Data analysis on two-point correlation function with sequential Bayesian method

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Motivation

- To find decay constants (e.g. $f_{B_{(s)}}$, $f_{D_{(s)}}$, $\cdots$) and semileptonic form factor (e.g. $h_{A_1}(w)$, $h_{A_2}(w)$, $\cdots$), the data analysis on the 2-point correlation function should be done first.

- The data analysis should determine not only the ground state but also the excited states.

Sequential Bayesian method

1. Step 1: Do the 1st fitting. ex) 1+0 fit (2 parameters)
2. Step 2: Feed the results as prior information for the 2nd fitting.
   ex) 1+1 fit (4 parameters + 2 prior information)
3. Step 3: Do stability test and find optimal prior information.
4. Step 4: Move the 2nd fitting results into the 1st fitting.
5. Step 5: Make the next fitting (e.g. $2+1$ fit) the 2nd fitting.
6. Step 6: Go back to "Step 2".
7. ex) $1+0 \to 1+1 \to 2+1 \to 2+2 \to 3+2 \to \cdots$

Measurement information

Motivation: we provide information on sea and valance quarks.

1. MILC HISQ ensemble with $N_f = 2 + 1 + 1$ [PRD **87** 054505]
   Example of this talk: a12m220 (Ensemble ID)

a (fm)	$N_s^3 \times N_t$	M_π (MeV)	am_l	am_s	am_c	N_{cfg}
0.1184(10)	$32^3 \times 64$	216.9(2)	0.00507	0.0507	0.628	1000

2. Hopping parameter of Oktay-Kronfeld action for valance b quark
   $\kappa_{\text{crit}} = 0.051218$
   $\kappa_b = 0.04070$

3. HISQ action parameter for valance light quark
   $m_x = m_s = 0.0507$

Correlator fit

Motivation: fitting functional form

- 2-point correlation function for B meson [PoS(LAT2019)050]

$$C(t) = \sum_{\alpha=1}^{4} \sum_{\mathbf{x}} \left\langle \mathcal{O}_\alpha^\dagger(t,\mathbf{x}) \mathcal{O}_\alpha(0) \right\rangle ,$$

$$\mathcal{O}_\alpha(t,\mathbf{x}) = \left[\bar{\psi}_b(t,\mathbf{x}) \gamma_5 \Omega(t,\mathbf{x})\right]_\alpha \chi_\ell(t,\mathbf{x}) ,$$

$$\Omega(t,\mathbf{x}) \equiv \gamma_1^{x_1} \gamma_2^{x_2} \gamma_3^{x_3} \gamma_4^{t} ,$$

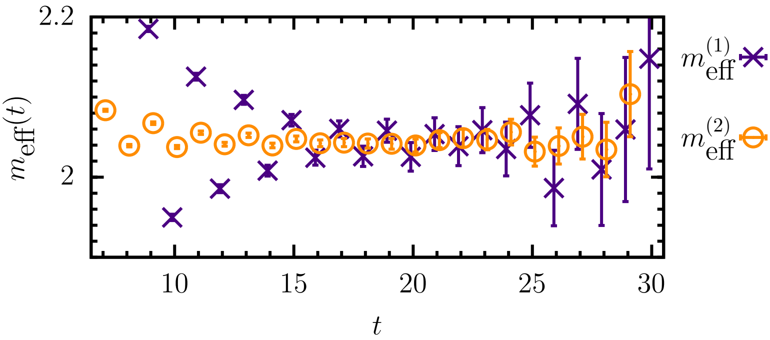
- Fitting function

$$f(t) = g(t) + g(T-t)$$

$$g(t) = A_0 e^{-E_0 t} \Big[1 + R_2 e^{-\Delta E_2 t} \left(1 + R_4 e^{-\Delta E_4 t} (1 + \cdots) \right) - (-1)^t R_1 e^{-\Delta E_1 t} \left(1 + R_3 e^{-\Delta E_3 t} (1 + \cdots) \right) \Big]$$

where $R_i = \dfrac{A_i}{A_{i-2}}$, $\Delta E_i = E_i - E_{i-2}$, $A_{-1} = A_0$ and $E_{-1} = E_0$.

Effective mass plot



$$m_{\text{eff}}^{(j)}(t) = \frac{1}{j} \ln\left(\frac{C(t)}{C(t+j)}\right)$$

Motivation: we determine fit range for 1+0 fit
$21 \leq t \leq 29.$

Results for 1+0 fit

Motivation: we provide fitting results for 1+0 fit.

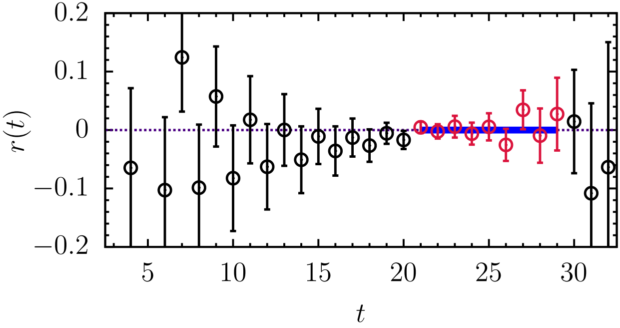
- Fitting function:

$$f(t) = g(t) + g(T - t)$$
$$g(t) = A_0 e^{-E_0 t}$$

- Fit results:

parameter	$\mathbf{p} = (0, 0, 0)$
A_0	0.0182(29)
E_0	2.0468(76)
χ^2/d.o.f.	0.319(14)

Residual plot of 1+0 fit



1. Blue solid line represents the fit range ($21 \leq t \leq 29$).

$$r(t) = \frac{C(t) - f(t)}{|C(t)|}$$

$C(t)$: correlator data , $f(t)$: fitting function

2. **Motivation:** we want to determine the next fitting. Data for $t \leq 21$ oscillate $\rightarrow$ 1+1 fit.

Step up from 1+0 fit to 1+1 fit

Motivation: we step up from 1+0 to 1+1 fit.

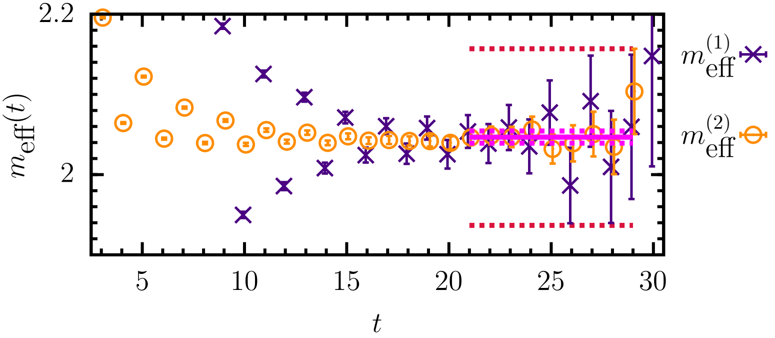
1. 1+1 fitting function

$$f(t) = g(t) + g(T - t)$$
$$g(t) = A_0 e^{-E_0 t}(1 - (-1)^t R_1 e^{-\Delta E_1 t})$$

where $\Delta E_1 = E_1 - E_0$ and $R_1 = \frac{A_1}{A_0}$.

2. Use the 1+0 fit results for A_0 and E_0 as the prior information for 1+1 fit.

Determination of prior width for 1+1 fit



1. Effective mass: $m^{(j)}_{\mathrm{eff}}(t) = \frac{1}{j} \ln \left(\frac{C(t)}{C(t+j)} \right)$
2. Magenta solid (dotted) line: average (error) of 1+0 fit.
3. Red dotted line: prior width for E_0.
4. **Motivation:** we choose the maximal fluctuation of effective mass as prior width for E_0.

Results for 1+1 fit

Motivation: we provide fitting results for 1+1 fit.

1. 1+1 fitting function:

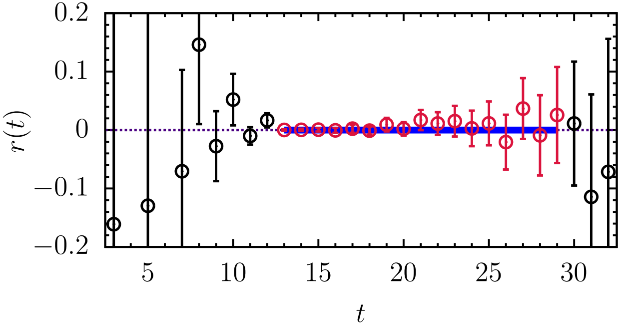
$$f(t) = g(t) + g(T - t)$$
$$g(t) = A_0 e^{-E_0 t}(1 - (-1)^t R_1 e^{-\Delta E_1 t})$$

where $\Delta E_1 = E_1 - E_0$ and $R_1 = \frac{A_1}{A_0}$.

2. Fit results:

parameter	$\mathbf{p} = (0,0,0)$	prior	width
A_0	0.01724(52)	0.0182(144)	5.0σ
E_0	2.0448(22)	2.0468(1101)	14.53σ
R_1	3.5(58)		
ΔE_1	0.36(12)		
χ^2/d.o.f.	0.2306(80)		

Residual plot of 1+1 fit



1. Blue solid line represents the fit range ($13 \leq t \leq 29$).

$$r(t) = \frac{C(t) - f(t)}{|C(t)|}$$

   $C(t)$: correlator data , $f(t)$: fitting function

2. **Motivation:** we want to determine the next fitting.
   Data for $t \leq 13$ oscillate $\rightarrow$ 2+2 fit.

Fitting function for 2+2 fit

Motivation: we provide 2+2 fitting function.

1. 2+2 fitting function

$$f(t) = g(t) + g(T - t)$$
$$g(t) = A_0 e^{-E_0 t}[1 + R_2 e^{-\Delta E_2 t} - (-1)^t R_1 e^{-\Delta E_1 t}(1 + R_3 e^{-\Delta E_3 t})]$$

   where $\Delta E_2 = E_2 - E_0$, $\Delta E_1 = E_1 - E_0$, $\Delta E_3 = E_3 - E_1$, $R_2 = \frac{A_2}{A_0}$, $R_1 = \frac{A_1}{A_0}$ and $R_3 = \frac{A_3}{A_1}$.

2. Use 1+1 fit results for A_0, E_0, R_1, ΔE_1 as the prior information for the 2+2 fit.

Numerical precision problem on covariance matrix

During the 1+1 fit data analysis, we found a problem.

1. Problem: covariance matrix inversion did not work with many time slices (e.g. $15 \leq t \leq 29$).
   - Inversed covariance matrix is used in χ^2.

$$\chi^2 = [C(t_i) - f(t_i)]\, V^{-1}(t_i, t_j)\, [C(t_j) - f(t_j)]$$

2. Reason:
   λ_L = largest eigenvalue of $V \cong 10^{-35}$
   λ_S = smallest eigenvalue of $V \cong 10^{-60}$

3. **Solution**:
   1) rescaling method
   2) correlation matrix method

Solution 1: rescaling method

1. Step 1: determine $R(t)$, the rescaling factor,

$$R(t) = A_0^{\text{rsc}} \exp[-E_0^{\text{rsc}} t] + A_0^{\text{rsc}} \exp[-E_0^{\text{rsc}}(T - t)]$$

   with a fit range (e.g. $23 \leq t \leq 29$).

2. Step 2: rescale the correlator value so that $\tilde{C}(t) = C(t)/R(t)$, and get rescaled covariance matrix $\tilde{V}(t_i, t_j)$.

3. Step 3: in the χ^2-minimizer, fitting function $f(t)$ should also be rescaled by $R(t)$,

$$\chi^2 = \left[\tilde{C}(t_i) - \tilde{f}(t_i)\right] \tilde{V}^{-1}(t_i, t_j) \left[\tilde{C}(t_j) - \tilde{f}(t_j)\right] ,$$

   that is, $R(t)$ must not change the final fitting result.

4. This method solves the numerical precision problem.

Solution 2: correlation matrix method

1. Step 1: For given covariance matrix $V(t_i, t_j)$, obtain correlation matrix

$$\rho(t_i, t_j) = \frac{V(t_i, t_j)}{\sigma(t_i)\sigma(t_j)}$$

   where $\sigma(t_i) = \sqrt{V(t_i, t_i)}$.

2. Step 2: The inversed covariance matrix is

$$V^{-1}(t_i, t_j) = \mathrm{diag}\left[\frac{1}{\sigma(t_i)}\right] \rho^{-1}(t_i, t_j)\ \mathrm{diag}\left[\frac{1}{\sigma(t_j)}\right]$$

3. This method also solves the numerical precision problem.

Comparison

Motivation: we compare the two methods.

- 1+1 fit result with fit range $13 \leq t \leq 29$

parameter	rescaling	correlation
A_0	0.01724(52)	0.01724(52)
E_0	2.0448(22)	2.0448(22)
R_1	3.5(58)	3.5(58)
ΔE_1	0.36(12)	0.36(12)
χ^2/d.o.f.	0.2306(80)	0.2306(80)
run time	**73.3s**	**72.8s**

- Both methods give the same fitting results.
- Correlation matrix method is slightly faster (0.7 %) than rescaling method but this difference is negligible.
- Both methods are good.

Fitting function for 2+2 fit

Motivation: we provide 2+2 fitting function.

1. Use 1+1 fit results on A_0, E_0, R_1, ΔE_1 as the prior information for the 2+2 fit.

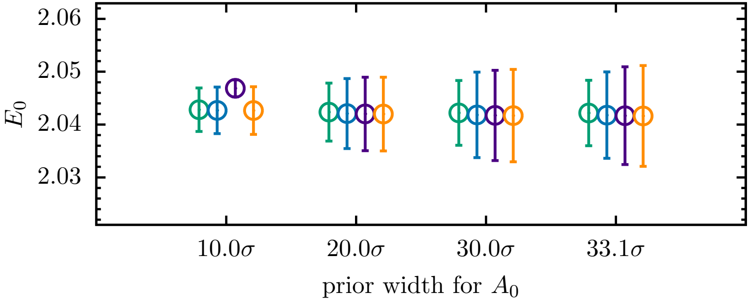
2. 2+2 fitting function

$$
\begin{aligned}
f(t) &= g(t) + g(T-t) \\
g(t) &= A_0 e^{-E_0 t}[1 + R_2 e^{-\Delta E_2 t} \\
&\qquad - (-1)^t R_1 e^{-\Delta E_1 t}(1 + R_3 e^{-\Delta E_3 t})]
\end{aligned}
$$

where $\Delta E_2 = E_2 - E_0$, $\Delta E_1 = E_1 - E_0$, $\Delta E_3 = E_3 - E_1$, $R_2 = \frac{A_2}{A_0}$, $R_1 = \frac{A_1}{A_0}$ and $R_3 = \frac{A_3}{A_1}$.

Stability test on A_0 and E_0 (at 2+2 fit)

- **Motivation:** we find stable prior width for A_0 and E_0.
- X-axis tics: $\sigma_{A_0}^{\text{prior}} = (10\sigma, 20\sigma, 30\sigma, 33.1\sigma(\text{signal cut: } N/S = 1))$
- symbol color: $\sigma_{E_0}^{\text{prior}} = (10\sigma, 20\sigma, 30\sigma, 40\sigma)$.
- Y-axis: 2+2 fit results for E_0 at $(\sigma_{A_0}^{\text{prior}}, \sigma_{E_0}^{\text{prior}})$ in $10 \leq t \leq 29$



- Stability test: find smallest possible σ^{prior} giving stable error.

$$\rightarrow (\sigma_{A_0}^{\text{prior}}, \sigma_{E_0}^{\text{prior}}) = (30\sigma, 20\sigma).$$

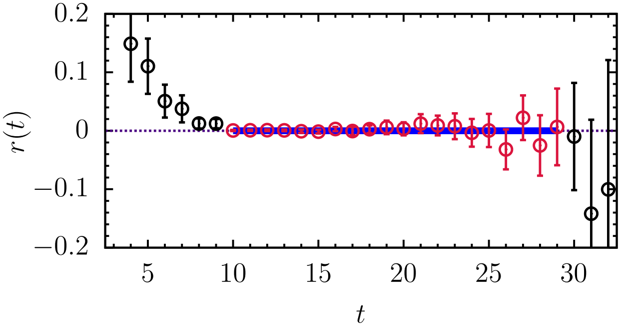
Results for 2+2 fit

Motivation: we provide fitting results for 2+2 fit.

parameter	$\mathbf{p} = (0,0,0)$	prior	width
A_0	0.0161(35)	0.01724(1562)	30.0σ
E_0	2.0418(81)	2.0448(449)	20.0σ
R_1	0.64(24)	3.5(35)	0.61σ
ΔE_1	0.240(31)	0.36(36)	2.96σ
R_2	0.24(21)		
ΔE_2	0.15(25)		
R_3	0.033(75)		
ΔE_3	0.5(36)		
χ^2/d.o.f.	0.3668(91)		

- Stability test: find smallest possible σ^{prior} giving stable error.
$$\rightarrow (\sigma^{\text{prior}}_{A_0}, \sigma^{\text{prior}}_{E_0}) = (30\sigma, 20\sigma).$$
- No stability test for R_1 and ΔE_1, because their N/S results are not good ($N/S > 1$)

Residual plot of 2+2 fit



1. Blue solid line represents the fitting range ($10 \leq t \leq 29$).

$$r(t) = \frac{C(t) - f(t)}{|C(t)|}$$

$C(t)$: correlator data , $f(t)$: fitting function

2. **Motivation:** we want to determine the next fitting. Data for $t \leq 10$ behave exponentially $\rightarrow 3 + 2$ fit.

Problem in initial guess (3+2 fit)

Motivation: we want to fix the problem on initial guess for the χ^2-minimizer.

1. **Problem:**
   - A. The old initial guess for χ^2 minimizer does not come from data.
   - B. If no prior is given, the old initial guess is

$$R_{2j} = 2.5\ j$$
$$R_{2j-1} = 0.025\ j$$
$$\Delta E_{2j} = \Delta E_{2j-1} = 0.1 E_0$$

   - C. For example, in 3+2 fit, $R_4 = 2.5 \times 2 = 5.0$, $\Delta E_4 = 0.1\ E_0$, regardless of the data. $\rightarrow$ slow down the fitting code.
   - D. In our fitting, $R_i \lesssim 1$. $\rightarrow$ The problem gets worse with larger j.
2. **Solution:** The new initial guess must come from data.
   $\rightarrow$ Newton's method

Newton's method for initial guess

Motivation: we use Newton's method for initial guess.

1. We have already determined eight fit parameters from 2+2 fit.
2. For 3+2 fit, two more fit parameters, R_4 and ΔE_4, should be determined by initial guess.
3. For given time slices t_1, t_2, we solve the following equations using Newton's method

$$f(t_1) = C(t_1)$$
$$f(t_2) = C(t_2)$$

where

$$f(t) = \text{fitting function for 3+2 fit}$$
$$C(t) = \text{correlator data}$$

4. **Solution:** the initial guess comes from data. $\leftarrow$ Newton's method

To-do List

- We have applied Newton's method to initial guess for 1+1 fit. $\rightarrow$ **speed up by** $\approx$ **10 %**.
- Similarly, we plan to implement Newton's method up to 3+3 fit.
- Hence, we want to obtain initial guess directly from the data.
- Results of this data analysis will be used as input parameters to fit data of decay constants and semileptonic form factors.
- We need to do the non-perturbative renormalization (NPR) for the OK action to get the physical results for the decay constants and semileptonic form factors.

Thank you so much for your attention!