The decoupling strategy for the determination of α_s

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Reminder of Λ-parameter and RGI quark mass

Given $\bar{g}_s(\mu)$, $\overline{m}_s(\mu)$ in mass independent scheme s, N_f quarks:

$$\Lambda_s^{(N_\mathrm{f})} = \mu \times \varphi_s^{(N_\mathrm{f})}(\bar{g}_s^{(N_\mathrm{f})}(\mu))\,,$$

$$\varphi_s^{(N_\mathrm{f})}(\bar{g}_s) = (b_0\bar{g}_s^2)^{-b_1/(2b_0^2)} \mathrm{e}^{-1/(2b_0\bar{g}_s^2)} \times \exp\left\{-\int_0^{\bar{g}_s} \mathrm{d}x \left[\frac{1}{\beta_s(x)} + \frac{1}{b_0 x^3} - \frac{b_1}{b_0^2 x}\right]\right\}$$

$$M = \overline{m}_s(\mu)\left[2b_0\bar{g}_s^2(\mu)\right]^{-\frac{d_0}{2b_0}} \exp\left\{-\int_0^{\bar{g}_s(\mu)} \left[\frac{\tau_s(x)}{\beta_s(x)} - \frac{d_0}{b_0 x}\right] dx\right\}\,,$$

- Scheme dependence: $\Lambda_s^{(N_\mathrm{f})}/\Lambda_{\overline{\mathrm{MS}}}^{(N_\mathrm{f})}$ *exactly* computable in a 1-loop calculation
- β-function, $\beta(g) = -b_0 g^3 - b_1 g^5 + \ldots$, and quark mass anomalous dimension $\tau(g) = -d_0 g^2 - d_1 g^4$ have leading coefficients

  $b_0 = (11 - \frac{2}{3}N_\mathrm{f})/(4\pi)^2, \qquad d_0/2b_0 = 4/(11 - \frac{2}{3}N_\mathrm{f}) \qquad (= 4/9$ if $N_\mathrm{f} = 3)$.
- We define $m_\star = \overline{m}_{\overline{\mathrm{MS}}}(\overline{m}_{\overline{\mathrm{MS}}})$

Decoupling in terms of Λ, M

Athenodorou et al (ALPHA '18) reformulate decoupling in terms of the Λ-parameter:

$$\underbrace{\frac{\Lambda^{(N_\mathrm{f})}_{\overline{\mathrm{MS}}}}{\mu_\mathrm{dec}}}_{\text{target}} \times \underbrace{P\left(\frac{M}{\mu_\mathrm{dec}} \frac{\mu_\mathrm{dec}}{\Lambda^{(N_\mathrm{f})}_{\overline{\mathrm{MS}}}}\right)}_{\text{pert. in } \alpha_{\overline{\mathrm{MS}}}(m_\star)} = \underbrace{\frac{\Lambda^{(0)}_{\overline{\mathrm{MS}}}}{\Lambda^{(0)}_s}}_{\text{known constant}} \times \underbrace{\varphi_s^{(0)}\Big(\underbrace{\sqrt{u_\mathrm{M}}}_{\text{massive } \bar{g}}\Big)}_{\text{step-scaling } N_\mathrm{f}=0} + O(M^{-k}), \quad k = ?$$

step-scaling $N_\mathrm{f} = 0$
(Dalla Brida & Ramos '19)
(Nada & Ramos '20)

Decoupling strategy (Dalla Brida et al, ALPHA '19)

- Gradient flow couplings in finite volume, $s' =$ GF and $s =$ GFT (s. below)
- Define scale μ_dec by $\bar{g}^2_{s'}(\mu_\mathrm{dec}) = C_\mathrm{dec}$

$\Rightarrow$ measure massive coupling $u_M = \bar{g}^2_s(\mu_\mathrm{dec}, M)$ at fixed μ_dec and large values of $z = M/\mu_\mathrm{dec}$.

- Approach to continuum and decoupling limit of u_M:
    - leading power k of $1/M$?
    - leading logarithmic corrections to a^2 and $1/M^k$?

Choice of GF/GFT couplings, boundary conditions

Gradient flow couplings:

- GF-scheme: SF boundary conditions at $x_0 = 0, T$; projection to $Q_{\text{top}} = 0$:

$$\bar{g}^2_{\text{GF}}(L) = \mathcal{N}^{-1} \sum_{i,j=1}^{3} \int d^3\mathbf{x}\, t^2 \left\langle \text{tr}\left\{G_{ij}(t,x)G_{ij}(t,x)\right\}\right\rangle \Bigg|_{T=L,\, M=0}^{x_0=T/2,\, c=\sqrt{8t}/L}$$

- GFT-scheme: same as GF but $T = 2L$, i.e. double the distance to the times boundaries at $x_0 = 0, T$
- flow-time t, fix the ratio $\sqrt{8t}/L = c$; in practice $c = 0.3,\ 0.42$
- Choice of schemes: GF to define μ_{dec}, GFT with $M \neq 0$ to define u_M.
- GF couplings are proportional to expectation values $\langle O_{\text{gf}} \rangle$.

Symanzik expansion

Assume non-perturbative O(a) improvement, obtain O(a^2) effects using the Symanzik expansion for gradient flow observable $O_{\rm gf}$ (ignore time boundaries):

$$\langle O_{\rm gf}\rangle_{\rm lat} = \langle O_{\rm gf}\rangle_{\rm cont} - a^2\langle O_{\rm gf} S_2\rangle_{\rm cont} + O(a^3)$$

- All a^2 corrections to gradient flow observables arise from insertion of

$$S_2 = \int d^4x \sum_{i=1}^{18} \omega_i O_i^{d=6}$$

- Effective theory can be treated in renormalized continuum PT ($\overline{\rm MS}$-scheme) at the scale $\mu = 1/a$, with $\omega_i = \omega_i(\alpha_{\overline{\rm MS}}^{(N_{\rm f})}(1/a))$.
- Given the 1-loop anomalous dimension **matrix**, γ_0, pass to RGI operators (Balog, Niedermayer, Weisz '09; Husung, Marquard, Sommer '19-21)

$$(O_{\rm R})_i(\mu = 1/a) \propto \sum_{j=1}^{18} \left([\alpha_{\overline{\rm MS}}^{(N_{\rm f})}(1/a)]^{\gamma_0/2b_0} \right)_{ij} O_j^{\rm RGI} \times \left(1 + O(\bar{g}^2)\right)$$

- Changing to an operator basis with diagonal γ_0 renders the leading $a^2[\alpha(1/a)]^{\hat{\Gamma}_i}$ terms explicit (cf. talk by N. Husung).
- Exponents $\hat{\Gamma}_i$ obtained as eigenvalues of $\gamma_0/2b_0$, shifted by 0 or 1, depending on leading order in expansion of ω_i's.
- Gradient flow observables: complete matrix γ_0 only known for Yang-Mills theory (Husung '21)

Quark mass dependence in QCD with $N_\mathrm{f} \geq 2$

Gluonic observables: quark mass dependence through fermion determinant:

- Ginsparg-Wilson regularization & index theorem (each quark flavour):

$$\det[D_{\mathsf{GW}}(-m)] = (-1)^{Q_{\mathsf{top}}} \det[D_{\mathsf{GW}}(m)]$$

$\Rightarrow$ must hold in continuum theory, too!

- Spurion analysis (either N_f even or $N_\mathrm{f} > 2$ odd with $Q_{\mathsf{top}} = 0$ projection):

$$\Rightarrow \qquad \langle O_{\mathsf{gf}} \rangle_{(m)} = \langle O_{\mathsf{gf}} \rangle_{(-m)}, \qquad \langle O_{\mathsf{gf}} S_2 \rangle_{(m)} = \langle O_{\mathsf{gf}} S_2 \rangle_{(-m)},$$

- Conclusion for GF couplings with Wilson quarks (ignoring boundaries):
  - no $1/M$ or a^2/M corrections;
  - no $a^2 M$, $a^2 M^3$ corrections
  - $a^2 M^2$ effects are the leading cutoff effects, from operators (in S_2):

$$m^2 \times \underbrace{\frac{1}{g^2}\,\mathsf{tr}\,(F_{\mu\nu}F_{\mu\nu})}_{\equiv \mathcal{B}_0}, \qquad m^3 \bar{\psi}\psi \qquad (\in S_2)$$

Decoupling expansion for $\langle O_{\text{gf}} \rangle$

- Effective decoupling theory, action:

$$S_{\text{dec}} = S_{\text{dec},0} + \frac{1}{m^2} S_{\text{dec},2} + O(1/m^4), \qquad S_{\text{dec},0} = \int d^4x \frac{1}{2g_0^2} \text{tr}\, (F_{\mu\nu}(x) F_{\mu\nu}(x))$$

- 2 dimension-6 pure gauge operators:

$$S_{\text{dec},2} = \int d^4x \left\{ \tilde{\omega}_1 \frac{1}{g^2} \text{tr}\, (D_\mu F_{\mu\nu} D_\rho F_{\rho\nu}) + \tilde{\omega}_2 \frac{1}{g^2} \text{tr}\, (D_\mu F_{\rho\nu} D_\mu F_{\rho\nu}) \right\}$$

  Their one-loop anomalous dimension matrix is known (Husung '21), eigenvalues: $\hat{\gamma}_{1,2} = 0,\ 7/11$

- In diagonal basis & using $M/m_\star \propto [\alpha(m_\star)]^{4/9}[1 + O(\alpha)]$:

$$\langle O_{\text{gf}} \rangle = \langle O_{\text{gf}} \rangle_{\text{dec}} + A_1 \frac{\alpha_{\overline{\text{MS}}}(m_\star)^{-8/9+n}}{M^2} \left(1 + A_2 \alpha_{\overline{\text{MS}}}(m_\star)^{7/11} + O[\alpha_{\overline{\text{MS}}}(m_\star)] \right)$$

- $A_{1,2}$ are constants and $\tilde{\omega}_{1,2}(\alpha) = k_{1,2}\alpha^n + O(\alpha^{n+1})$.
- Note that $\alpha_{\overline{\text{MS}}}^{(N_\text{f})}(m_\star) = \alpha_{\overline{\text{MS}}}^{(N_\text{f}=0)}(m_\star) + O(\alpha^3)$, so no distinction required.

Decoupling expansion of a^2 terms

- Expanding $\langle O_{\text{gf}} \underbrace{O_i(\mu)}_{\in S_2} \rangle_{\text{cont}}$ proceeds in 2 steps:

  1. RG evolve from $\mu = 1/a$ to $\mu = m_\star$ using leading order γ_0, mixing matrix:

  $$\left([\alpha_{\overline{\text{MS}}}^{(N_{\text{f}})}(1/a)]/\alpha_{\overline{\text{MS}}}^{(N_{\text{f}})}(m_\star)]\right)^{\gamma_0}$$

  2. At scale $\mu = m_\star$ match to effective $N_{\text{f}} = 0$ theory:
     1. effective field for observable: $[O_{\text{gf}} O_i(\mu)]_{\text{dec}} = \Phi_0 + \Phi_1/m_\star^2 + \ldots$;
        Φ_0: 3 possible operators: $O_{\text{gf}} \times B_{1,2}$ and $m_\star^2 \times O_{\text{gf}} \times B_0$.
        $\langle \Phi_0 \rangle_{\text{dec}} \to a^2 M^2,\ a^2$
     2. insertion $\langle \Phi_0 S_{\text{dec},2} \rangle_{\text{dec}} \to a^2 M^2 / M^2 = a^2$
     3. $\langle \Phi_1 \rangle_{\text{dec}}$: new terms $\propto a^2/M^2 \longrightarrow$ neglect!

- Continuum extrapolation <u>at fixed values</u> $m_{\star,i}$ (with constants c_i, $p_{1,2}$):

$$u_{M_i} = c_i + p_{1,i} \times a^2 M_i^2 [\alpha_{\overline{\text{MS}}}^{(N_{\text{f}})}(1/a)]^{\hat{\Gamma}_1} + p_{2,i} \times \frac{a^2}{L^2} [\alpha_{\overline{\text{MS}}}^{(N_{\text{f}})}(1/a)]^{\hat{\Gamma}_2} + \ldots$$

- Leading exponents $\hat{\Gamma}_{1,2}$ not yet known but likely in the range $[-1, 1]$ for the CLS action.

Boundary effects

SF boundary conditions $\Rightarrow S_{\text{dec}}$ gets additional boundary term at order $1/m$:

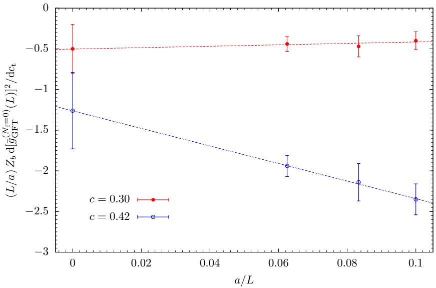
$$S_{\text{dec},1} = \omega_b \int d^3\mathbf{x}\,\text{tr}\,\{F_{0k}(0,\mathbf{x})F_{0k}(0,\mathbf{x}) + F_{0k}(T,\mathbf{x})F_{0k}(T,\mathbf{x})\}$$

Define & estimate contribution to $O_{\text{gf}} =$ GFT coupling

$$\begin{aligned}\Delta(z) &= \langle O_{\text{gf}}\rangle - \langle O_{\text{gf}}\rangle_{\text{dec}} = -\frac{1}{m_\star}\langle O_{\text{gf}} S_{1,\text{dec}}\rangle_{\text{dec}} + \ldots, \qquad z = ML_{\text{dec}} \\ &= \frac{\omega_b}{z}\frac{M}{m_\star}\left(\lim_{a\to 0}\frac{L_{\text{dec}}}{a} Z_b \frac{d}{dc_{\text{t}}}\left[\bar{g}^{(N_{\text{f}}=0)}_{\text{GFT}}(L_{\text{dec}})\right]^2\right)\end{aligned}$$

- $\omega_b = -0.054 \times N_{\text{f}} \times \alpha_{\overline{\text{MS}}}(m_\star) + \ldots$ from SF coupling (S.& Sommer '95):
- insertion of bare lattice Hamiltonian measurable as c_{t}-derivative, use 1-loop renormalization Z_b.
- Using $z = 4$ and $z = 6$ and flowtime parameters $c = 0.3, 0.42$

$\Rightarrow$ $1/M$ "contamination" in GFT coupling u_M is reasonably small compared to statistical error $\longrightarrow$ negligible!



Conclusions

- Theoretical framework of decoupling for gradient flow observables well-developed, based on effective Symanzik and decoupling theories.
- PT only used at the scales $\mu = m_\star$ (effective decoupling theory) and $\mu = 1/a$ (Symanzik effective theory)
- With non-perturbatively O(a) improved Wilson quarks and O(a^2) improved gradient flow (Zeuthen flow) we find
    - No odd terms in M in the bulk and up to O(a^2); Leading correction terms:
        - $\propto a^2M^2$, a^2 in the continuum limit;
        - $\propto 1/M^2$ in the decoupling limit.
    - The decoupling limit (like the continuum limit) receives logarithmic correction to $1/M^2$, which take the form $\alpha(m_\star)^{\hat{\Gamma}}/M^2$ with fractional exponents $\hat{\Gamma}$.
- SF b.c's: contaminations by O($1/M$) boundary terms below statistical error.
- Given $u_M = u_\infty + \Delta u$ with $\Delta u = [\alpha(m_\star)]^{-8/9+n}/M^2 + ...$, translate to Λ-parameter:

$$\varphi_s^{(0)}(\sqrt{u_M}) = \varphi_s^{(0)}(\sqrt{u_\infty}) + \Delta_u \times \frac{\varphi_s^{(0)}(\sqrt{u_\infty})}{-2\sqrt{u_\infty}\beta_s(\sqrt{u_\infty})} + \ldots$$

($\rightarrow$ talk by Roman Höllwieser)

Thank you!