Finite volume renormalization schemes and the fermionic gradient flow

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Non perturbative renormalization in lattice QCD

Computation of QCD parameters such as Λ, M or any other RGI operators can be separate into 3 different energy ranges:

$$\text{High energy} \longrightarrow \text{Intermediate energy} \longrightarrow \text{Low energy}$$

Non-perturbative renormalization requires connection between low energy hadronic inputs and the perturbative high energy regime.

- High energy is where we can make use of Perturbation Theory (PT)
- The running between high energy all the way down to low energy is performed by using some recursive finite volume renormalization methods in the Schrödinger Functional (SF) renormalization scheme.
- At low energy the Gradient Flow (GF) scheme is the suitable one to monitor the errors because of its small fluctuations.

Our proposal: improve the renormalization conditions at low energy by adding the **GF for both gauge and quark fields in a finite volume scheme** and set up suitable boundary conditions:

- ✓ Nice properties of composite operators defined at positive flow time ($t > 0$) under renormalization.

One application: compute the renormalization factor Z_P for the psuedo-scalar current $P^{ij}(x) = \overline{\psi}_i(x)\gamma_5\psi_j$ which is related to the **quark mass renormalization**.

Gradient flow equations for gauge and fermion fields

The gradient flow evolves fields as functions of the flow time $t \geq 0$, the evolution starts from the initial conditions (unsmeared fields)

$$B_\mu(t,x)|_{t=0} = A_\mu(x)$$

$$\chi(t,x)|_{t=0} = \psi(x) \quad \overline{\chi}(t,x)|_{t=0} = \overline{\psi}(x),$$

time-dependent fields $B_\mu(t,x)$, $\chi(t,x)$ and $\overline{\chi}(t,x)$ are determined from gradient flow differential equations for **gauge fields** [Narayanan & Neuberger 2006] [Lüscher, 2010]

$$\boxed{\partial_t B_\mu = D_\nu G_{\nu\mu}} \quad \longrightarrow \quad [t] = \text{length}^2$$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu + [B_\mu, B_\nu] \qquad D_\mu = \partial_\mu + [B_\mu, \cdot]$$

and for the **quark fields** [Lüscher, 2013]

$$\boxed{\partial_t \chi = \Delta \chi \qquad \partial_t \overline{\chi} = \overline{\chi} \overleftarrow{\Delta}}$$

$$\Delta = \nabla_\mu \nabla_\mu \qquad \nabla_\mu = \partial_\mu + B_\mu .$$

- If the underlying theory, defined as a path integral over the unsmeared fields, is renormalized, then correlation functions of the gauge fields require no further renormalization [Lüscher & Weisz 2011].
- At finite flow time the smeared fermion fields require one multiplicative wave function renormalization [Lüscher, 2013]: $\chi = Z_\chi^{-1/2} \chi_R$, $\overline{\chi} = Z_\chi^{-1/2} \overline{\chi}_R$.

Given such composite operators

$$\mathcal{O}_\Gamma^{ij}(x) = \overline{\psi}_i(x)\Gamma\psi_j(x), \quad (i,j\text{: flavour indices})$$
$$\mathcal{Q}_\Gamma^{ij}(t,x) = \overline{\chi}_i(t,x)\Gamma\chi_j(t,x), \quad (\Gamma\text{: some gamma matrix})$$

the renormalization for the flowed one is obtained by counting the number of fermionic insertions ($Z_\chi^{-1/2}$ for each fermion field) while the non-flowed renormalizes with the appropriate factor

$$\left(\mathcal{O}_\Gamma^{ij}(x)\right)_R = Z_{\mathcal{O}_\Gamma}\mathcal{O}_\Gamma^{ij}(x) \qquad \left(\mathcal{Q}_\Gamma^{ij}(t,x)\right)_R = Z_\chi \mathcal{Q}_\Gamma^{ij}(t,x).$$

Then we can define the renormalization constant $Z_{\mathcal{O}_\Gamma}$ for every non-flowed operator in the following ways (**wave function renormalization cancellation**)

$$i) \qquad Z_{\mathcal{O}_\Gamma} \left.\frac{t^{3/2}\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_1}^{ij}(t,x)\mathcal{O}_{\Gamma_1}^{ji}(y)\rangle}{\sqrt{t^{3/2}\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_2}^{ij}(t,x)\mathcal{Q}_{\Gamma_2}^{ji}(t,y)\rangle}}\right|_{\mathbf{y}=\mathbf{0}} \equiv \left.\frac{t^{3/2}\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_1}^{ij}(t,x)\mathcal{O}_{\Gamma_1}^{ji}(y)\rangle}{\sqrt{t^{3/2}\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_2}^{ij}(t,x)\mathcal{Q}_{\Gamma_2}^{ji}(t,y)\rangle}}\right|_{\mathbf{y}=\mathbf{0},\ \text{tree level}},$$

$$ii) \qquad Z_{\mathcal{O}_\Gamma} \left.\frac{\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_1}^{ij}(t,x)\mathcal{O}_{\Gamma_1}^{ji}(y)\rangle}{\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_2}^{ij}(t,x)\mathcal{Q}_{\Gamma_2}^{ji}(0,y)\rangle}\right|_{\mathbf{y}=\mathbf{0}} \equiv \left.\frac{\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_1}^{ij}(t,x)\mathcal{O}_{\Gamma_1}^{ji}(y)\rangle}{\sum_{\mathbf{x}}\langle \mathcal{Q}_{\Gamma_2}^{ij}(t,x)\mathcal{Q}_{\Gamma_2}^{ji}(0,y)\rangle}\right|_{\mathbf{y}=\mathbf{0},\ \text{tree level}},$$

- $Z_{\mathcal{O}_\Gamma}$ is equal to 1 at tree level in the continuum limit,
- Because of the Gaussian damping factor $\sim e^{-tp^2}$ in the propagator due to the flow, we have no finiteness problem at $x = y$,
- Fixed values of t, x_0 and y_0 correspond to different renormalization conditions.

When we calculate correlation functions in **finite volumes** of size $\frac{T}{a} \times \left(\frac{L}{a}\right)^3$ we fix some of the scales:

$$\{\mu\ ,\ T\ ,\ L\ ,\ a\ ,\ x_0\ ,\ y_0\ ,\ t\ ,\ m_q\}$$

- renormalization scale $\mu = 1/L$ (SF naturally allows it) ,
- time coordinates $\frac{x_0}{T} = \frac{y_0}{T} = \frac{1}{2}$,
- aspect ratio $\rho \equiv \frac{T}{L} = 1$,
- SF is a **mass independent RS**: all computations with **massless quarks** ($\boldsymbol{m_q = 0}$) $\to$ no quark mass dependence in the renormalization factors Z,
- the flow time t to
$$t = c^2 \frac{L^2}{8},$$
c is a dimensionless parameter labelling (finite volume) renormalization schemes.

On the lattice everything will be expressed in terms of c and a/L

$$Z_{\mathcal{O}_\Gamma}(g_0^2, c, a/L), \quad (g_0 \text{ is the bare gauge coupling constant}).$$

In order to investigate the best scheme with the smallest cutoff effects at tree level we are going to "play" with the following ingredients

- definition of the fermion flow $\partial_t \chi(t,x) = \boxed{\ ?\ }\ \chi(t,x)$;
- finite volume requires suitable boundary conditions for fermion fields;
- different ren. conditions for $Z_{\mathcal{O}_\Gamma}$ and choices for the auxiliary operator $\mathcal{Q}(t,x)$.

Definition of the fermion flow

The original formulation for fermions is the one that makes use of the Laplacian

$$\begin{cases} \partial_t \chi = \Delta \chi \\ \chi|_{t=0} = \psi \end{cases}$$

But other possible solutions that can be tested on the lattice are

- $\partial_t \chi = D_W^\dagger D_W \chi$
- $\partial_t \chi = \Delta_I \chi$

where

$$\Delta = \sum_{\mu=0}^{3} \nabla_\mu^* \nabla_\mu, \quad (\nabla_\mu^*, \nabla_\mu : \text{backward and forward derivative})$$

$$D_W = \sum_{\mu=0}^{3} \left\{ \gamma_\mu \left(\frac{\nabla_\mu + \nabla_\mu^*}{2} \right) - \frac{a}{2} \nabla_\mu^* \nabla_\mu \right\}, \quad (\text{Wilson Dirac operator})$$

$$\Delta_I = \sum_{\mu=0}^{3} \nabla_\mu^* \nabla_\mu \left(1 - \frac{a^2}{12} \sum_{\nu=0}^{3} \nabla_\nu^* \nabla_\nu \right), \quad (\text{"improved" Laplacian: } \Delta_I \xrightarrow{a \to 0} \Delta^{\text{cont}} + O(a^4))$$

(similar equations for $\overline{\chi}$).

Boundary conditions for fermion fields

Finite volume schemes require boundary conditions for fields, we have two types of fermion

- $\boldsymbol{\psi}, \overline{\boldsymbol{\psi}}$ **satisfy Dirac equation** (first order diff. eq.) with SF b.c. (similar for $\overline{\psi}$)

$$P_+\psi(x)|_{x_0=0} = 0 \quad P_-\psi(x)|_{x_0=T} = 0 \quad (P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_0))$$

- $\boldsymbol{\chi}, \overline{\boldsymbol{\chi}}$ do not satisfy ~~Dirac equation~~, they **satisfy flow equation** (second order diff. eq.) natural choice can be Dirichlet b.c. (similar for $\overline{\chi}$)

$$\chi(t,x)|_{x_0=0} = 0 = \chi(t,x)|_{x_0=T}$$

Possible variants for boundary conditions:

- Using the Dirac operator D_W with the SF b.c. automatically implements the SF b.c. for the χ fields and it also fixes the complementary half components with Neumann b.c.

$$\textbf{SF b.c. for } \boldsymbol{\psi}, \overline{\boldsymbol{\psi}}, \boldsymbol{\chi}, \overline{\boldsymbol{\chi}}.$$

- A modified formulation of standard SF exists, called **chirally rotated SF** (**$\boldsymbol{\chi}$SF**), as in the continuum limit they are related to the standard SF by a chiral field rotation (flavour structure induced): similar to the SF case, D_W with χSF b.c. automatically implements

$$\boxed{\boldsymbol{\chi}\textbf{SF b.c. for } \boldsymbol{\psi}, \overline{\boldsymbol{\psi}}, \boldsymbol{\chi}, \overline{\boldsymbol{\chi}}}\ .$$

Nice property of χSF:

- **Automatic $\boldsymbol{O(a)}$ improvement** ([Sint, 2005]), while it fails with standard SF b.c's., We used the following renormalization conditions for Z_P

$$\textbf{i)}\ Z_P(g_0^2,c,a/L)\ \frac{t^{3/2}\sum_x\langle P(t,x)P(0,y)\rangle}{\sqrt{t^{3/2}\sum_x\langle P(t,x)P(t,y)\rangle}}\Bigg|_{\mathbf{y=0}} = \frac{t^{3/2}\sum_x\langle P(t,x)P(0,y)\rangle}{\sqrt{t^{3/2}\sum_x\langle P(t,x)P(t,y)\rangle}}\Bigg|_{\mathbf{y=0},\ \text{tree level}} \to Z_P$$

$$\textbf{ii)}\ Z_P(g_0^2,c,a/L)\ \frac{\sum_x\langle P(t,x)P(0,y)\rangle}{\frac{1}{3}\sum_k\sum_x\langle A_k(t,x)A_k(0,y)\rangle}\Bigg|_{\mathbf{y=0}} = \frac{\sum_x\langle P(t,x)P(0,y)\rangle}{\frac{1}{3}\sum_k\sum_x\langle A_k(t,x)A_k(0,y)\rangle}\Bigg|_{\mathbf{y=0},\ \text{tree level}} \to \frac{Z_P}{Z_A}$$

$$\textbf{iii)}\ Z_P(g_0^2,c,a/L)\ \frac{\sum_x\langle P(t,x)P(0,y)\rangle}{\frac{1}{3}\sum_k\sum_x\langle V_k(t,x)\tilde{V}_k(0,y)\rangle}\Bigg|_{\mathbf{y=0}} = \frac{\sum_x\langle P(t,x)P(0,y)\rangle}{\frac{1}{3}\sum_k\sum_x\langle V_k(t,x)\tilde{V}_k(0,y)\rangle}\Bigg|_{\mathbf{y=0},\ \text{tree level}} \to Z_P\ (Z_{\tilde{V}}=1)$$

$P(t,x)=\overline{\chi}(t,x)\gamma_5\chi(t,x)$, $A_\mu(t,x)=\overline{\chi}(t,x)\gamma_\mu\gamma_5\chi(t,x)$, $V_\mu(t,x)=\overline{\chi}(t,x)\gamma_\mu\chi(t,x)$, $\tilde{V}_\mu(t,x)$: point-split vector current.

Technical details of the numerical investigation:

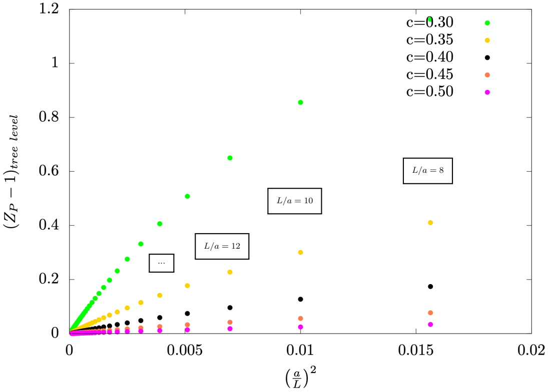
- **Tree level** computation.
- $\boldsymbol{D_W^\dagger D_W}$ for the fermion flow with $\boldsymbol{\chi}$**SF b.c.** for both ψ and χ fields.
- Fixed parameters: **massless quarks** $m_q=0$, $\frac{x_0}{T}=\frac{y_0}{T}=\frac{1}{2}$, $\frac{T}{L}=1$.
- Free parameters: a/L and $c=\frac{\sqrt{8t}}{L}\in[0.3,0.5]$.
- $Z_P(c,a/L)\big|_{\text{tree level}}\sim 1+\cancel{O(g)}+\underline{\text{cut-off effects } O(a^2)}$.

$$\downarrow$$

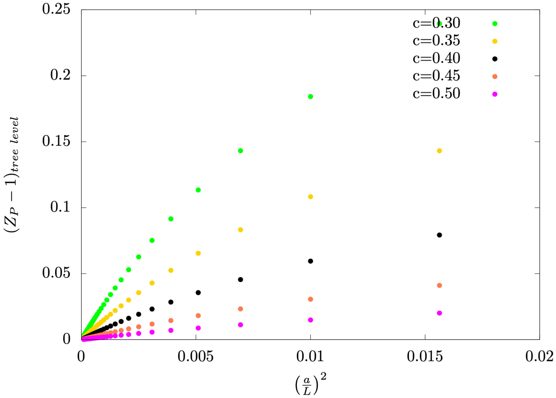
Finally let us see which case shows the smallest discretization effects!

Preliminary tests and results (1/3)

- Vector sector looks better than the Axial, which has discretization effects ~ 4 times bigger.

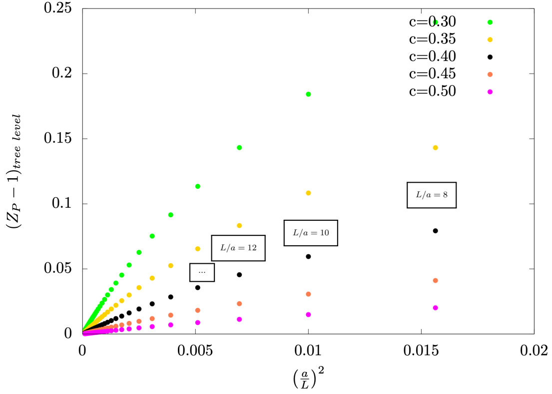


(a) ii) Ren. condition with axial current

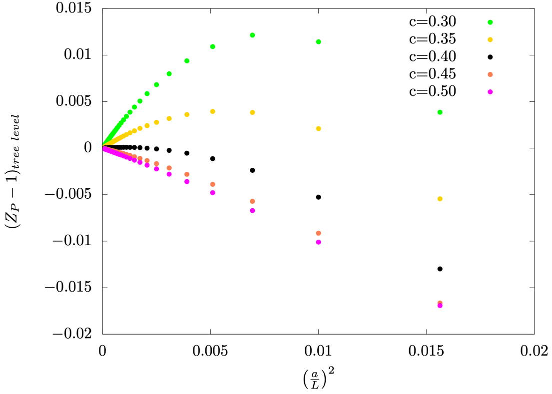


(b) iii) Ren. condition with vector current

- Using **pseudo-scalar correlation function** also at the denominator seems to be the case with the **smallest discretization effects** (a difference of one order of magnitude).



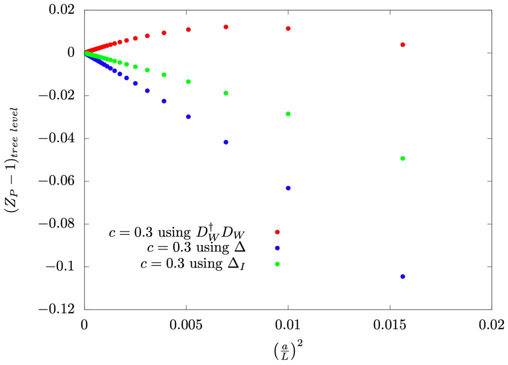
(c) iii) Ren. condition with vector current



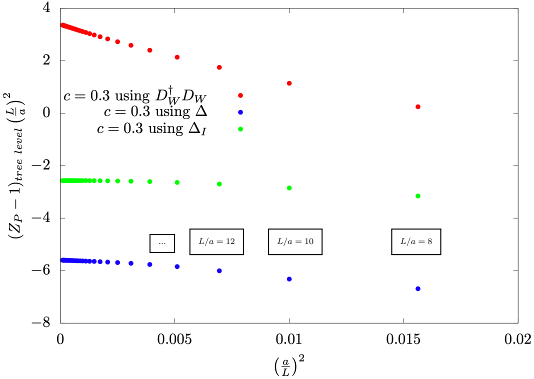
(d) i) Ren. condition with pseudo-scalar current

Preliminary tests and results (3/3)

- We investigated how order $O(a^2)$ terms in the flow affect the total cut-off effects for Z_P,
- these three operators at tree level have 3 different implementations which differ of order $O(a^2)$ terms.



(e) i) Ren. condition with pseudo-scalar current



(f) i) Ren. condition: quadratic coefficients

We found some suitable renormalization conditions that could work at tree level.

What we are doing now:

- Investigate also the Laplacian with the standard SF b.c.'s for both ψ and χ fields (at tree level).

What we are going to do in the next future:

- test these schemes non-perturbatively, implementing the fermionic flow equations on some simulation program (openQCD).
- the one which looks promising already at tree level will be the most interesting to investigate.

If successful, the new method will improve the renormalization of composite operators and its applications such as the running of the quark mass renormalization constant.

Thanks for your attention!

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