretical predictions have achieved high accuracy.

- **Theoretical challenge：**

High precision calculation of long distance contribution associated with non-perturbative QCD

↓

lattice QCD plays an increasing role in recent years



1. Introduction: $K \to l\nu_l l'^+ l'^-$ decay width

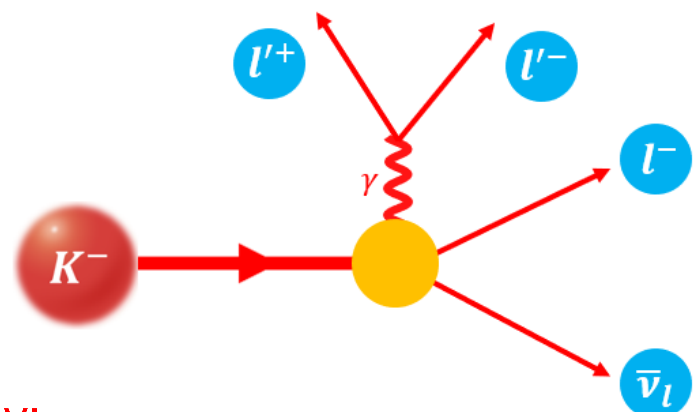
Channel	m_{ee} cut	ChPT[1]/10^{-8}	Exp/10^{-8}
$K \to e\nu_e e^+ e^-$	> 140 MeV	3.39	2.91(23)[2]
$K \to \mu\nu_\mu e^+ e^-$	> 140 MeV	8.51	7.93(33)[2]
$K \to e\nu_e \mu^+ \mu^-$	–	1.12	1.72(45)[3]
$K \to \mu\nu_\mu \mu^+ \mu^-$	–	1.35	–

[1]J. Bijnens, G. Ecker and J. Gasser *Nucl. Phys. B.* 396:81-118(1993) arXiv: hep-ph/9209261
[2]A. A. Poblaguev et al. *Phys. Rev. Lett.* 89:061803(2002) arXiv: hep-ex/0204006
[3]H. Ma et al., *Phys. Rev. D.,* 73:037101(2006) arXiv: hep-ex/0505011

This talk: Solving challenges in lattice calculation of $K \to l\nu l' l'$ decay width

Euclidean space-time $\neq$ Minkowski space-time

Finite volume $\neq$ Infinite volume



The same question has also been studied by:
Filippo Mazzetti's talk (5:30 am, Wednesday, at Hadron structure)

2. Lattice calculation techniques

Lattice calculation procedures

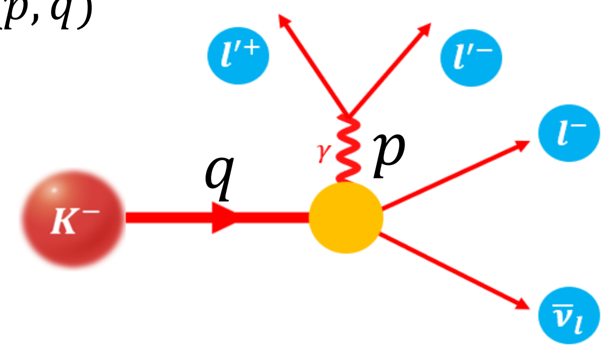
Euclidean space hadronic function $H_E^{\mu\nu}(x,Q)$ → Minkowski space hadronic function $H_M^{\mu\nu}(p,q)$ → Decay amplitude → Decay width

General steps：

complicated but no obstacle

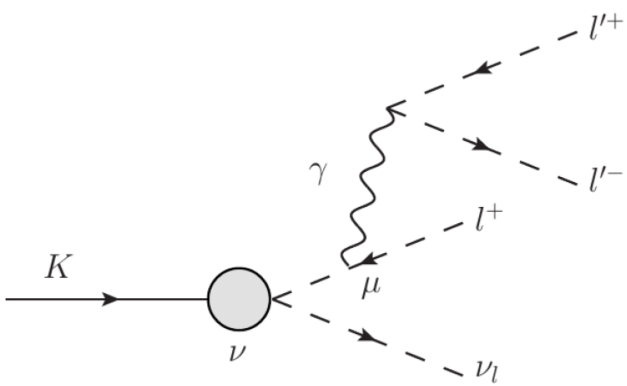
Definition of Minkowski hadronic function $H_M^{\mu\nu}(p,q)$

$$H_M^{\mu\nu}(p,q) = i\left\langle 0\left|T\left\{J^{\mu}_{em}(p)J^{\nu}_{W}(q-p)\right\}\right|K(q)\right\rangle$$

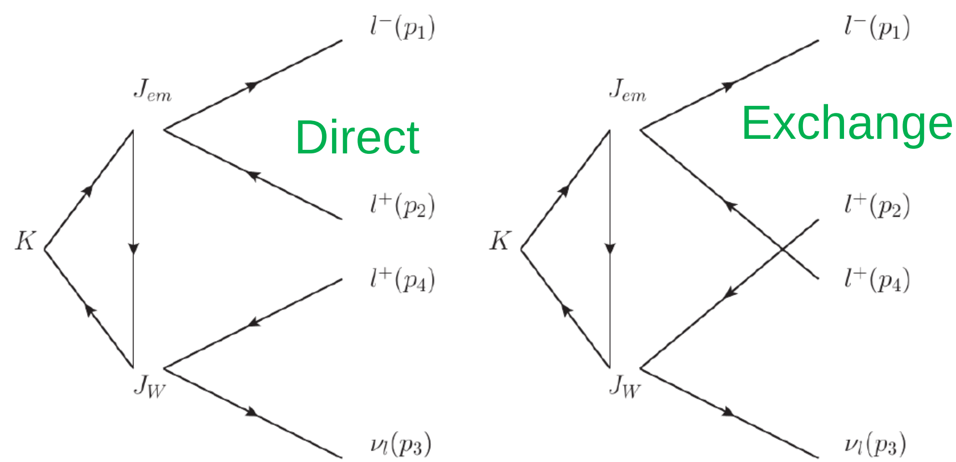


General steps in Minkowski space

Euclidean space hadronic function $H_E^{\mu\nu}(x, Q)$ → Minkowski space hadronic function $H_M^{\mu\nu}(p, q)$ → Decay amplitude → Decay width



Leptonic radiative contribution



Hadronic contribution

General steps in Minkowski space

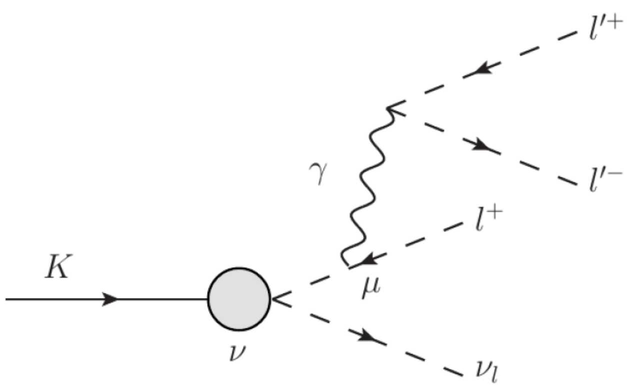
Euclidean space hadronic function $H_E^{\mu\nu}(x,Q)$ → Minkowski space hadronic function $H_M^{\mu\nu}(p,q)$ → Decay amplitude → Decay width

$$i\mathcal{M}_D = -i\frac{G_F e^2 V_{us}^*}{\sqrt{2}s_{12}}\left[f_K L^\mu(p_1,p_2,p_3,p_4) - H_M^{\mu\nu}(p_{12},q)\,l_\nu(p_3,p_4)\right]\left[\bar{u}(p_1)\gamma_\mu v(p_2)\right]$$

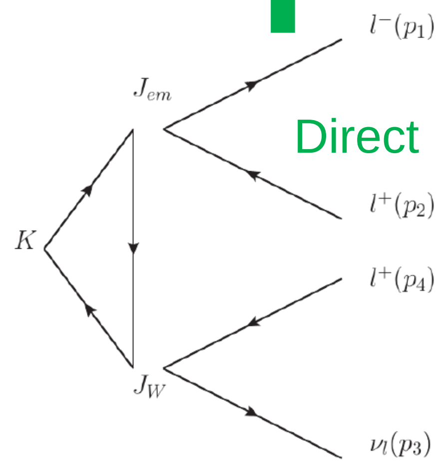
Direct

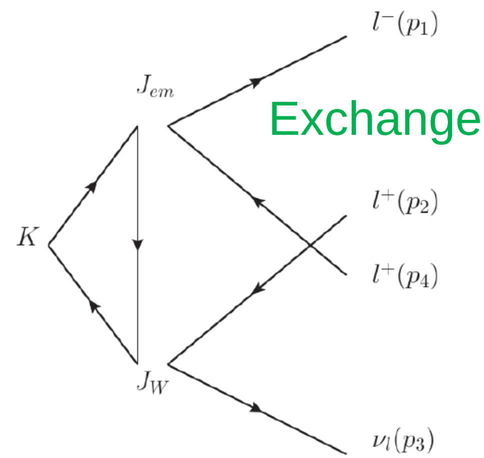
$$i\mathcal{M}_E = i\frac{G_F e^2 V_{us}^*}{\sqrt{2}s_{14}}\left[f_K L^\mu(p_1,p_4,p_3,p_2) - H_M^{\mu\nu}(p_{14},q)\,l_\nu(p_3,p_2)\right]\left[\bar{u}(p_1)\gamma_\mu v(p_4)\right]$$

Exchange



Leptonic radiative contribution





Hadronic contribution

General steps in Minkowski space

Euclidean space hadronic function $H_E^{\mu\nu}(x, Q)$ ➡ Minkowski space hadronic function $H_M^{\mu\nu}(p, q)$ ➡ Decay amplitude ➡ Decay width

- Four body phase space (p_1, p_2, p_3, p_4):

  16(total variables)-4(on shell condition)-4(energy momentum conservation)-3(rotational symmetry)=5 independent variables

- Convention: $(x_{12}, x_{34}, y_{12}, y_{34}, \phi)$

Karol Kampf et al., *Phys. Rev. D.*100(9):094509 (2019)

$$d\Phi_4 = \frac{S\lambda M^4}{2^{14}\pi^6} dx_{12} dx_{34} dy_{12} dy_{34} d\phi$$

- Phase space integration:

$$\frac{\Gamma}{\Gamma_K} = \frac{1}{2m_K\Gamma_K}\int d\Phi_4 \left(|\mathcal{M}_D|^2 + |\mathcal{M}_E|^2 + 2\mathrm{Re}\mathcal{M}_D\mathcal{M}_E^*\right)$$

Lattice calculation procedures

Euclidean space hadronic function $H_E^{\mu\nu}(x,Q)$?→ Minkowski space hadronic function $H_M^{\mu\nu}(p,q)$ → Decay amplitude → Decay width

- Define: Euclidean space hadronic function, from Lattice 3-pt function

$$H_E^{\mu\nu}(x,Q) = \langle 0 | T\left\{J_{em}^{\mu}(x) J_W^{\nu}(0)\right\} | K(Q)\rangle$$

$$H_E^{\mu\nu}(P,Q) = \int_{-T}^{T} dt \int d^3x e^{Et - i\vec{p}\cdot\vec{x}} H_E^{\mu\nu}(x,Q) \qquad P = (iE,\vec{p}), \quad Q = (im_K,\vec{0}),$$

- Goal: Minkowski space hadronic function

$$H_M^{\mu\nu}(p,q) = i\langle 0 | T\left\{J_{em}^{\mu}(p) J_W^{\nu}(q-p)\right\} | K(q)\rangle \qquad p = (E,\vec{p}), \quad q = (m_K,\vec{0})$$

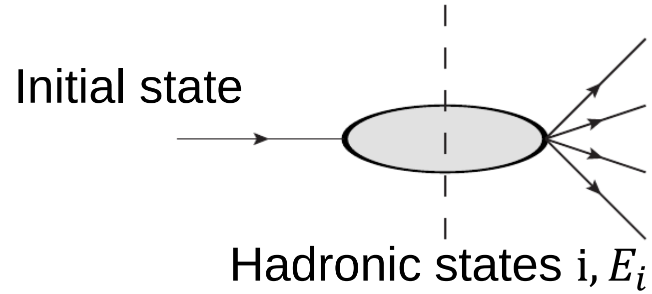
- Differences:

Euclidean space-time $\neq$ Minkowski space-time

Finite volume $\neq$ Infinite volume

Challenge 1: Euclidean ≠ Minkowski

- Insert complete basis of hadronic states:



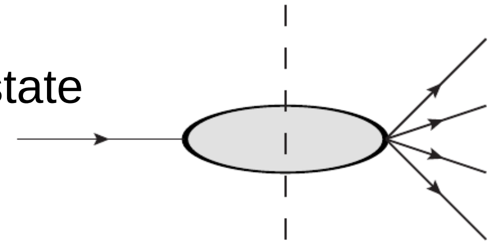
(E_0 initial or final state energy)

⇒ Extra term in Euclidean space-time $e^{-(E_i-E_0)T}$

Challenge 1： Euclidean ≠ Minkowski

- Insert complete basis of hadronic states：

Initial state



Final state

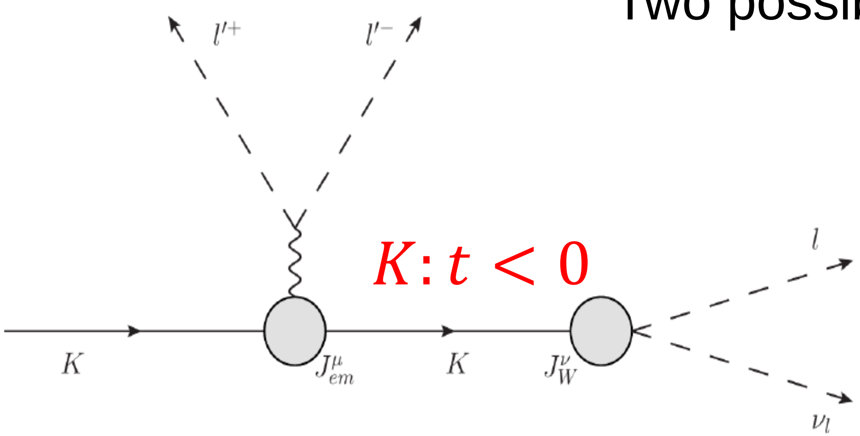
Hadronic states i, E_i

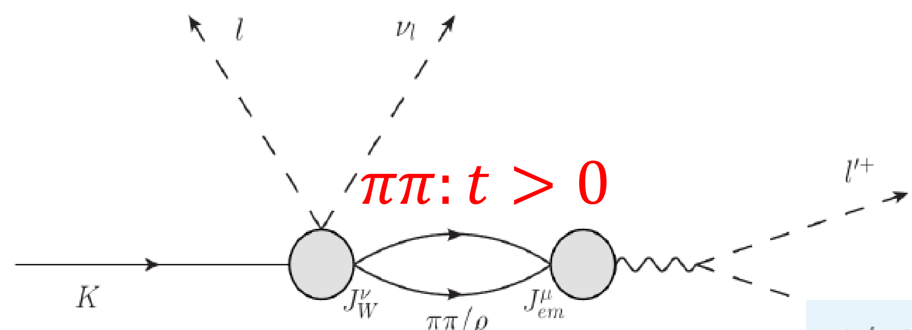
(E_0 initial or final state energy)

→ Extra term in Euclidean space-time $e^{-(E_i - E_0)T}$

- light states dominance
  - Suppressed for heavy states: $E_i \gg E_0$
  - Enhanced for light states: $E_i \sim E_0$

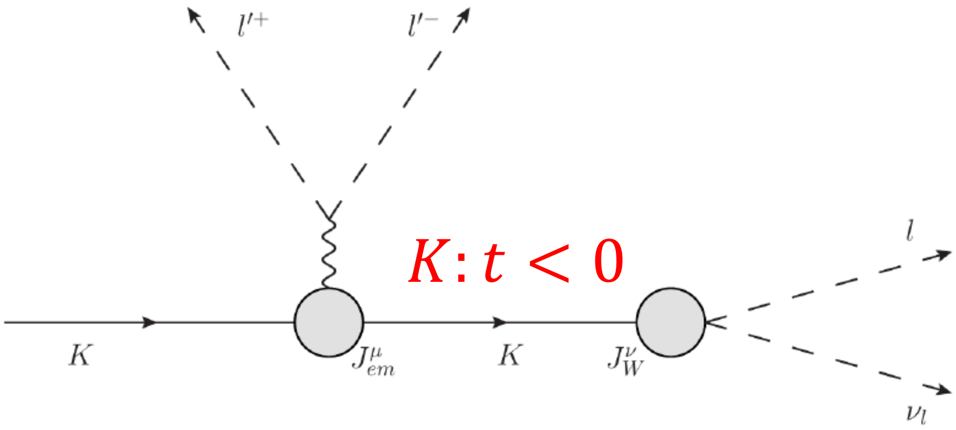
Two possible light states





Challenge 1: Euclidean ≠ Minkowski

Start with Kaon state.



Extra term $e^{-(E_K+E_\gamma-m_K)T}$

↓

Strongly enhanced for $E_K + E_\gamma - m_K \to 0$

↓

Soft photon region, nearly on shell Kaon

$$H_E^{\mu\nu}(P,Q) \propto \frac{\langle 0|J_W^\nu|K\rangle_{E\,E}\langle K|J_{em}^\mu|K\rangle\left(1 - e^{-(E+E_K-m_K)T}\right)}{E + E_K - m_K}$$

$$H_M^{\mu\nu}(p,q) \propto \frac{\langle 0|J_W^\nu|K\rangle_{M\,M}\langle K|J_{em}^\mu|K\rangle}{E + E_K - m_K - i\epsilon}$$

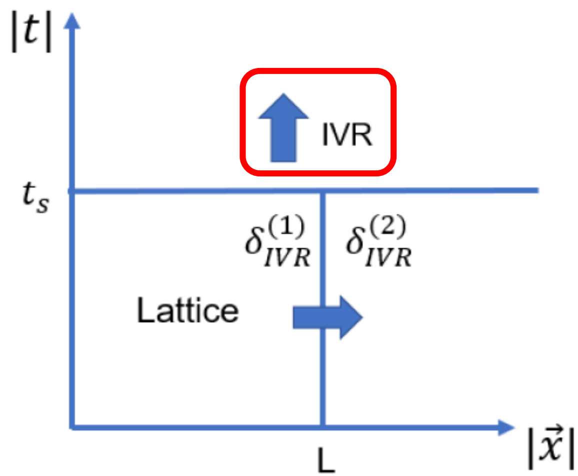
Challenge 1：Euclidean ≠ Minkowski

- Solution: infinite volume reconstruction $e^{-(E_K+E_\gamma-m_K)T}$

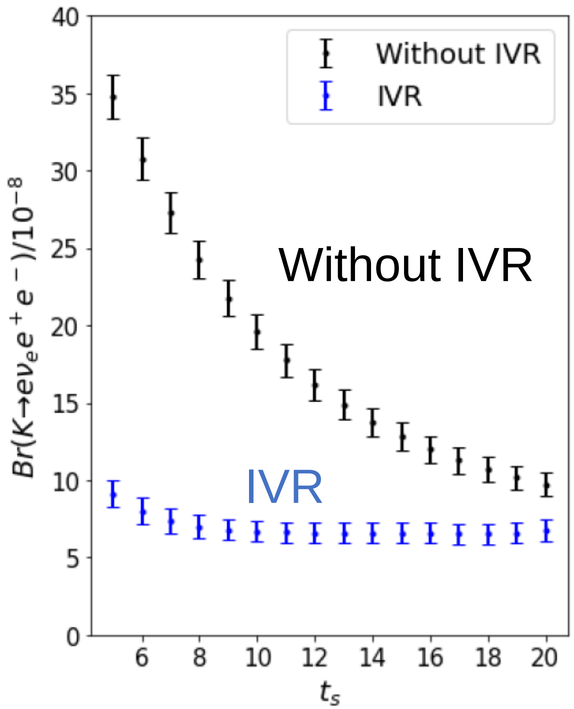
Choose some moderate t_s:

$H_E^{\mu\nu}(x,Q)$ with $|t|>t_s$: reconstruct from $|t|=t_s$ due to Kaon state dominance

Equivalently, $T\to\infty$, $e^{-(E_K+E_\gamma-m_K)T}\to 0$

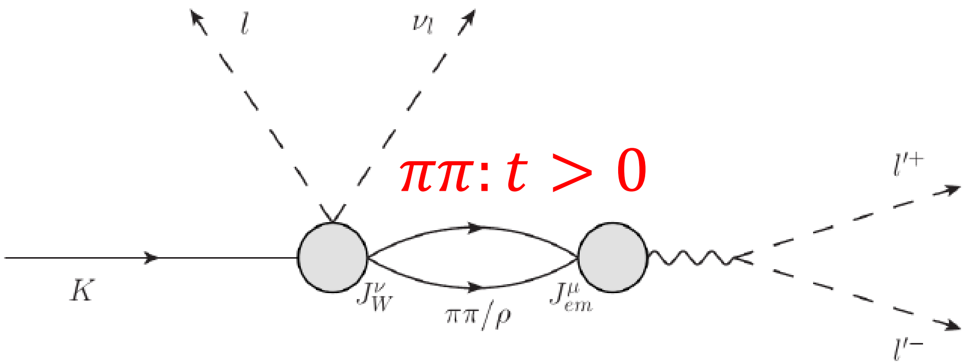


$H^{(L),\mu\nu}(x,Q)\to H^{\mu\nu}(x,Q)$



Challenge 1：Euclidean ≠ Minkowski

$\pi\pi$ states



Extra term $e^{-(E_{\pi\pi}-E_\gamma)T}$

Exponentially growing with on shell $\pi\pi$ states $E_{\pi\pi} < E_\gamma < m_K$

$$H_E^{\mu\nu}(P,Q) \propto \frac{\langle 0|J_{em}^\mu|\pi\pi\rangle_E\ {}_E\langle\pi\pi|J_W^\nu|K\rangle\left(1-e^{(E-E_{\pi\pi})T}\right)}{E-E_{\pi\pi}}$$

$$H_M^{\mu\nu}(p,q) \propto \frac{\langle 0|J_{em}^\mu|\pi\pi\rangle_M\ {}_M\langle\pi\pi|J_W^\nu|K\rangle}{E-E_{\pi\pi}+i\epsilon}$$

Fortunately, strongly suppressed:

phase space $K \to \pi\pi\ell\nu_\ell \to \ell\nu_\ell\ell'\ell'$	$\ll$	total phase space $K \to \ell\nu_\ell\ell'\ell'$

- Formulas under development, already confirm the phase space suppression by reconstruction of low-lying $\pi\pi$ states.

Challenge 2：general finite volume effects

- From periodic boundary condition in finite volume:

$$\int_V d^4x e^{Et-i\vec{p}\cdot\vec{x}} H^{(L),\mu\nu}(x,Q)$$

Discrete Fourier transform with lattice momentum ✓

Fourier transform with arbitrary momentum in phase space ✗

⇓

Arbitrary momentum: Study momentum dependence of form factors

Challenge 2：general finite volume effects

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$$\int_V d^4x e^{Et-i\vec{p}\cdot\vec{x}} H^{(L),\mu\nu}(x,Q)$$

Discrete Fourier transform with lattice momentum ✓

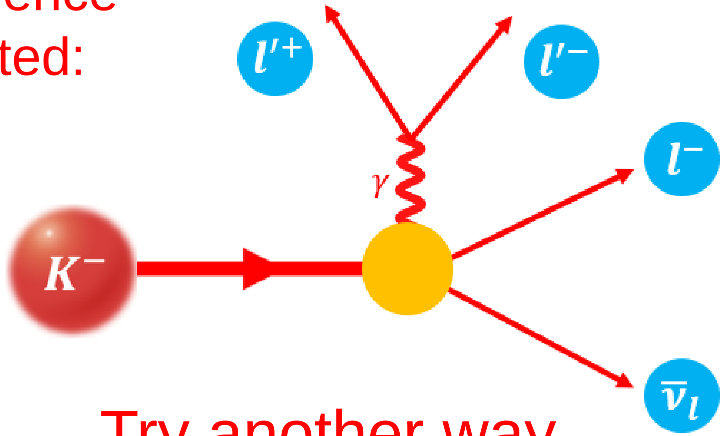
Fourier transform with arbitrary momentum in phase space ✗

↓

Arbitrary momentum: Study momentum dependence of form factors

- However, study momentum dependence of $K\to l\nu l'l'$ amplitude is complicated:

1. 4 form factors, complicated 4-body phase space
2. Depend on parameterization of form factors
   (VMD model? Taylor expansion?)



Try another way

Challenge 2： general finite volume effects

- Idea: View the errors of using "arbitrary momentum" as general finite volume effects

$$\int d^4x e^{Et-i\vec{p}\cdot\vec{x}} H^{\mu\nu}(x,Q) = \int_V d^4x e^{Et-i\vec{p}\cdot\vec{x}} H^{(L),\mu\nu}(x,Q) + \int d^4x e^{Et-i\vec{p}\cdot\vec{x}} \left[H^{\mu\nu}(x,Q) - H^{(L),\mu\nu}(x,Q)\right]$$

infinite volume $H^{\mu\nu}(x,Q)$

finite volume $H^{(L),\mu\nu}(x,Q)$ with "arbitrary momentum"

General finite volume effects

Challenge 2: general finite volume effects

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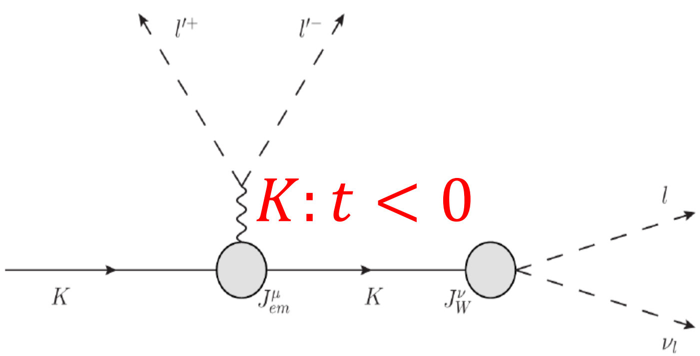
$$\int d^4x e^{Et-i\vec{p}\cdot\vec{x}} H^{\mu\nu}(x,Q) = \int_V d^4x e^{Et-i\vec{p}\cdot\vec{x}} H^{(L),\mu\nu}(x,Q) + \int d^4x e^{Et-i\vec{p}\cdot\vec{x}} \left[H^{\mu\nu}(x,Q) - H^{(L),\mu\nu}(x,Q) \right]$$

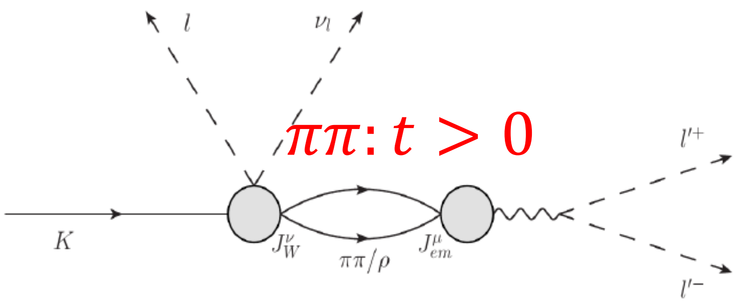
infinite volume $H^{\mu\nu}(x,Q)$

finite volume $H^{(L),\mu\nu}(x,Q)$ with "arbitrary momentum"

General finite volume effects

- General FV in decay width: light states dominance ➡ Solve explicitly





Challenge 2：general finite volume effects

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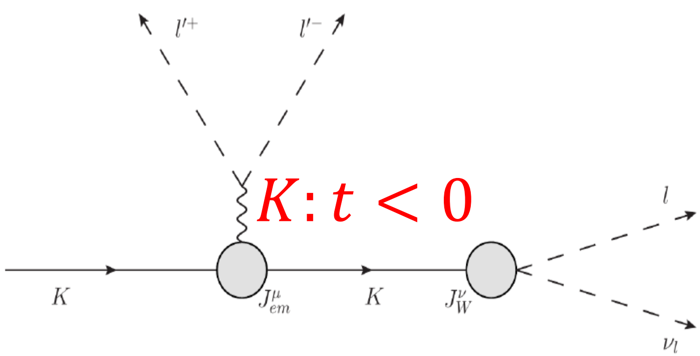
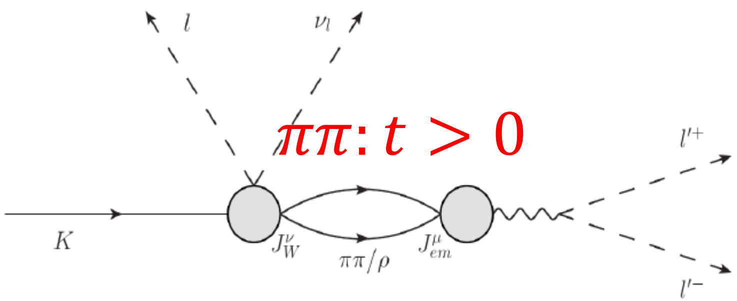
infinite volume $H^{\mu\nu}(x,Q)$ | finite volume $H^{(L),\mu\nu}(x,Q)$ with "arbitrary momentum" | General finite volume effects

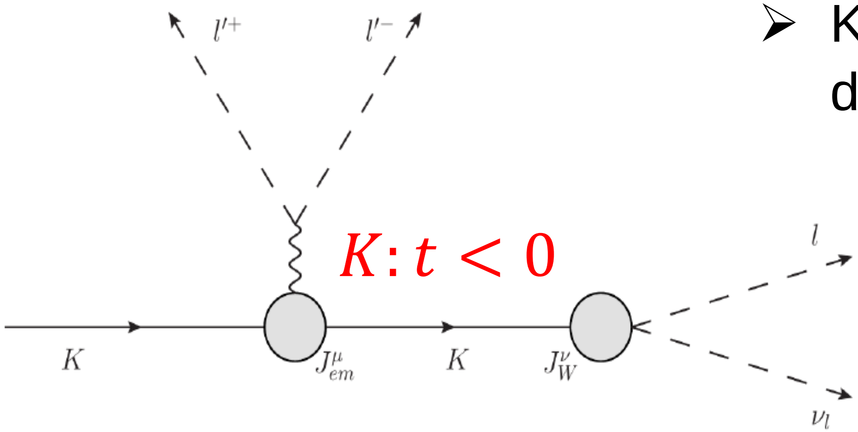
- General FV in decay width: light states dominance ➡ Solve explicitly





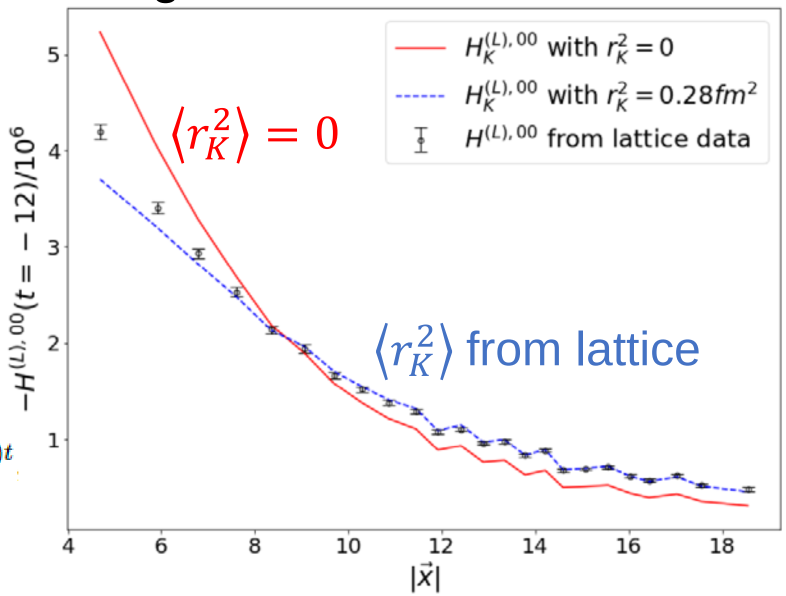
- Alternative way: use twisted boundary condition to study momentum dependence of form factors
  Filippo Mazzetti's talk (5:30 am, Wednesday, at Hadron structure)

Challenge 2：general finite volume effects

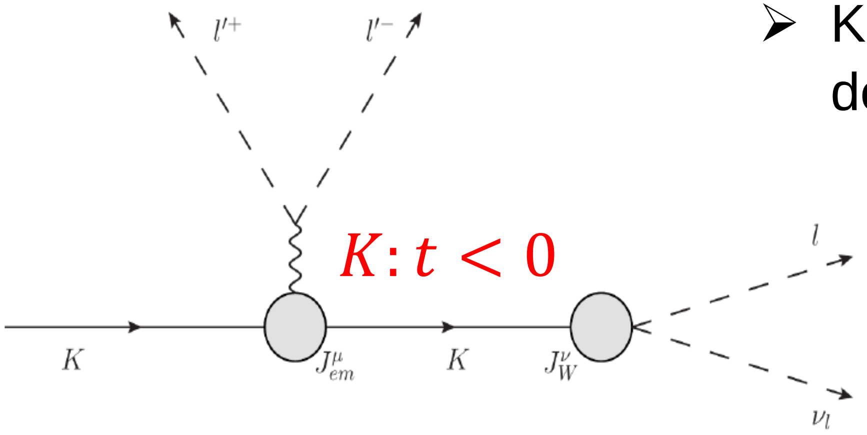


$$H_K^{(L),\mu\nu}(x,Q) = \frac{1}{L^3}\sum_{\vec{p}} \frac{1}{2E} f_K P^\nu (P+Q)^\mu F^{(K)}(q^2) e^{-i\vec{p}\cdot\vec{x}} e^{(E-m_K)t}$$

- Kaon contribution (input: $\langle r_K^2 \rangle$ from Lattice) describe long distance well:



Challenge 2: general finite volume effects



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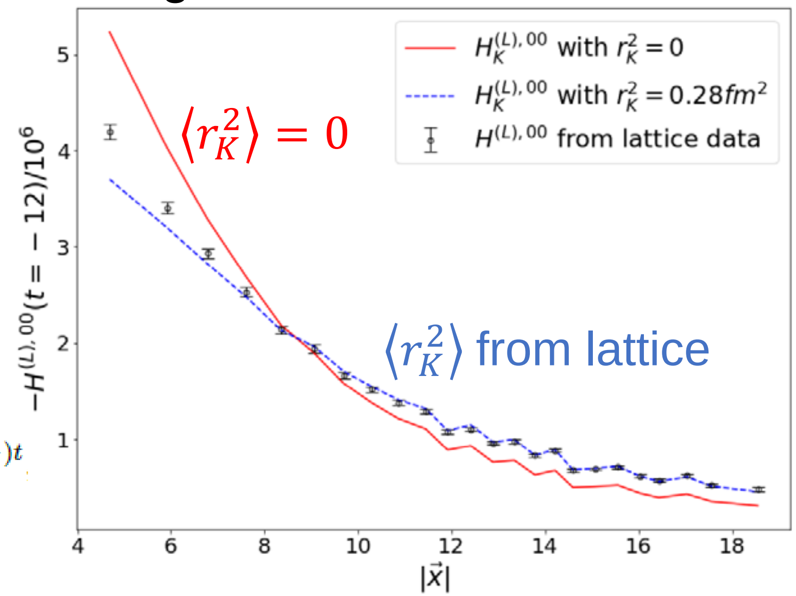


- General FV effects mainly come from long distance:

$$\int d^4x e^{Et - i\vec{p}\cdot\vec{x}} \left[H^{\mu\nu}(x,Q) - H^{(L),\mu\nu}(x,Q) \right]$$

Kaon dominance

$$\int d^4x e^{Et - i\vec{p}\cdot\vec{x}} \left[H_K^{\mu\nu}(x,Q) - H_K^{(L),\mu\nu}(x,Q) \right]$$

Challenge 2：general finite volume effects

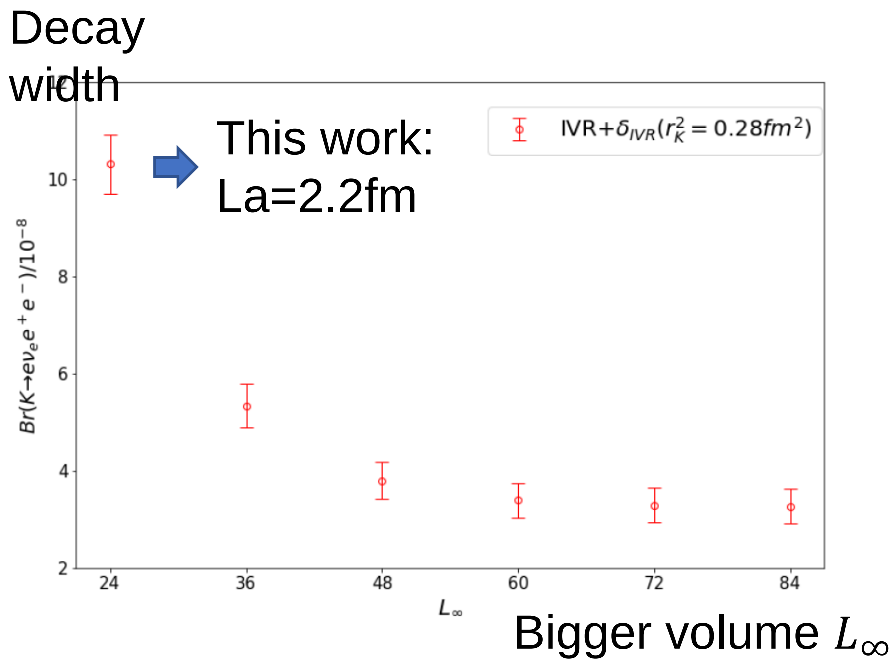
- We can prove: $O(e^{-m_K L})$ finite volume effects in calculation of total decay width.

Estimate FV correction from Kaon dominance:

$$\int d^4x e^{Et-i\vec{p}\cdot\vec{x}}\left[H_K^{\mu\nu}(x,Q)-H_K^{(L),\mu\nu}(x,Q)\right]$$

$$\approx\int d^4x e^{Et-i\vec{p}\cdot\vec{x}}\left[H_K^{(L_\infty),\mu\nu}(x,Q)-H_K^{(L),\mu\nu}(x,Q)\right]$$

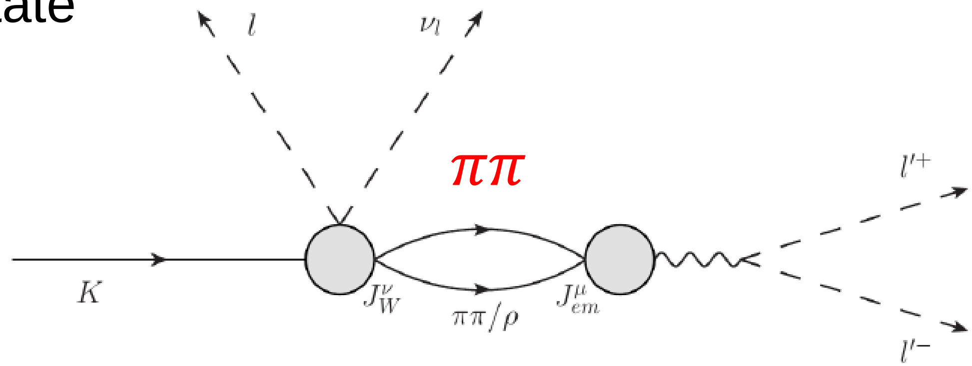
Kaon contribution in bigger volume L_∞



- This work: La=2.2fm is small, we need to correct this general FV effects.
- Future: It can be ignored in larger lattice volume.

Challenge 2：general finite volume effects

$t > 0$, $\pi\pi$ state



FV effects in $\pi\pi$ state contribution

- Power-law FV effects from on shell pion — Strongly suppressed → phase space $K \to \pi\pi\ell\nu_\ell \to \ell\nu_\ell\ell'\ell'$ $\ll$ total phase space $K \to \ell\nu_\ell\ell'\ell'$
- General finite volume effects: similar as in Kaon case, with $O(e^{-m_\rho L})$

➢ These FV correction formulas are under development.

3. Results in twisted mass ensemble

Lable	$L^3 \times T$	a^{-1}	N_{conf}	m_π	m_K	ΔT
cA211b.53.24	$24^3 \times 48$	2.12GeV	51	0.3515(15)GeV	0.5071(14)GeV	12

based on: **arxiv: hap-lat/2103.11331**

Channel	m_{ee} cut	Unphysical m_π	Physical m_π	
		Lattice/10^{-8}	ChPT/10^{-8}	Exp/10^{-8}
$K \to e\nu_e e^+e^-$	> 140 MeV	3.29(35)	3.39	2.91(23)
$K \to \mu\nu_\mu e^+e^-$	> 140 MeV	11.08(39)	8.51	7.93(33)
$K \to e\nu_e \mu^+\mu^-$	–	0.94(8)	1.12	1.72(45)
$K \to \mu\nu_\mu \mu^+\mu^-$	–	1.52(7)	1.35	-

Already correct systematic effects with Kaon states

Residue errors:

1. Unphysical quark mass
2. FV effects from $\pi\pi$
3. Lattice artifacts

→

Future work:

1. Develop FV formulas of $\pi\pi$
2. Ensembles with physical π mass

4. Conclusion

- We solve several technical problems in calculation of $K \to l\nu l'l'$ decay width.

- We present a first calculation result in twisted mass ensemble with $m_\pi = 352MeV$;

- Future work is to develop the FV formulas of $\pi\pi$ contribution, and to calculate in physical ensembles.

Backup slides

Why is general FV $\sim O(e^{-mL})$

$$\int d^4x e^{Et - i\vec{p}\cdot\vec{x}} \left[H^{\mu\nu}(x,Q) - H^{(L),\mu\nu}(x,Q) \right]$$

If we ignore power-law finite volume effects from multi-particle intermediate states (considered separately):

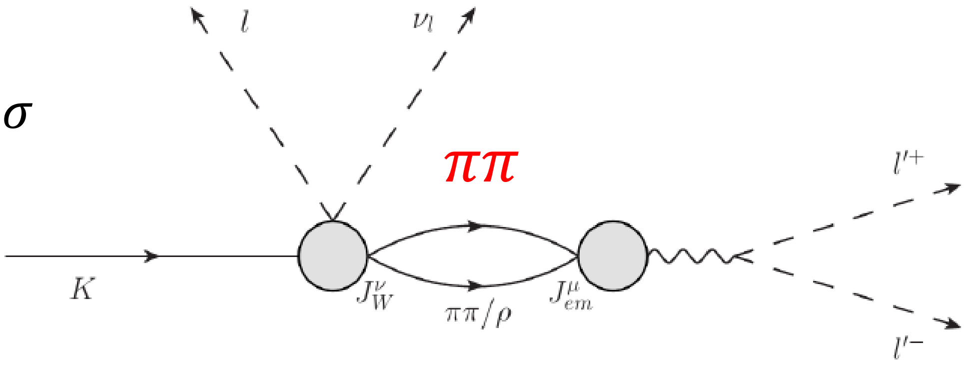
$$H^{\mu\nu}(x) - H^{(L),\mu\nu}(x) = \begin{cases} \sum_{\vec{l}\neq 0} H^{\mu\nu}(\vec{l}L - \vec{x}, t) < O(e^{-mL}) & \text{Inside of lattice box} \\ H^{\mu\nu}(\vec{x}, t) < O(e^{-mL}) & \text{Outside of lattice box} \end{cases}$$

Exponentially suppressed of FV effects of $H^{\mu\nu}(x)$ → Exponentially suppressed of general finite volume effects

How to correct $\pi\pi$ effects

$\pi\pi$ effects
In decay width

- Exponentially growing term $\ll 1\sigma$
- FV effects $\sim 1\sigma$



Idea: reconstruction

form factors (from lattice/GEVP/exp), $\pi\pi$ phase shift

Lellouch-Luscher formula

Finite volume results

infinite volume results

Effects at roughly 1σ level of decay width, parameterization dependence of form factors can be ignored