Investigation of the Perturbative Expansion of Heavy Quark Correlators for $N_f = 0$

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Perturbative Expansions

- α renormalizes, necessary to introduce renormalization scale μ in perturbation theory

$$\alpha_{\text{bare}} \to \alpha_{\text{R}}(\mu)$$

- This scale is unphysical, observables do not depend on it
- **Asymptotic** expansion for small α

$$\mathcal{M}_n(Q) = \sum_{i\geq 0} c_n^{(i)}(\mu/Q)\,\alpha^i_{\overline{\text{MS}}}(\mu) \quad \Longrightarrow \quad \mathcal{M}_n(Q,\mu) = \sum_{i\geq 0}^{L} c_n^{(i)}(\mu/Q)\,\alpha^i_{\overline{\text{MS}}}(\mu) + \mathcal{O}(\alpha^{L+1})$$

Spurious μ dependence

Truncating means introducing a hard-to-estimate, possibly sizeable systematic error

- How small must α/how big must μ be for PT to correctly describe physics?
- μ has to be close to *Q*, scale of the problem, or coefficients grow and spoil series behavior
- variation of μ often used to assess truncation error

The Window Problem

Main Idea: compare PT and lattice, non-perturbative moments at different values of quark mass, thus assessing down to what energy (or α value) PT behaves well for this observables.

Need a quantity where **both PT** and some **non-perturbative method** work well: **moments of heavy quark correlators**

$$a^{-1} \gg Q \gg \Lambda_{\mathrm{QCD}} \gg L^{-1}$$

- **The larger** the **scale** of the problem *Q*, the **smaller** the **truncation error**
- Large *Q* means **large *aQ***, i.e. large **discretization effects**. Necessary to have small *a*.
- But there are also **finite volume** effects on the lattice. There is a **window of energies where one may be able to use both methods**

Moments in Momentum Space

Moments method, pioneered by Bochkarev, de Forcrand [*hep-lat/9505025*] and HPQCD in 2008 [*hep-lat/0805.2999*].

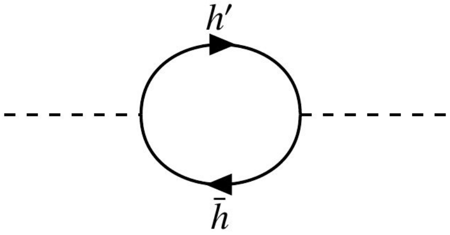
The observables are derivatives of the vacuum polarization with heavy quarks (*h, h'*) at CoM energy $q^2 = 0$

$$\Pi(q^2, m) = i \int \mathrm{d}^4 x\, e^{iq\cdot x} \langle 0 | \mathscr{T} \left\{ J^\dagger(x) J(0) \right\} | 0 \rangle$$

$$\mathcal{M}_n(M_{RGI}) = \frac{1}{n!} \left(\frac{\partial}{\partial q^2} \right)^n \Pi(q^2, m) \Big|_{q^2=0}$$

Object known to high (4-loop) orders in PT

[Maier, Maierhöfer, Marquard, Smirnov '09]



$$[\mathcal{M}_n] = \mathrm{En.}^{4-n} \quad \Rightarrow \quad \mathcal{M}_n(M_{RGI}) = \overline{m}_{\overline{\mathrm{MS}}}(\mu)^{4-n} \sum_{i \geq 0} c_n^{(i)}(\mu/\overline{m}_{\overline{\mathrm{MS}}}(\mu))\, \alpha^i_{\overline{\mathrm{MS}}}(\mu)$$

$$c_n^{(i)}(\mu/\overline{m}_{\overline{\mathrm{MS}}}(\mu)) = c_n^{(i,0)} + \sum_{k=1}^{i-1} c_n^{(i,k)} \log^k(\mu/\overline{m}_{\overline{\mathrm{MS}}}(\mu))$$

Pseudoscalar, smaller statistical errors:

$$J_{PS}(x) = i\, m_h \overline{\psi}_h(x) \gamma_5 \psi_{h'}(x)$$

Moments in Position Space

- Doing the spatial integral, one can easily show the moments to be:

$$\mathcal{M}_n(M_{RGI}) = \int_{-\infty}^{\infty} \mathrm{d}t\, t^n G(t,m)$$

- The **lattice transcription** of the moments is:

$$\mathcal{M}_n(aM_{RGI}, \sqrt{8t_0}M_{RGI}) = \lim_{T\to\infty, L\to\infty} a \sum_t^T t^n \left(\frac{a}{L}\right)^3 a^3 \sum_{\vec{x},\vec{y}}^{L-a} \mu_{tm}^2 \langle J(t,\vec{x}) J^\dagger(0,\vec{y})\rangle$$

Doublet of mass-degenerate twisted mass Wilson fermions, at full twist

- $J_{PS}(x) = i\, m_h \overline{\psi}_h(x)\gamma_5\psi_{h'}(x)$ does not renormalize in certain regularizations: $Z_P Z_\mu = 1$

OPE: $J_A(x)J_B(0) \underset{x\to 0}{\sim} \sum_l O_l C_{A,B}^{(l)}(x)$

$$O_1 = \mathbb{1}\,,\ C_{A,B}^{(1)} \sim \frac{1}{|x|^6} \text{ up to logs} \implies G(t) \underset{t\to 0}{\sim} \frac{1}{|t|^3} \implies \text{for } n>3 \quad \exists \lim_{t\to 0}\left\{G(t)\,t^n\right\} \Rightarrow \boxed{n = 4,\ 6,\ 8,\ 10, \ldots}$$

Lattice Setup

[1] Lüscher, Sint, Sommer, Weisz, Wolff. (*arXiv:hep-lat/9609035*)

- **Plaquette** gauge action
- P.b.c. in space, **open b.c. in time** to avoid **frozen topological charge** at small *a*
- Full twist doublet, with **non-perturbative c_{sw}** to reduce cutoff effects
- Stochastic evaluation of trace and sum over space with **U(1) noise sources**
- **Source** placed at **1 fm from boundary**, checked absence of boundary effects
- Full twist, set **κ to its critical value** [1]
- Autocorrelation analysis done with **Γ-method**
- Scale set through **gradient flow** t_0

  [Lüscher, *arXiv:hep-lat/1006.4518*]
- Physical **Volume of (2 fm)^3**, time direction

  about **T≃6 fm**

- At every *a* we tune the bare mass in order to keep the RGI mass fixed: $M_{RGI} = \lim_{\mu\to\infty} \overline{m}_X(\mu) \left[2b_0\, \overline{g}_X^2(\mu)\right]^{-d_0/(2b_0)}$

$$z = \sqrt{8t_0} M_{RGI} = \frac{\sqrt{8t_0}}{a} \underbrace{\frac{M_{RGI}}{\overline{m}_{SF}(\mu)}} a\mu_{tm} \left(\underbrace{Z_P^{SF}(a\mu, g_0)}\right)^{-1}$$

[1] Capitani, Lüscher, Sommer, Wittig (*arXiv:hep-lat/9810063*)

Measurements

Table 1: Gauge run details, $l = L/a$, $t = T/a$.

Run Name	β	$l^3 \times t$	N_{cnfg}	t_0/a^2	$a[fm]$	$\tau_{\text{int}}(t_0)[\text{cfg}]$
q_beta616	6.1628	$32^3 \times 96$	128	5.376(10)	0.071	0.78
q_beta628	6.2885	$36^3 \times 108$	137	7.790(22)	0.059	1.37
q_beta649	6.4956	$48^3 \times 144$	109	13.778(51)	0.044	1.55
sft4	6.7859	$64^3 \times 192$	50(200)	29.39(10)	0.030	1.00
sft5	7.1146	$96^3 \times 320$	80	67.74(23)	0.020	0.55
sft6	7.3600	$128^3 \times 320$	98	124.21(91)	0.015	1.03
sft7	7.7	$192^3 \times 480$	55			

[Ensembles sft from:

Husung, Krah, Sommer

arXiv:hep-lat/1711.01860]

We measure for a range of masses:

$$M/M_{\text{charm}} \simeq 3.48\,,\ 2.32\,,\ 1.55\,,\ 1.16\,,\ 0.77\,.$$

Tackling Cutoff Effects

- ★ We compute the **finite volume, finite *a* tree-level** (TL) **analytically** and divide the moments by it:
- ★ The TL is computed with $a\mu_{tm} = a\overline{m}_h(\overline{m}_h)$ not with the bare parameter value of the non-perturbative simulation:

$$R_n(\sqrt{8t_0}M_{RGI}, aM_{RGI}) = \begin{cases} \dfrac{\mathcal{M}_n(\sqrt{8t_0}M_{RGI}, aM_{RGI})}{\mathcal{M}_n^{TL}(a\mu_{tm})}, & n = 4 \\ \left(\dfrac{\mathcal{M}_n(\sqrt{8t_0}M_{RGI}, aM_{RGI})}{\mathcal{M}_n^{TL}(a\mu_{tm})}\right)^{\frac{1}{n-4}}, & n > 4 \end{cases}$$

Leading cutoff effects suppressed by a power of the coupling

$$\mathcal{M}_n(a\mu_{tm}) = \mathcal{M}_n^{(0)}(a\mu_{tm}) + \alpha\mathcal{M}_n^{(1)}(a\mu_{tm}) + \ldots \rightarrow R_n(a\mu_{tm}) = 1 + \alpha R_n^{(1)}(a\mu_{tm}) + \ldots$$

$O(a^2\mu_{tm}^2) \rightarrow O(\alpha a^2\mu_{tm}^2)$ up to logs (see Husung, Marquard, Sommer [*hep-lat/1912.08498*] + Husung's talk at 13:45 CEST)

- ★ For *n>4* take ratios of moments to get rid of strong mass dependence and mitigate some error sources:

$$\lim_{a\to 0} \frac{R_n(\sqrt{8t_0}M_{RGI}, aM_{RGI})}{R_{n+2}(\sqrt{8t_0}M_{RGI}, aM_{RGI})} = \sum_{i\geq 0}^{L} c_n^{(i)}(\mu/\overline{m}_{\overline{\rm MS}}(\mu))\, \alpha_{\overline{\rm MS}}^i(\mu) + O(\alpha^{L+1})$$

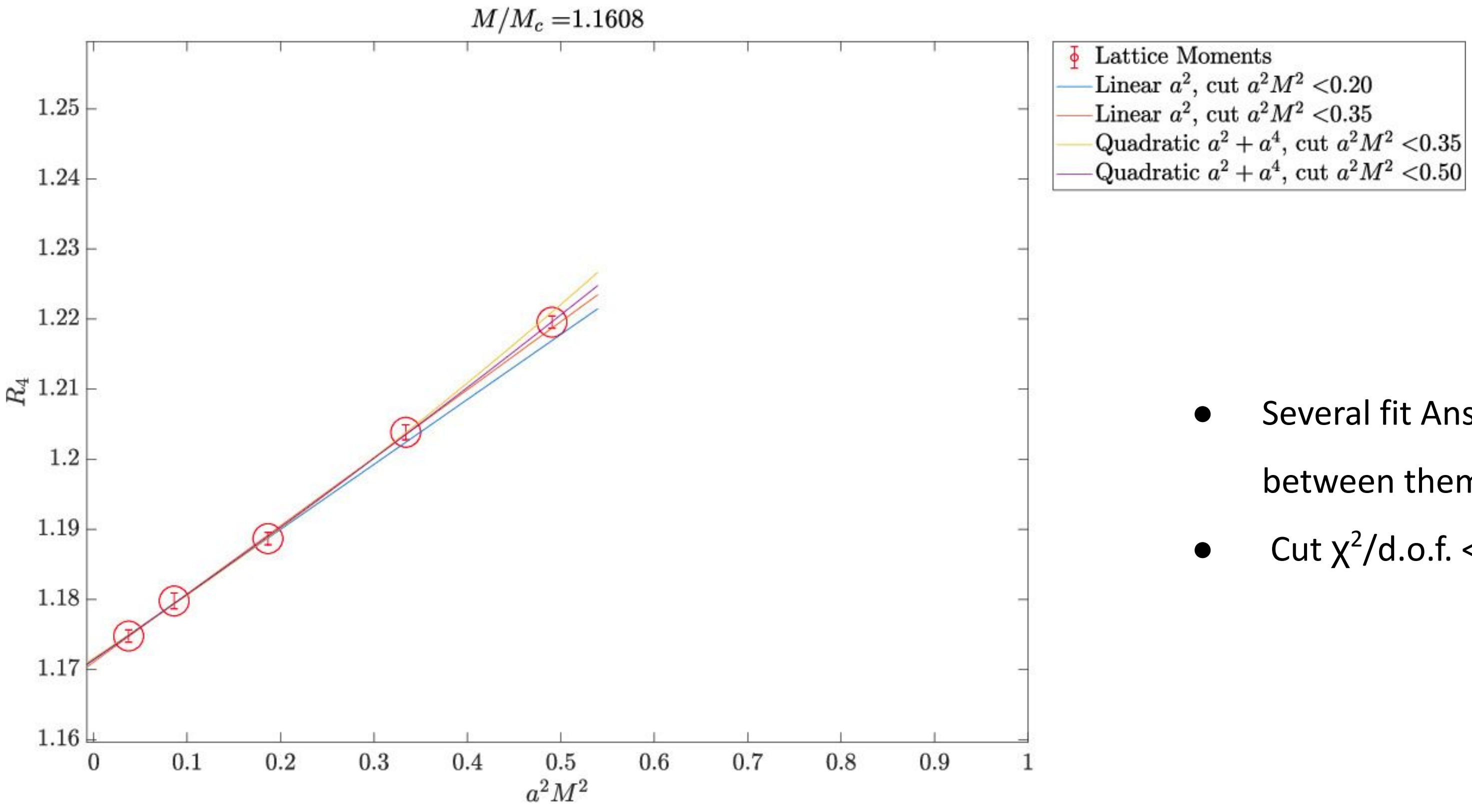
Results

Disclaimer:

the numerical results that will be shown shortly

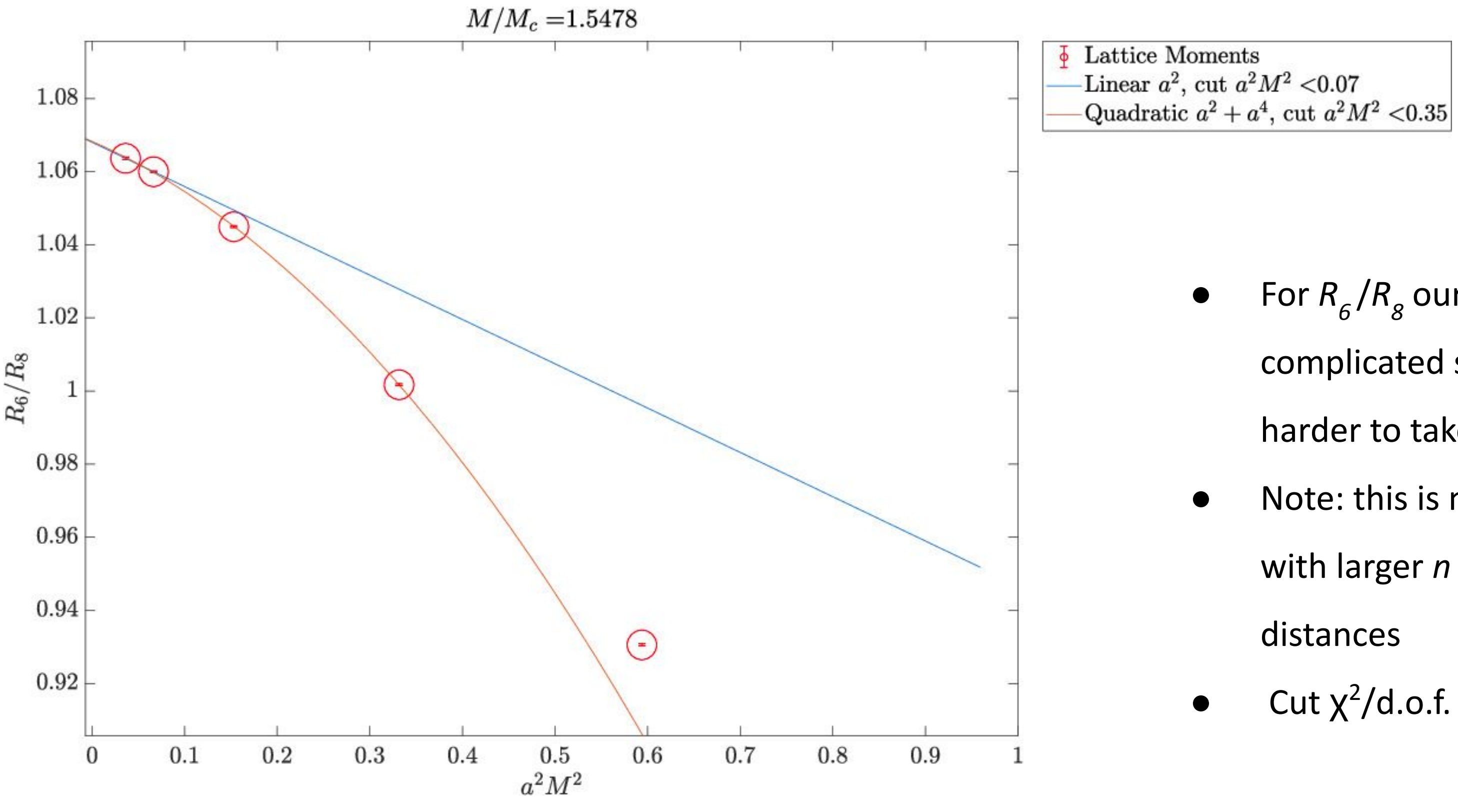
are still to be considered preliminary!

Continuum Extrapolations - I



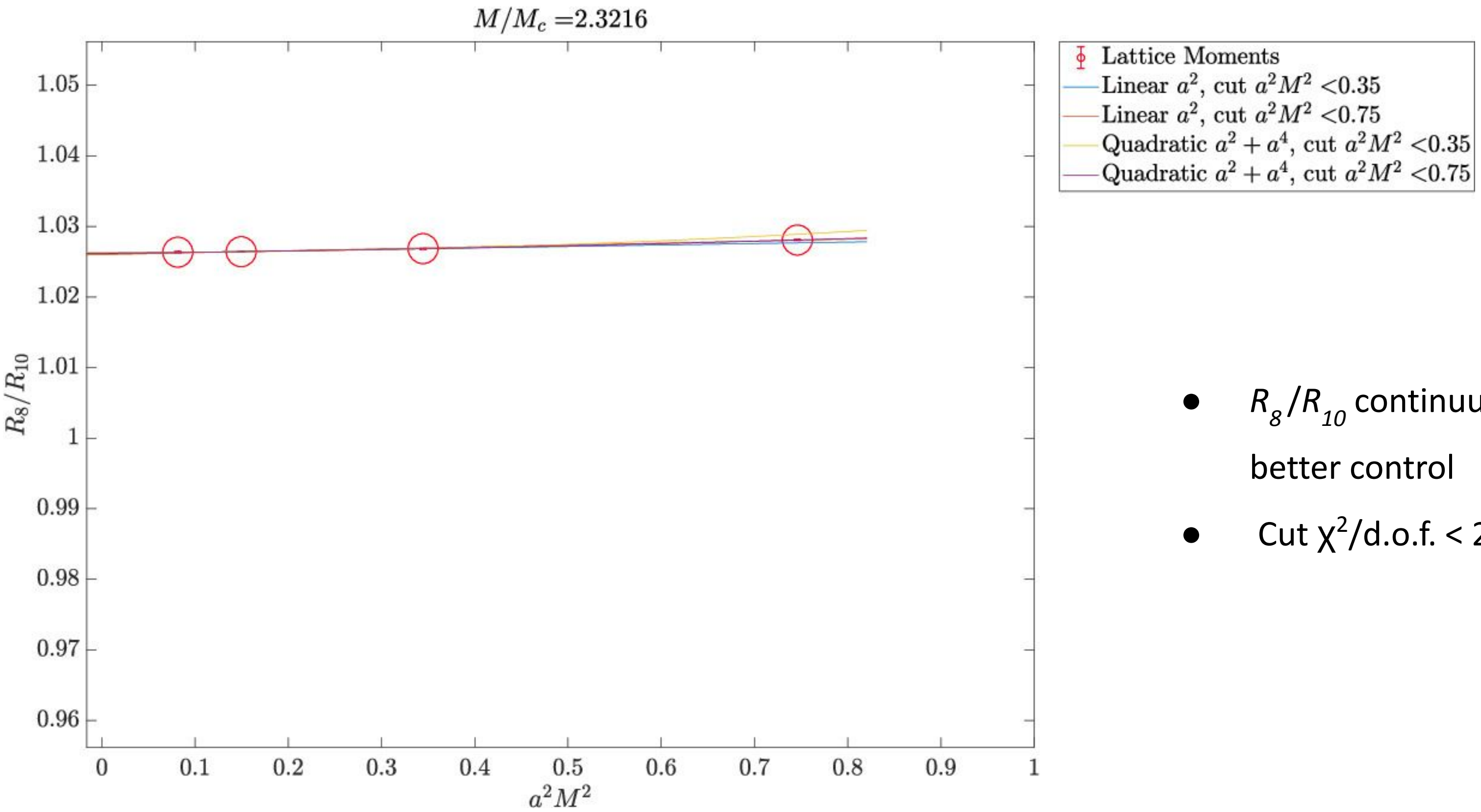
- Several fit Ansätze, good agreement between them
- Cut χ^2/d.o.f. < 2

Continuum Extrapolations - II



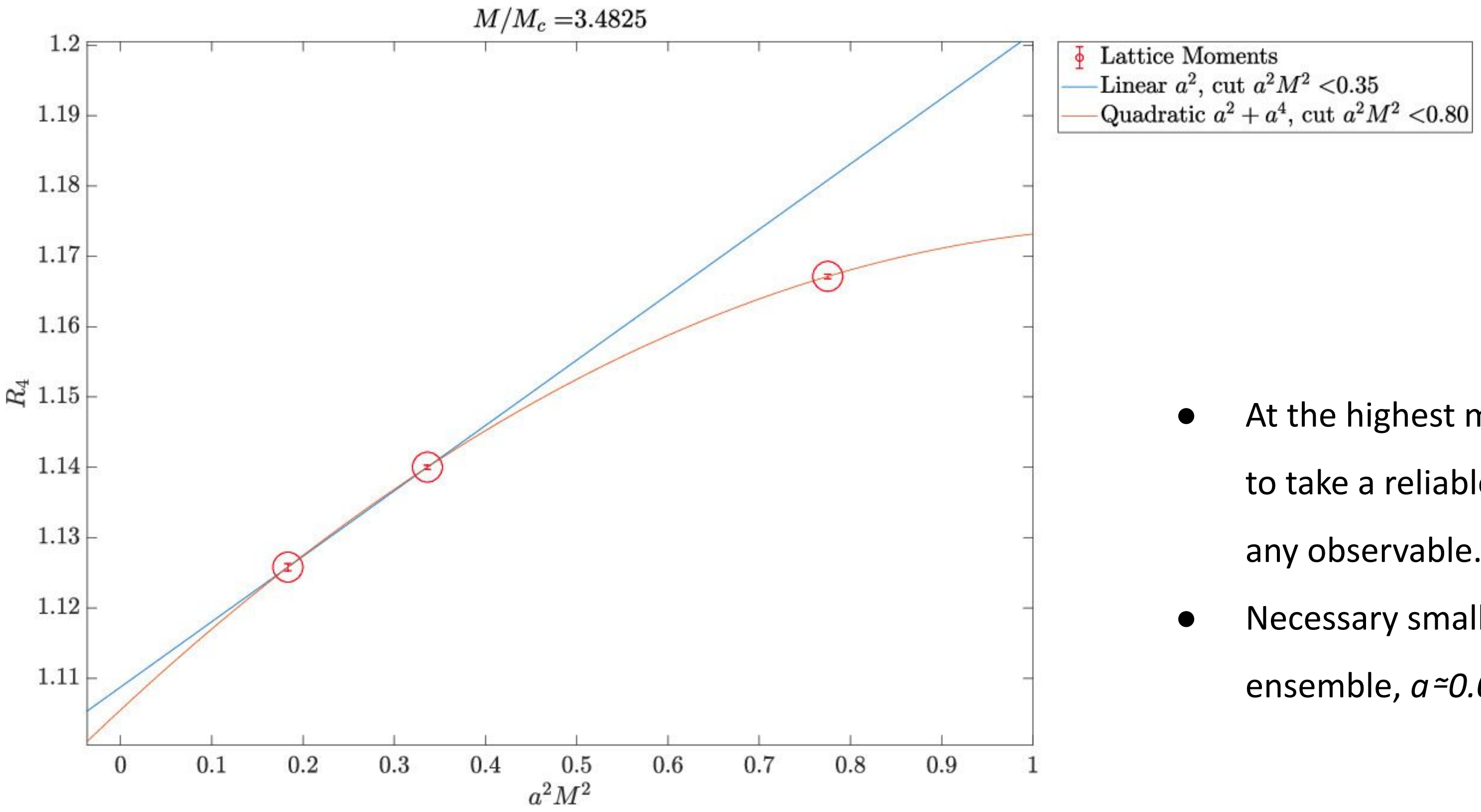
- For R_6/R_8 our results have more complicated scaling violations, harder to take continuum limit
- Note: this is not expected, moments with larger *n* are dominated by *large* distances
- Cut χ^2/d.o.f. < 2

Continuum Extrapolations - III



- R_8/R_{10} continuum limit under better control
- Cut χ^2/d.o.f. < 2

Continuum Extrapolations - IV



- At the highest mass we are not able yet to take a reliable continuum limit for any observable.
- Necessary smaller lattice spacing, sft7 ensemble, *a≈0.01* fm

Extracting α and Λ

- Perturbative expansion of moments with known coefficients:

$$\lim_{a\to 0} R_4(\sqrt{8t_0}M_{RGI}, aM_{RGI}) = \sum_{i\geq 0}^{L} r_4^{(i)}(\mu/\overline{m}_{\overline{\mathrm{MS}}}(\mu))\, \alpha^i_{\overline{\mathrm{MS}}}(\mu) + O(\alpha^{L+1})$$

- Fix scale $\mu_* = s\overline{m}_{\overline{\mathrm{MS}}}(\mu_*)$ and and invert to obtain α at this scale:

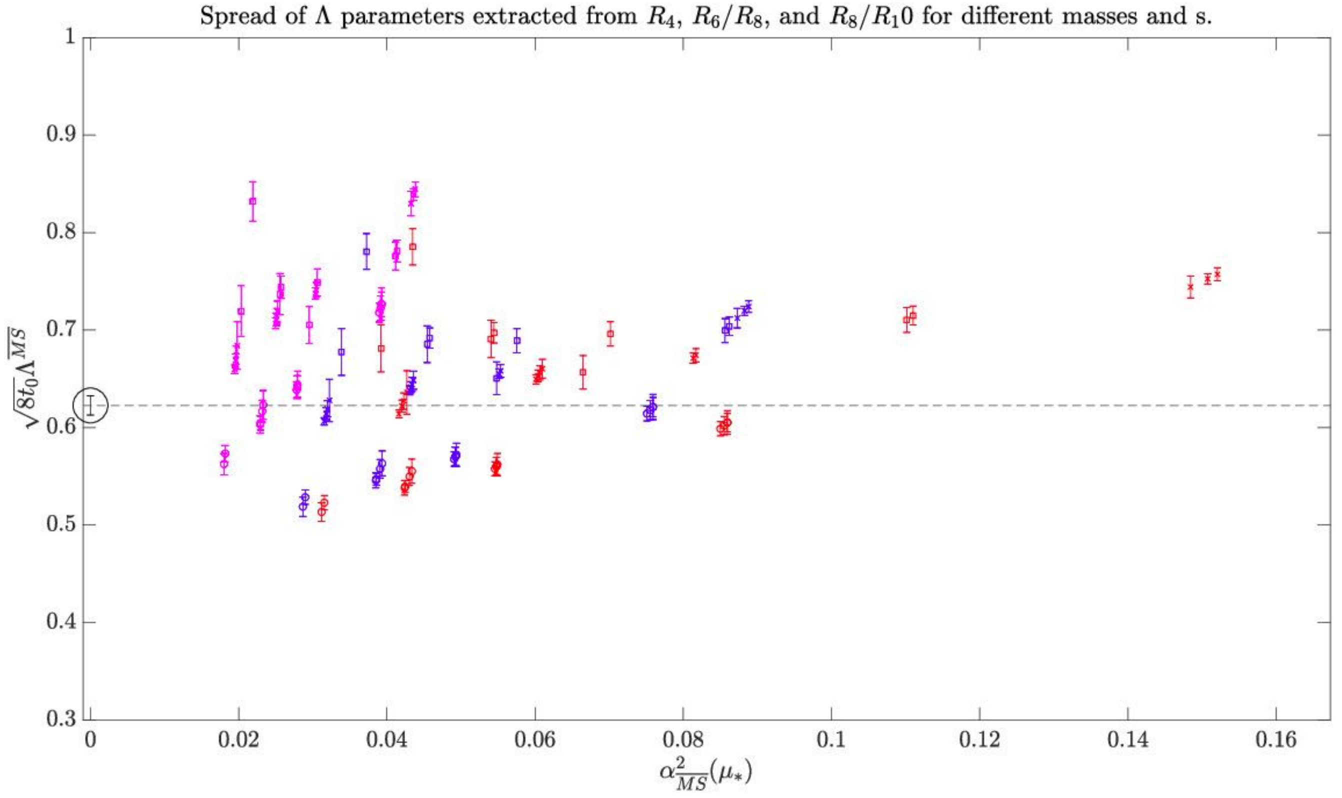
$$\lim_{a\to 0} R_4(\sqrt{8t_0}M_{RGI}, aM_{RGI}) = \sum_{i\geq 0}^{L} r_4^{(i)}(s)\, \alpha^i_{\overline{\mathrm{MS}}}(\mu_*) + O(\alpha^{L+1})$$

- Run to infinite energy via 5L beta function and 4L mass anomalous dimension:

$$\frac{\sqrt{8t_0}\Lambda^{\overline{\mathrm{MS}}}_{RGI}}{\sqrt{8t_0}M_{RGI}} = s\frac{(b_0 g_{\overline{\mathrm{MS}}}(\mu_*)^2)^{-b_1/(2b_0^2)}}{(2b_0 g_{\overline{\mathrm{MS}}}(\mu_*)^2)^{-d_0/(2b_0)}} \exp\left\{-\frac{1}{2b_0 g_{\overline{\mathrm{MS}}}(\mu_*)^2} - \int_0^{g_{\overline{\mathrm{MS}}}(\mu_*)} \mathrm{d}x \left[\frac{1-\tau_{\overline{\mathrm{MS}}}(x)}{\beta_{\overline{\mathrm{MS}}}(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x} + \frac{d_0}{b_0 x}\right]\right\}$$

- Remember, variation of *s* often used to estimate PT's truncation error

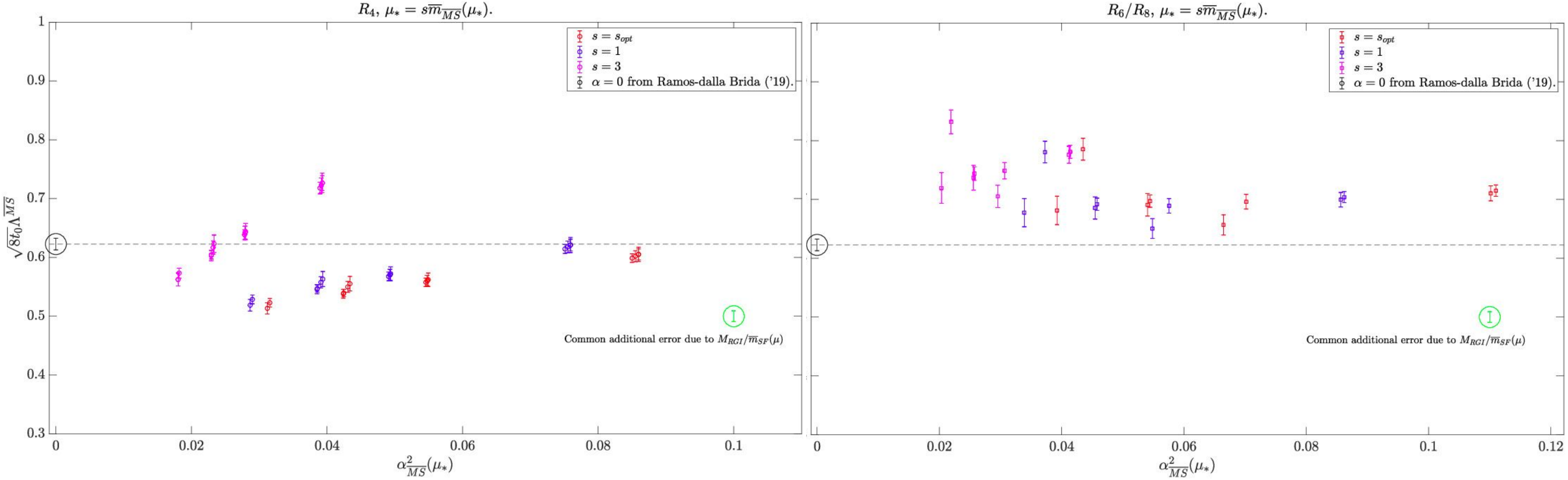
Λ for all Moments, Masses and Scales



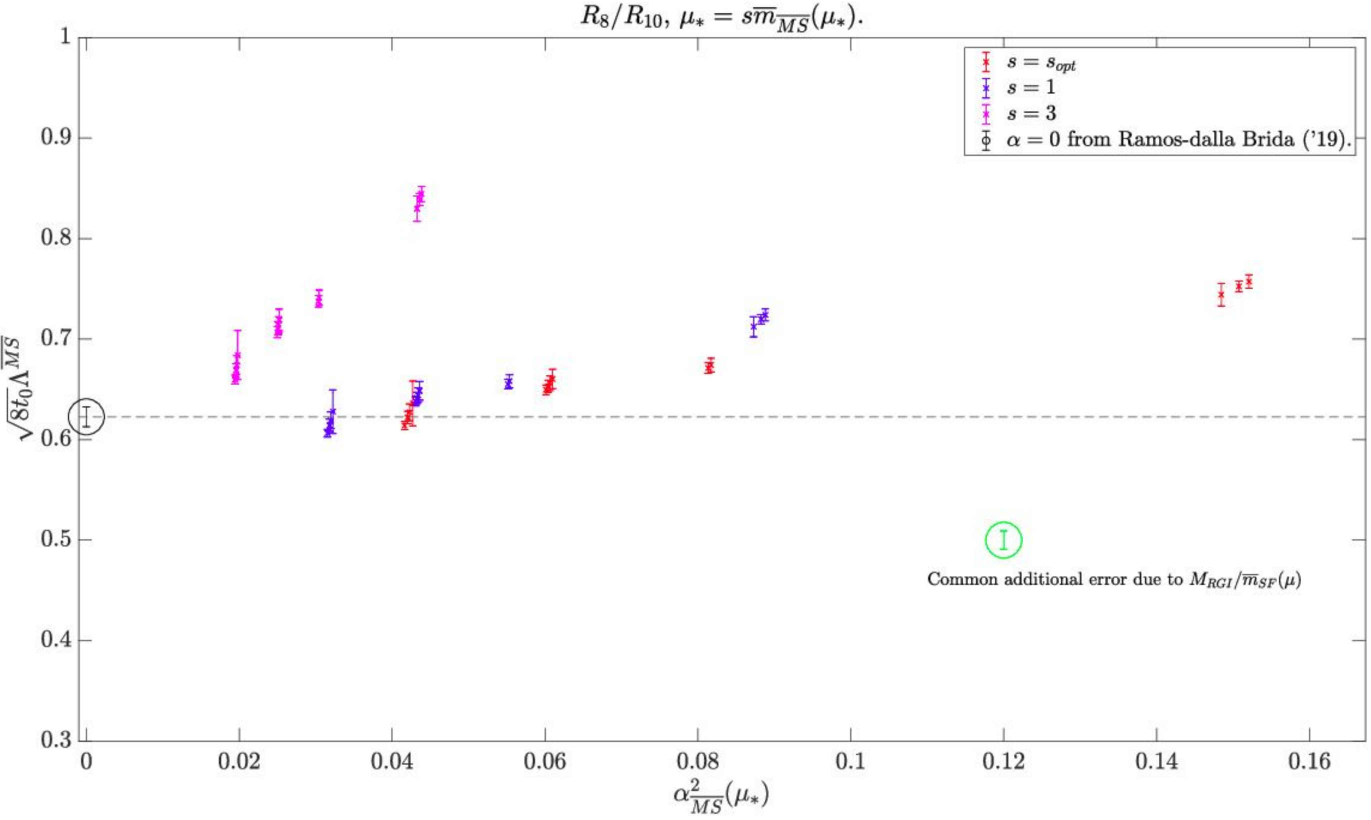
- The spread of Λ is very large for different *n, s, M*.
- Need to analyze systematically at fixed *n, s.*

$$\Lambda^{\text{eff}}_{RGI} = \Lambda_{RGI} + \mathcal{O}\left(\alpha^2_{\overline{\text{MS}}}(\mu_*)\right)$$

Lambda Parameter I



Lambda Parameter II



Summary and Conclusions

- Even with non-perturbative c_{sw} and fully twisted, quenched Wilson fermions taking the continuum limit is very challenging.
- We really need to go to the last available lattice spacing, $a \approx 0.01$ fm (which is $L/a \times T/a = 192 \times 480$), to consolidate the results.
- We also want to include the leading logarithmic correction.
- We want to attempt global fits of all M, a together
- Extracting the Λ-parameter from R_n with $n=4,6,8,10$ with controlled errors is very demanding. It might be beyond our capabilities in the pure gauge theory.
- Window problem looks tough for moments method.

Thank You!

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Questions Please!

Backup 1: tm ren. factors

- For some scheme S, renormalization scale μ, we have for a doublet of mass -degenerate Wilson, twisted mass fermions out of full twist:

$$\overline{m}_S(\mu) = \lim_{a\to 0}\left[Z_P^S(a\mu, g_0)\right]^{-1}\sqrt{\mu_{tm}^2 + Z_A^2(g_0)m_{PCAC}^2}$$

$$= \lim_{a\to 0}\left[Z_P^S(a\mu, g_0)\right]^{-1}\sqrt{\mu_{tm}^2 + Z_A^2(g_0)Z^2(g_0)m_q^2}$$

$$m_q = m_0 - m_{cr.}$$

$$Z(g_0) = \frac{Z_m^S(a\mu, g_0)Z_P^S(a\mu, g_0)}{Z_A(g_0)}$$

Backup 2: Position Space def. of Moments

once $q_\mu = q^\mu = (q_0, \vec{0})$

$$\Pi(q^2, m) = i \int \mathrm{d}^4 x \, e^{iq\cdot x} \langle 0| \mathcal{T} \left\{ J^\dagger(x) J(0) \right\} |0\rangle \qquad q_0^2\, \Pi(q_0^2) = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{itq_0} \int \mathrm{d}^3 \vec{x} \langle J(x) J(0)\rangle = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{itq_0} G(t)$$

$$\mathcal{M}_n = \left(\frac{\partial}{\partial i q_0}\right)^n q_0^2\, \Pi(q_0^2) \Big|_{q_0=0} = \left(\frac{\partial}{\partial i q_0}\right)^n \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{itq_0} G(t) \Big|_{q_0=0} =$$

$$= \int_{-\infty}^{\infty} \mathrm{d}t\, t^n \mathrm{e}^{itq_0} G(t) \Big|_{q_0=0} = \int_{-\infty}^{\infty} \mathrm{d}t\, t^n G(t)\,,$$

Backup 3: tm PS and Disconnected Diagrams

$$P^i(x) = i m_0 \overline{\psi}(x)\gamma_5 \frac{\tau^i}{2}\psi(x)$$

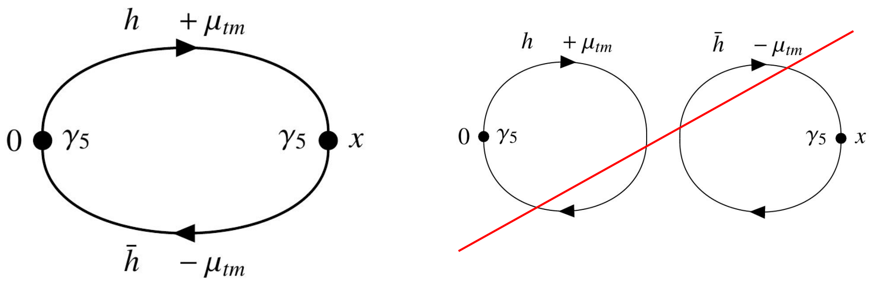
$$P^\pm(x) = P^1(x) \pm i P^2(x)$$

$$P^{1,2}(x) \xrightarrow{\text{chiral rotation}} P^{1,2}(x)$$

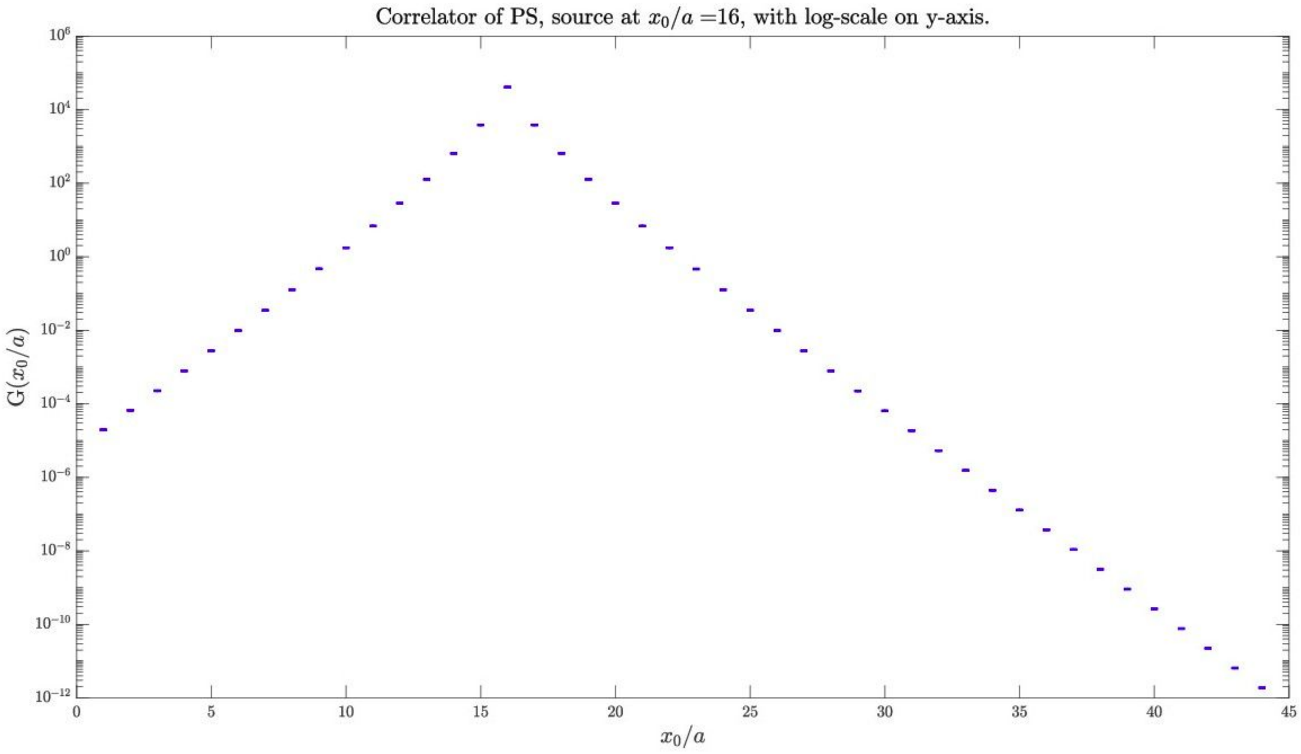
$$P^\pm(x) \xrightarrow{\text{chiral rotation}} P^\pm(x)$$

$$G(t) = -\mu_{tm}^2 \sum_{\vec{x}} a^3 \left\langle \mathrm{tr}\left[S_h(x,0)\gamma_5 S_{h'}(0,x)\gamma_5\right]\right\rangle$$

The 2 different flavors mean no disconnected diagrams are present:

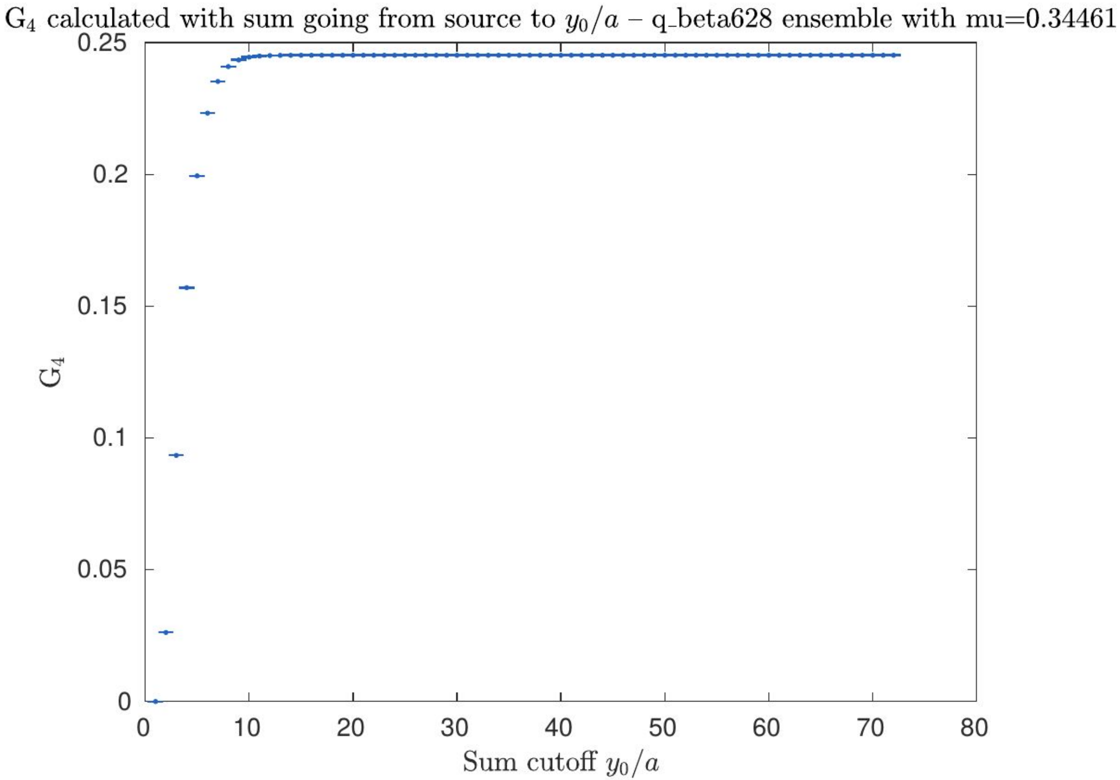


Backup 4: Boundary Effects



Boundary effects are smaller than the statistical error, the correlator around the source is symmetric up to very close to the boundary

Backup 5: Saturation of Sum



- The moments saturate after a certain number of time slices have been summed up.
- The sum is cut off when the plateau has been reached.

Backup 6: Integrand of Moment