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# Renormalisation constants of quark bilinear operators in QCD with dynamical $u, d, s$ and $c$ quarks

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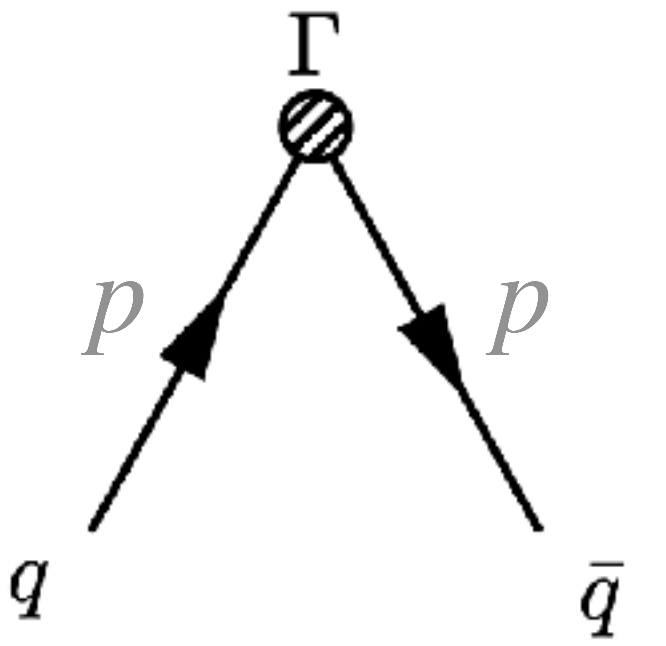
# Renormalisation in the RI'-MOM scheme

## Numerical procedure:

- **Building irreducible vertices** from propagators
- **Reduction of cut-off effects**
  - selection of "democratic momenta"
  - subtraction of $O(g^2 a^\infty)$ effects **[new]**
- **Chiral extrapolation** of vertices
- **Perturbative evolution** to a common scale
- **Extraction** of the RCs through appropriate **fit procedure**

Two independent **RI'-MOM analyses** currently ongoing with complementary lattice setups and procedures




$$O_\Gamma = \bar{q}_1 \, \Gamma \, q_2$$

$$\Gamma \;=\; \mathbf{1}, \; \gamma_5, \; \gamma_\mu, \; \gamma_\mu\gamma_5, \; \sigma_{\mu\nu}$$

$S \quad P \quad V \quad A \quad T$

$$\langle p | O_\Gamma^R | p \rangle \big|_{p^2=\mu^2} = \langle p | O_\Gamma | p \rangle_{\text{tree}}$$

$$Z_q^{-1} \, Z_O \;\; \text{Tr}[\Lambda_O P_O]\big|_{p^2=\mu^2} = 1$$

with

$$Z_q = \begin{cases} -\frac{i}{4} \, \text{Tr}\left[\frac{\partial S^{-1}(p)}{\partial \not{p}}\right]_{p^2=\mu^2} & \textbf{RI} \\ -\frac{i}{4} \, \text{Tr}\left[\frac{\not{p} S^{-1}(p)}{p^2}\right]_{p^2=\mu^2} & \textbf{RI'} \end{cases}$$

G. Martinelli *et al.* - NPB 445 (1995) 81

# Hadronic contaminations in RI-MOM RC estimators

Let us study the **large** $p^2$ behaviour of the 2-fermion Green function ($p^2 \gg \Lambda^2_{\mathrm{QCD}}$)

$$G_O(p^2) = \int \mathrm{d}^4x\, \mathrm{d}^4y\, e^{-ip(x-y)} \langle q_1(x)\, O^{\Gamma}_{12}(0)\, \bar{q}_2(y)\rangle$$

$$\sim S_q(p)\, \Lambda_O(p^2)\, S_q(p) + \sum_H w^{\Gamma}_H \frac{\Lambda^4_{\mathrm{QCD}}}{(p^2)^2\, m^2_H}$$

contr. from **quark** states
$|x| \sim 1/|p|$ **OR** $|y| \sim 1/|p|$

contr. from **hadronic** states
$|x-y| \sim 1/|p|$
$|x+y| \sim$ large

$$O^{\Gamma}_{12}(0) = \bar{q}_1(0)\Gamma q_2(0)$$

$$w^{\Gamma}_H \sim \frac{\langle 0|O^{\Gamma}_{21}|H\rangle\langle H|O^{\Gamma}_{12}|0\rangle}{\Lambda^4_{\mathrm{QCD}}}$$ (dimensionless)

$S_q(p) \sim 1/\not{p}$ quark propagator

**Hadronic contaminations** in $\Lambda_O$ are suppressed as $\mathcal{O}(\Lambda^2_{\mathrm{QCD}}/p^2)$ but have **different chiral behaviour**:

$\Gamma = \gamma_5$

$$\frac{1}{p^2}\frac{\Lambda^4_{\mathrm{QCD}}}{m^2_\pi}$$

*Goldsone boson pole*

$\Gamma = \gamma_\mu\gamma_5$

$$\frac{1}{p^2}\frac{m^2_\pi}{\Lambda^2_{\mathrm{QCD}}}\frac{\Lambda^4_{\mathrm{QCD}}}{m^2_\pi} \to \frac{\Lambda^2_{\mathrm{QCD}}}{p^2}$$

$\Gamma \neq \{\gamma_5, \gamma_\mu\gamma_5\}$

$$\frac{\Lambda^2_{\mathrm{QCD}}}{p^2}$$

G. Martinelli *et al.* - NPB 445 (1995) 81
+ ETMC *(in preparation)*

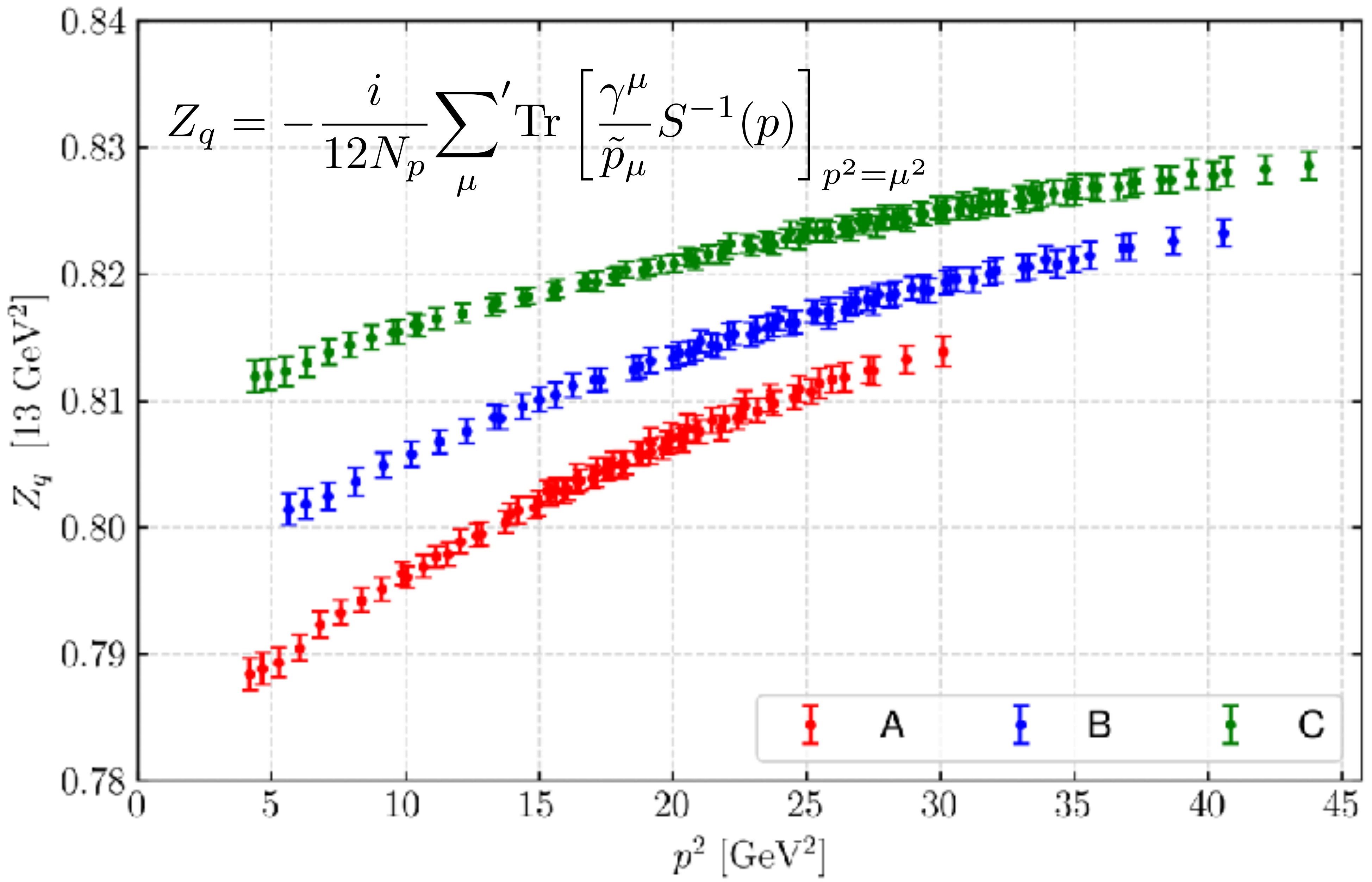
# Determination of $Z_q$

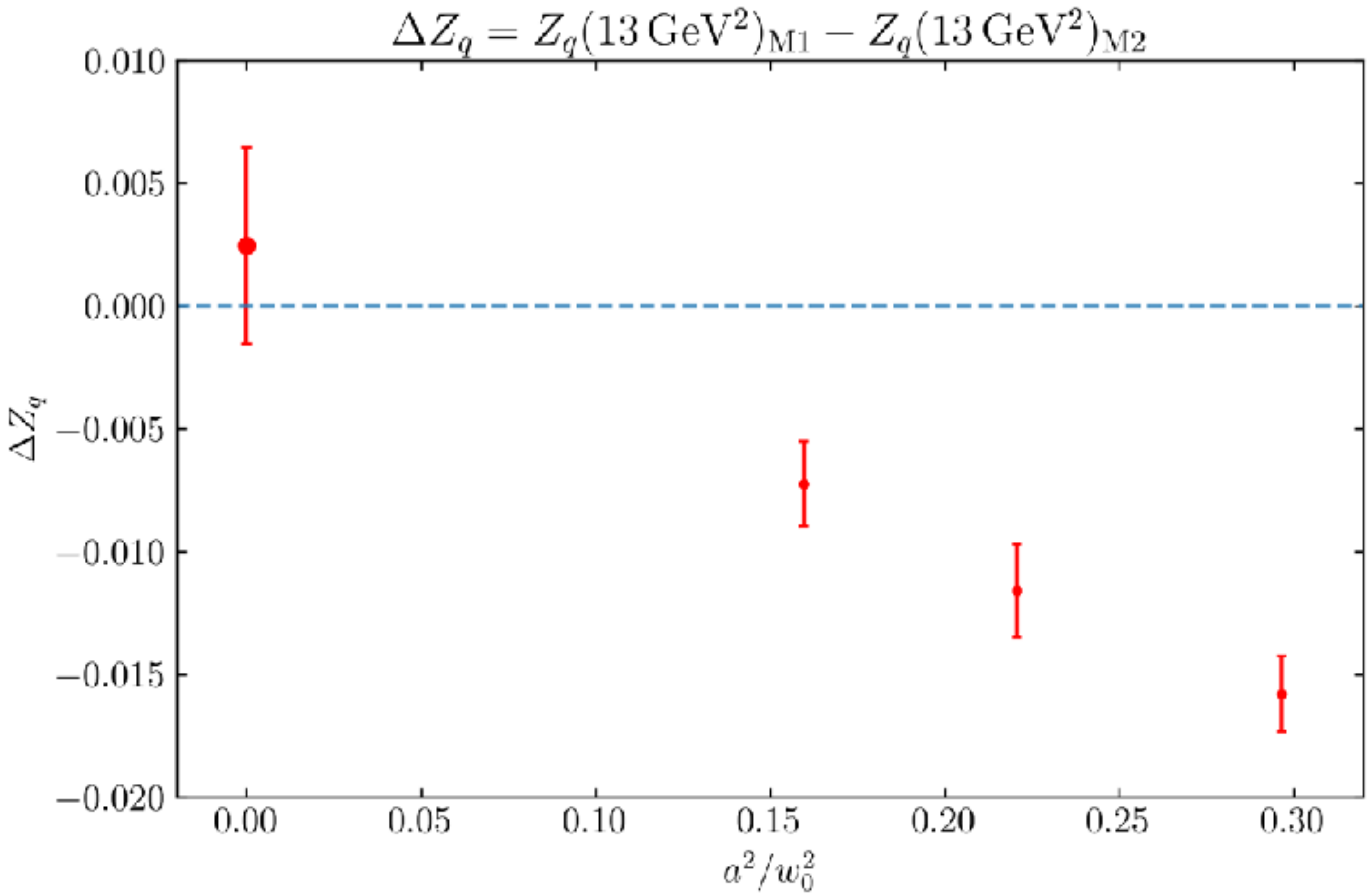
- **no** hadronic contaminations & **mass dependence**: extremely mild (⇒ constant fit)
- clean extraction using the following **fit methods**:

$$\zeta_q(p^2_{\rm ref};\, a^2p^2) \;=\; Z_q(p^2_{\rm ref}) \;+\; d_2^{(q)}\,(a^2p^2)$$

**M1:** linear fit of $\zeta_O$ in $p^2 \in \mathscr{I}_{\rm M1}(p^2_{\rm ref})$

**M2:** constant fit of $\zeta_O$ in $p^2 \in \mathscr{I}_{\rm M2}(p^2_{\rm ref})$



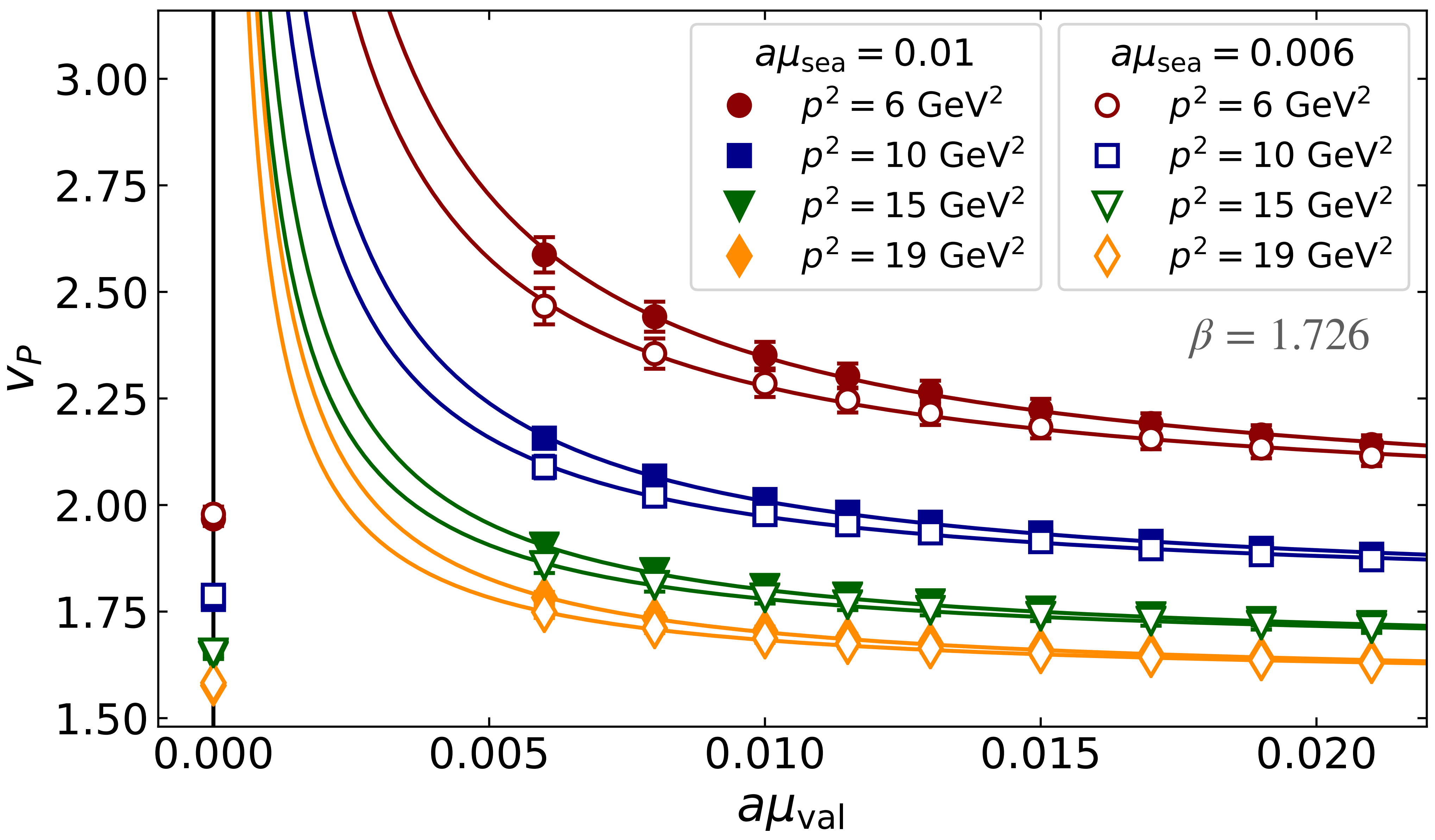
# Determination of $Z_P$

## Chiral extrapolation & the Goldsone pole

In our **partially quenched setup** $[m_\pi^2]_{\rm val} \sim \mu_{\rm val}$ (at LO in PQ-$\chi$PT)

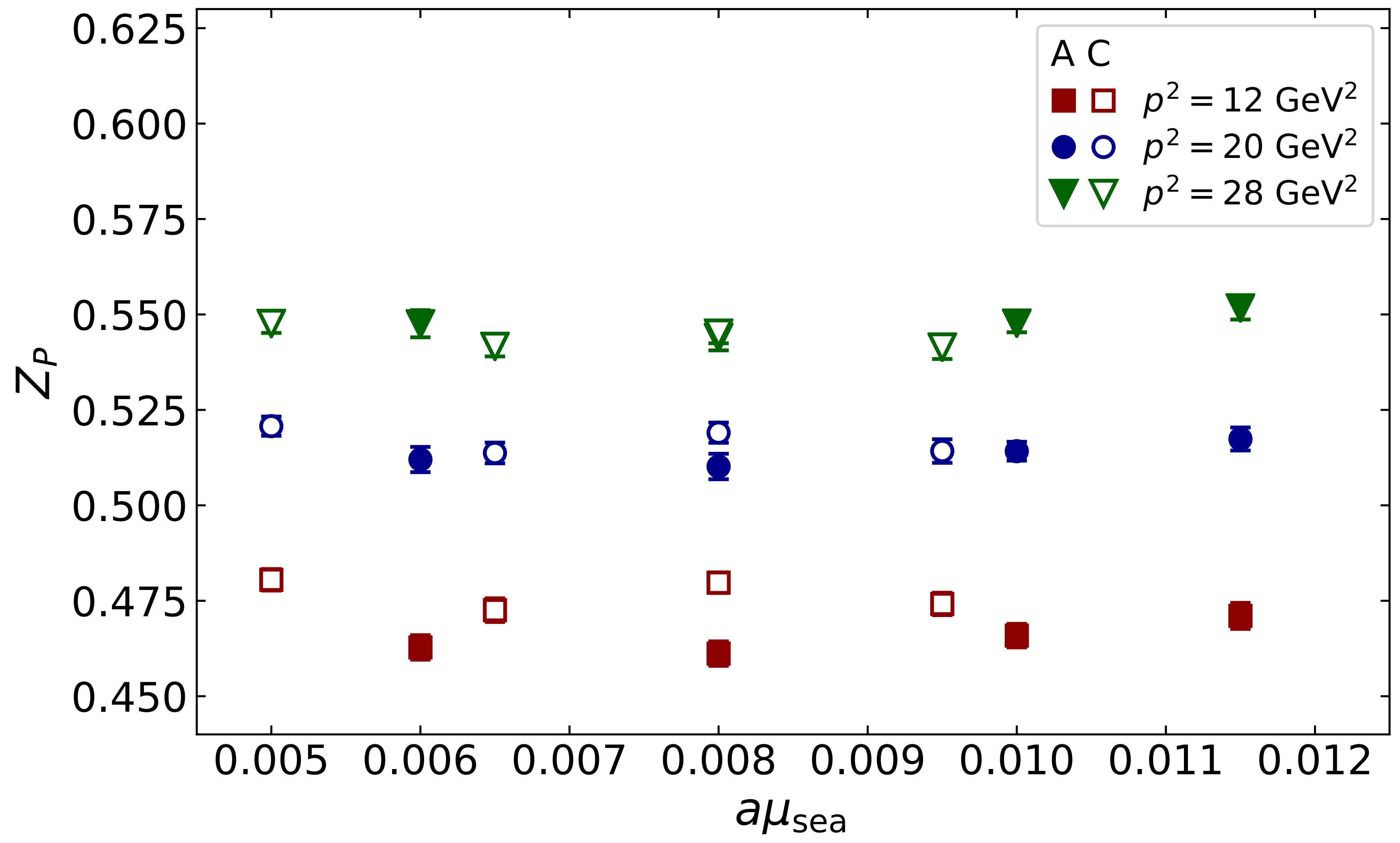
$$v_P(p^2, \mu_{\rm val}, \mu_{\rm sea}) = \mathcal{V}_P(p^2, \mu_{\rm sea}) + \frac{K'}{p^2} + \frac{K}{p^2}\frac{1}{\mu_{\rm val}} + \frac{K''}{p^2}\mu_{\rm val} + \dots$$

cannot be disentangled — pion pole — mostly $\mathcal{O}(a^2)$




C. Alexandrou *et al.* (ETM) - arXiv: 2104.13408 [hep-lat]

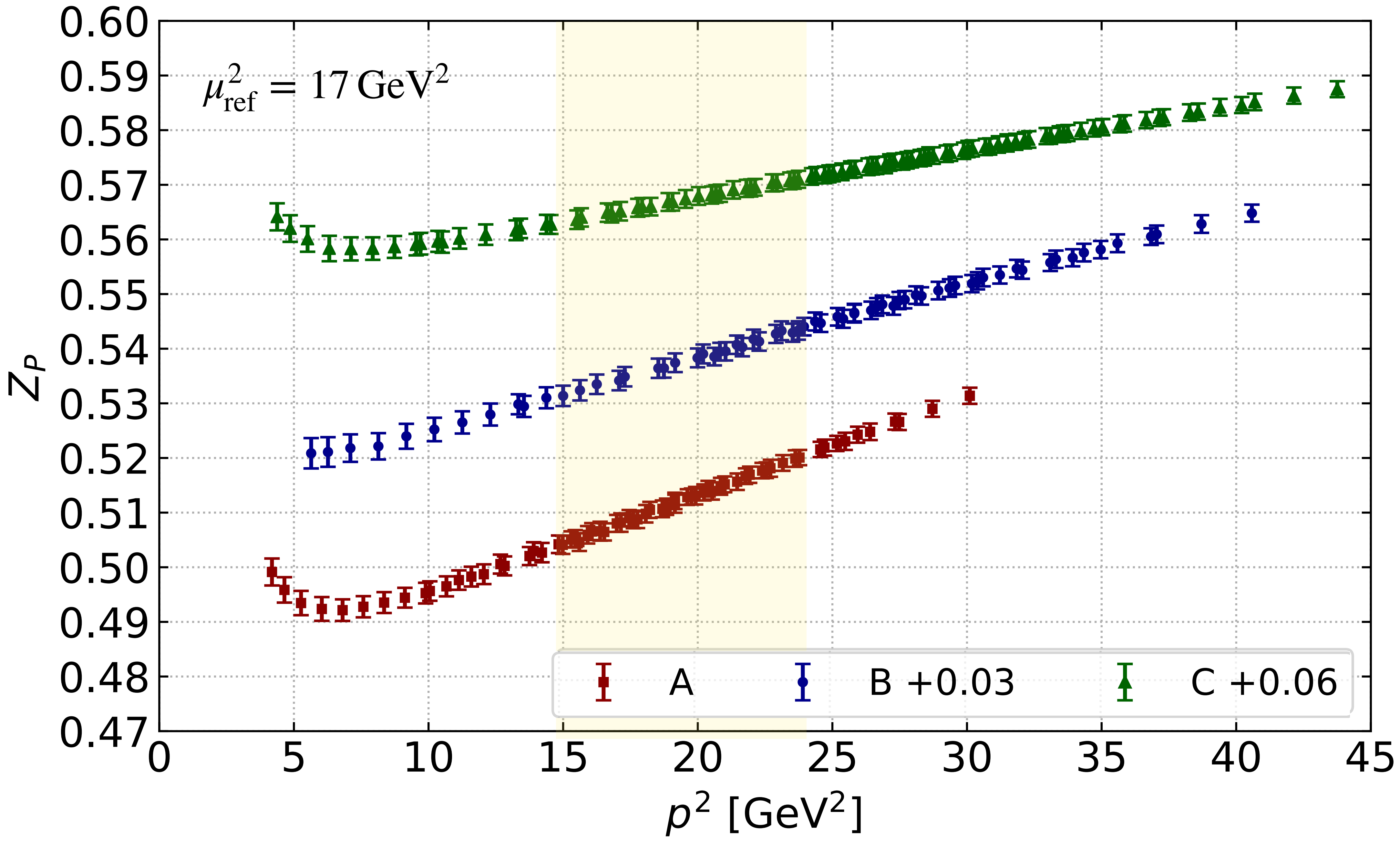
- **extrapolate** to $\mu_{\rm val} = 0$ **without** including $K''$ in the fit:
  $\Rightarrow$ absorb the $O(a^2)$ [+ tiny $O(a^0)$] into $Z_P$ $\leftrightarrow$ better statistical precision
- residual dependence on $\mu_{\rm sea}$ $\leftrightarrow$ statistical fluctuations
  $\Rightarrow$ **average** of $Z_P(\mu_{\rm sea}, p^2)$ over $\mu_{\rm sea}$
- get estimate of $Z_P(p^2)$ in the (full) chiral limit



# Determination of $Z_P$

## After **chiral extrapolation & evolution**

Residual dependence on $p^2$ due to **lattice artefacts** & possibly also to $O(a^0)$ residual **hadronic contaminations** $\sim 1/p^2$




**Safety checks against hadronic contaminations:**

- **Fitting** the pole

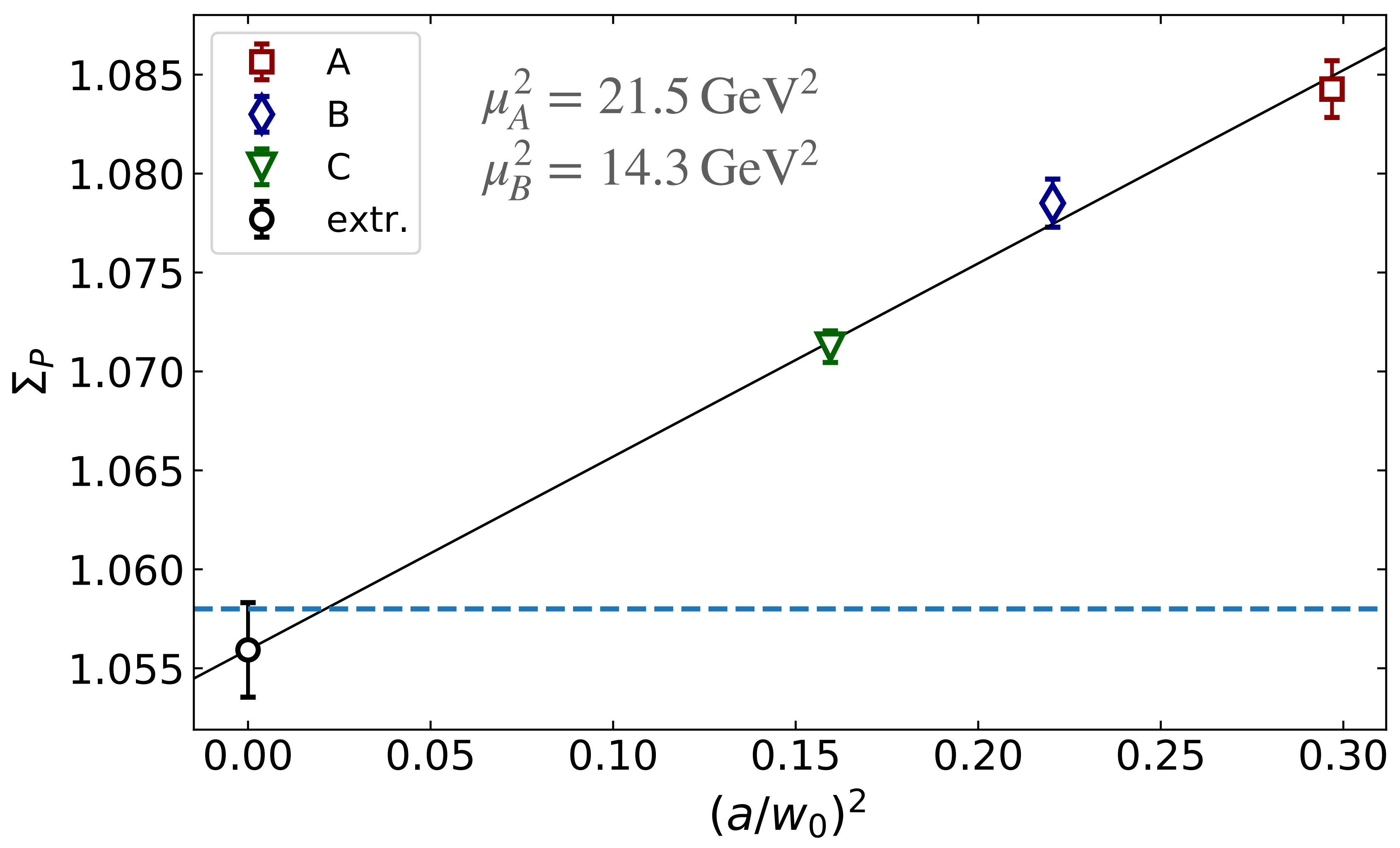
$$Z_P(\mu^2_{\rm ref}) = z_0 + z_1 p^2 + \frac{z_{-1}}{p^2}$$

| $\beta$ | $z_{-1}$ |
|---|---|
| 1.726 | -0.011 (43) |
| 1.778 | 0.033 (42) |
| 1.836 | -0.021 (30) |

$\Rightarrow$ corresponding parameter $z_{-1}$ compatible with **zero**

- **Step scaling function** $\Sigma_P(\mu_A^2, \mu_B^2) = Z_P(\mu_A^2)/Z_P(\mu_B^2)$

$\Rightarrow$ small $O(a^2)$ and **agreement** with continuum PT ($N^3LO$)



# Determination of $Z_P$

## Final results

**RI'-MOM results:**

| RI'-MOM | | | | |
|---|---|---|---|---|
| $\beta$ | M1a | M2a | M1b | M2b |
| 1.726 | 0.4774(24) | 0.5079(24) | 0.4917(26) | 0.5301(24) |
| 1.778 | 0.4812(32) | 0.5042(26) | 0.4944(27) | 0.5255(23) |
| 1.836 | 0.4899(26) | 0.5053(23) | 0.5046(27) | 0.5240(24) |

**@17 GeV$^2$** $p^2 \in (15, 19)$ GeV$^2$

**@21 GeV$^2$** $p^2 \in (18, 24)$ GeV$^2$

**Perturbative evolution intermediate scale:**

| RI'-MOM, $\mu^2_{\rm ref}$=19 GeV$^2$ | | | | |
|---|---|---|---|---|
| $\beta$ | M1a | M2a | M1b | M2b |
| 1.726 | 0.4849(24)(35) | 0.5159(24)(37) | 0.4851(26)(32) | 0.5229(24)(34) |
| 1.778 | 0.4888(33)(35) | 0.5121(26)(37) | 0.4877(27)(32) | 0.5184(23)(34) |
| 1.836 | 0.4976(26)(36) | 0.5133(23)(37) | 0.4978(27)(33) | 0.5169(24)(34) |

**Perturbative conversion to $\overline{\rm MS}$ scheme:**

| $\overline{\rm MS}$, $\mu^2_{\rm ref}$=19 GeV$^2$ | | | | |
|---|---|---|---|---|
| $\beta$ | M1a | M2a | M1b | M2b |
| 1.726 | 0.569(3)(5) | 0.605(3)(5) | 0.569(3)(5) | 0.614(3)(5) |
| 1.778 | 0.574(4)(5) | 0.601(3)(5) | 0.572(3)(5) | 0.608(3)(5) |
| 1.836 | 0.584(3)(5) | 0.602(3)(5) | 0.584(3)(5) | 0.607(3)(5) |

Detailed analysis & results reported in recent ETMC paper on **quark masses**:

**C. Alexandrou et al. (ETM) - arXiv: 2104.13408 [hep-lat]**

(see also **C. Alexandrou's talk** - Wed 28/7 - 13:00 local time)

**Final error budget:**

RI'-MOM error $\lesssim$ **0.5 %** + evolution & conversion to $\overline{\rm MS}$ $\boldsymbol{O(0.5\% - 0.9\%)}$

⇒ Systematic uncertainty on **perturbative formula** for evolution and matching to $\overline{\rm MS}$ dominant, but can be improved

⇒ Reduction in RI'-MOM error possible thanks to the **analysis discussed** and to the **subtraction of** $O(g^2a^\infty)$ **effects**

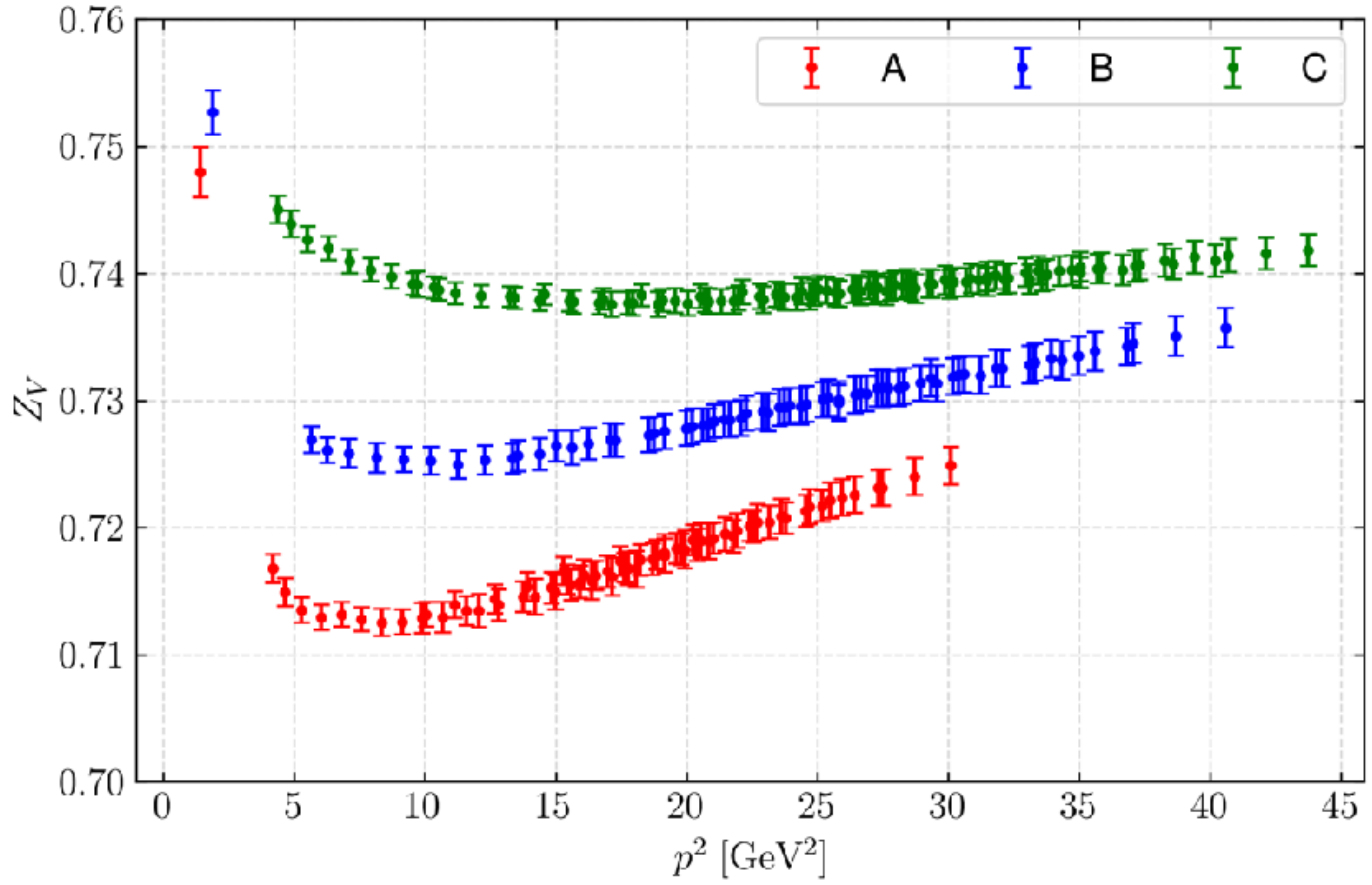
C. Alexandrou *et al.* (ETM) - PRD 95 (2017) 3

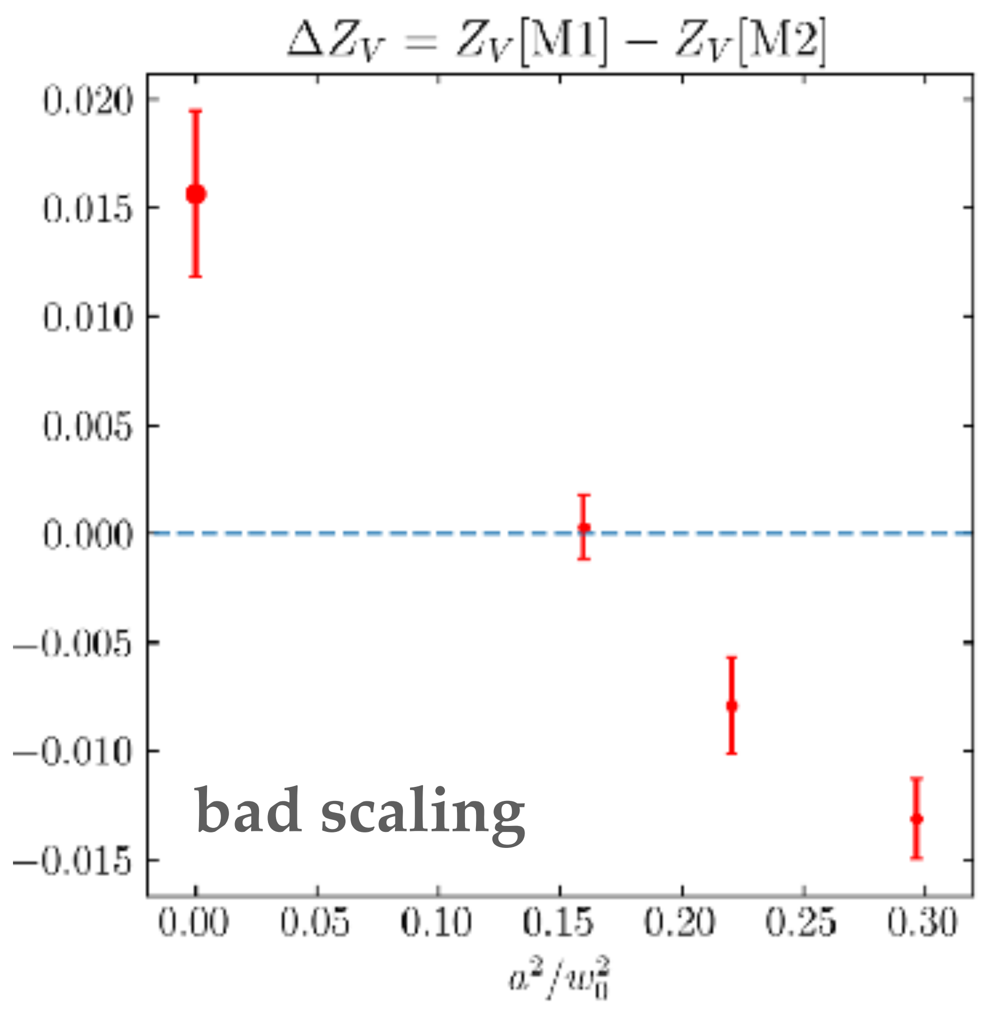
# Determination of $Z_V$

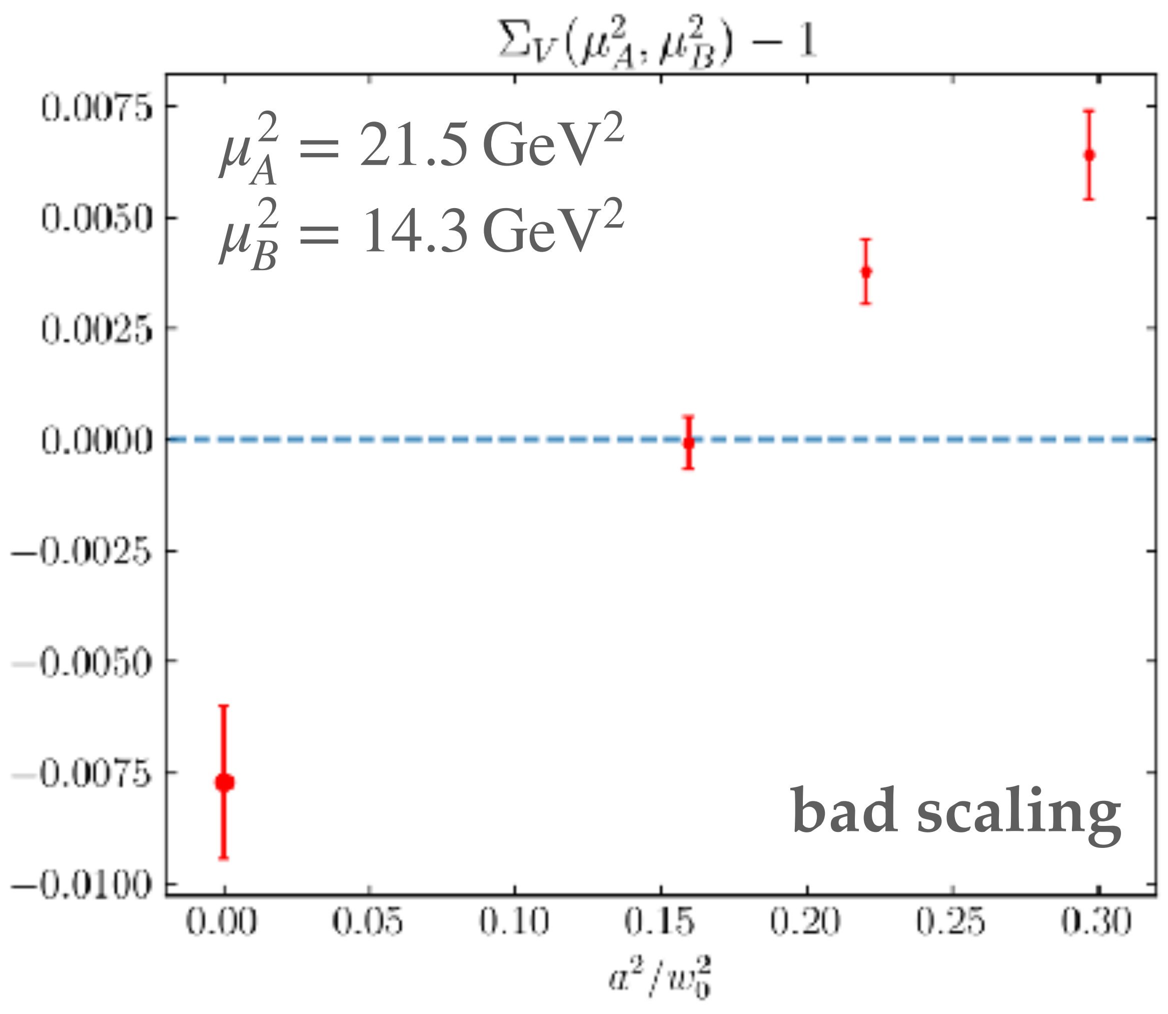
$$v_V(p^2,\mu_{\rm val},\mu_{\rm sea}) = \mathcal{V}_V(p^2,\mu_{\rm sea}) + \frac{C'}{p^2} + \frac{C''}{p^2}\mu_{\rm val} + \dots$$

- **Chiral extrapolation**: linear mass dependence tiny
  - ⇒ constant fit in $\mu_{\rm val}$ and $\mu_{\rm sea}$ improve stat. uncertainty
  - + tiny $O(a^0)$ mass dependence quoted as systematics ( $\lesssim 1‰$ )
- **Hadronic contaminations** $\sim 1/p^2$ are *finite* in the chiral limit and are **relevant** even at high $p^2$
  - ⇒ methods **M1** and **M2** not efficient ↔ bad scaling!
  - ⇒ need to fit directly the $\sim 1/p^2$ pole on a large range of $p^2$

$$\textbf{M3}: \quad \zeta_V(p^2) = Z_V + d_2^{(V)}(a^2p^2) + \frac{\epsilon_V}{p^2}$$



# Determination of $Z_V$

## Results & comparison with WTI

$$\textbf{M3}: \quad \zeta_V(p^2) = Z_V + d_2^{(V)}(a^2p^2) + \frac{\epsilon_V}{p^2}$$

Preliminary

| $\beta$ | $Z_V$ | $d_2^{(V)}$ | $\epsilon_V$ [GeV$^2$] |
|---|---|---|---|
| 1.726 | 0.6989 (15) (10) | 0.0038 (3) | 0.057 (6) |
| 1.778 | 0.7148 (22) (10) | 0.0032 (4) | 0.051 (11) |
| 1.836 | 0.7306 (13) (10) | 0.0018 (4) | 0.061 (6) |

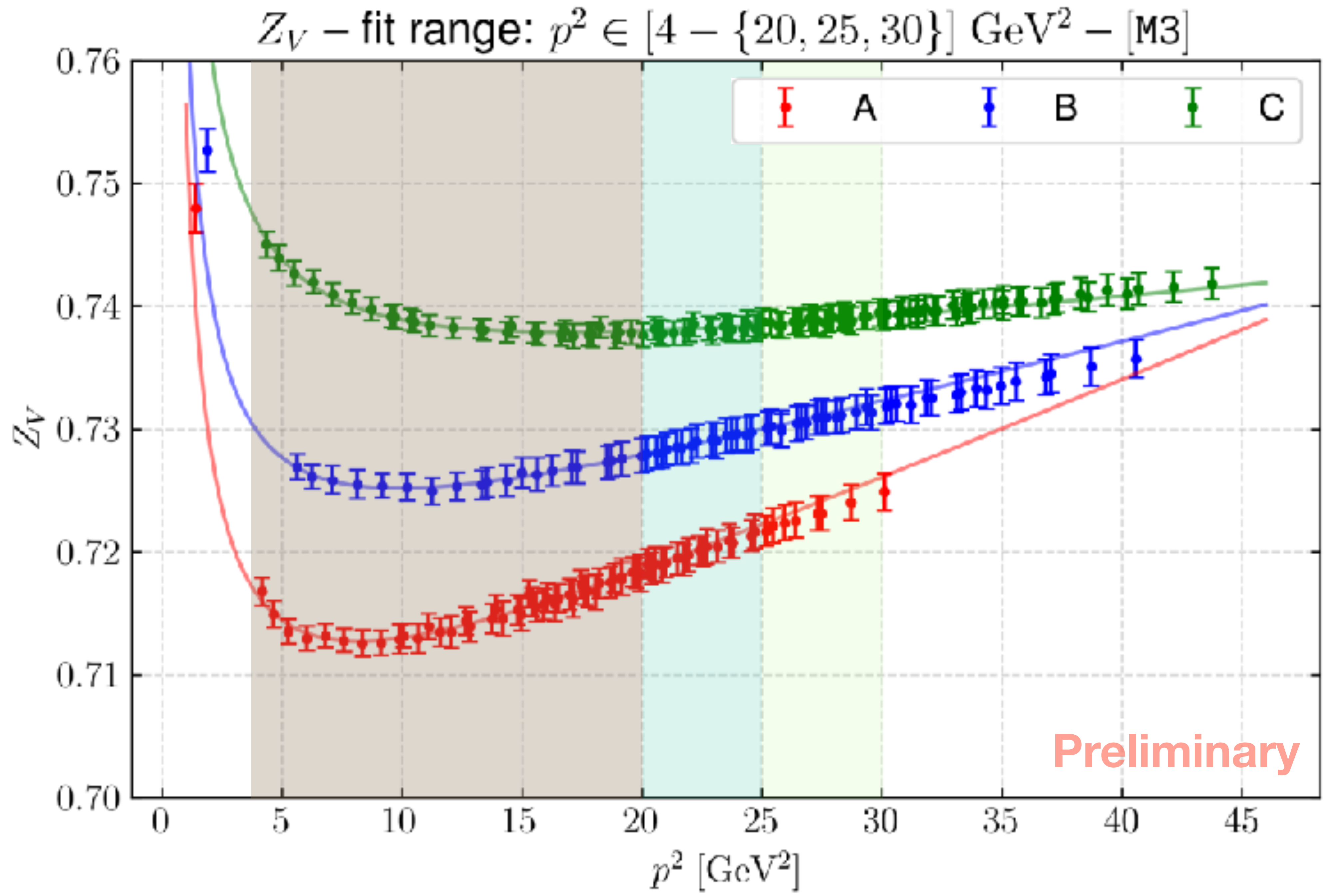
$Z_V$ can also be evaluated via the **PCAC Ward-Takahashi identity:**

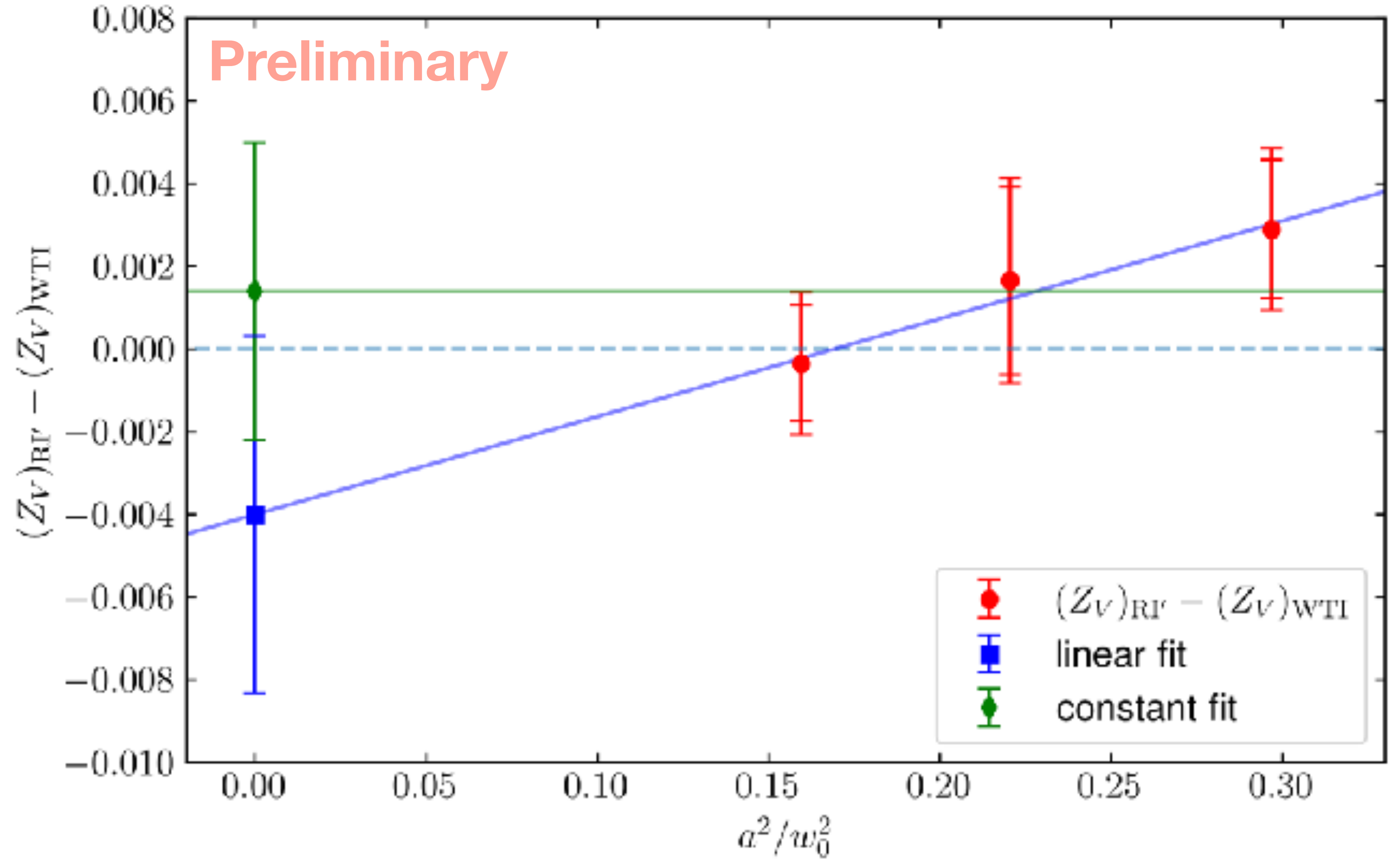
$$Z_V \sum_{\mathbf{x}} \langle \tilde{\partial}_0 V_0(t,\mathbf{x}) P_5^\dagger(0)\rangle^{(x)} = (\mu_1+\mu_2) \sum_{\mathbf{x}} \langle P_5(t,\mathbf{x}) P_5^\dagger(0)\rangle^{(x)}$$

$$(A_R)_\mu = Z_V\, A_\mu^{(q)} = -i\, Z_V\, V_\mu^{(x)}$$

Preliminary

| $\beta$ | $Z_V$ [WTI] |
|---|---|
| 1.726 | 0.6960 (7) |
| 1.778 | 0.7131 (6) |
| 1.836 | 0.7310 (5) |



# Determination of $Z_A/Z_V$
## & comparison with alternative methods

**M3 :** $\zeta_{A/V}(p^2) \simeq \frac{Z_A}{Z_V} + d_2^{(A/V)}(a^2p^2) + \frac{\epsilon_{A/V}}{p^2}$

$Z_A/Z_V$ can also be evaluated using different regularisations of the fermionic action:

$$\begin{cases} q_f = \exp(i\gamma_5 r_f \pi/4)\,\chi_f \\ \bar{q}_f = \bar{\chi}_f\, \exp(i\gamma_5 r_f \pi/4) \end{cases}$$

$$\langle\alpha|(O_\Gamma)_R|\beta\rangle = Z_{O_{\tilde{\Gamma}}}\,\langle\alpha|O_{\tilde{\Gamma}}|\beta\rangle^{\mathbf{tm}} + \mathcal{O}(a^2) = Z_{O_{\hat{\Gamma}}}\,\langle\alpha|O_{\hat{\Gamma}}|\beta\rangle^{\mathbf{OS}} + \mathcal{O}(a^2)$$

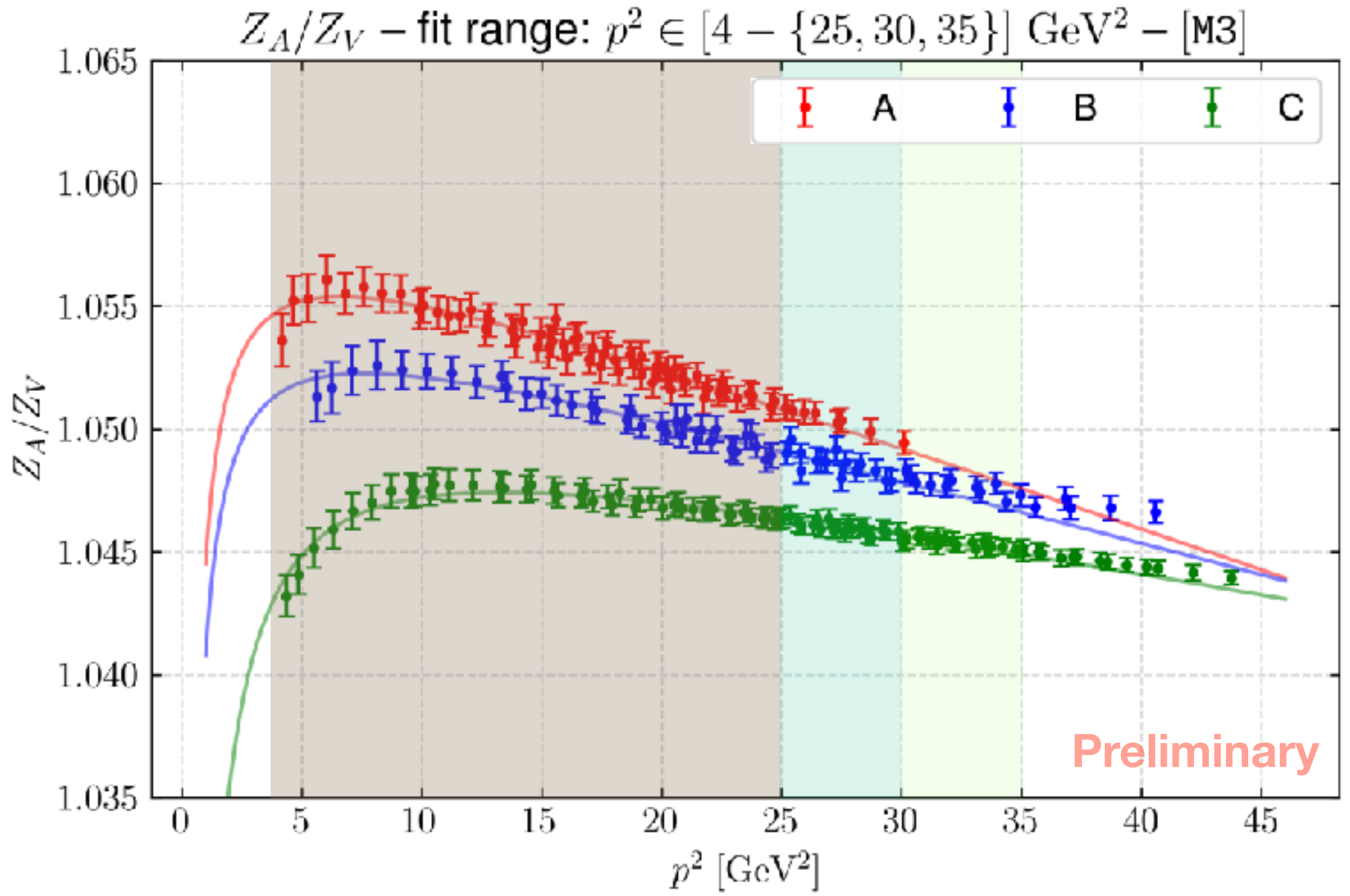
$r_u = -r_d = 1$ $\qquad$ $r_u = r_d = 1$

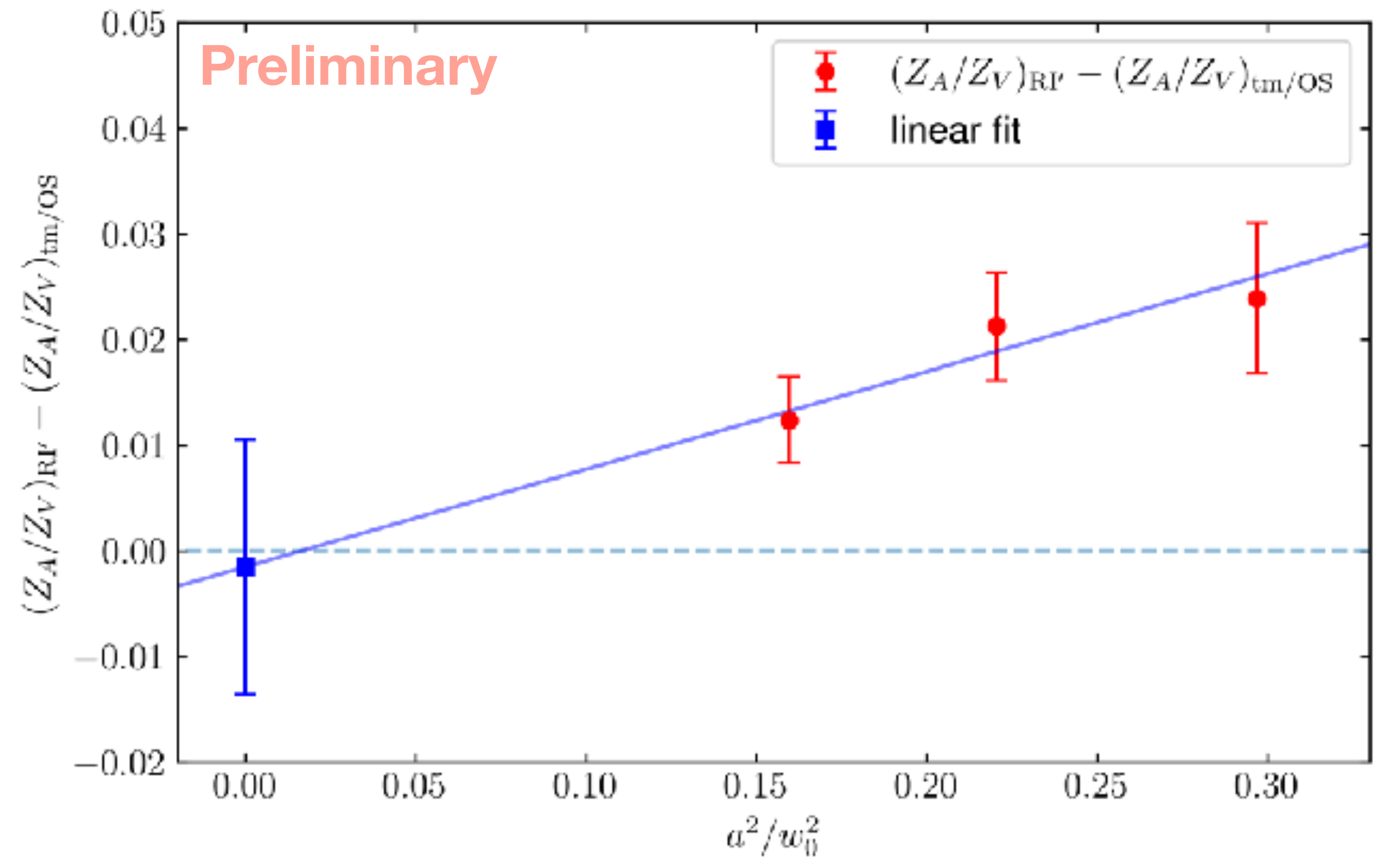
$$\langle 0|(A_0)_R|\pi\rangle \Rightarrow Z_V\,\langle\alpha|A_0|\beta\rangle^{\mathbf{tm}} = Z_A\,\langle\alpha|A_0|\beta\rangle^{\mathbf{OS}} + \mathcal{O}(a^2)$$

$A_\mu = \bar{q}_u \gamma_\mu \gamma_5 q_d$

Preliminary

| $\beta$ | $Z_A/Z_V$ [RI′] | $Z_A/Z_V$ [tm/OS] |
|---|---|---|
| 1.726 | 1.0599 (13) | 1.036 (7) |
| 1.778 | 1.0563 (11) | 1.035 (5) |
| 1.836 | 1.0524 (8) | 1.040 (4) |



# Indirect evaluation of $Z_A$ and $Z_S$

## Combining RI-MOM and alternative methods

We can extract $Z_A$ and $Z_S$ with higher precision by **combining** RI'-MOM results and the ones from alternative methods:

$$Z_A = \left(\frac{Z_A}{Z_V}\right)_{\mathrm{RI'}} \cdot (Z_V)_{\mathrm{WTI}}$$

$\boldsymbol{O}$**(0.1%)** dominated by $(Z_A/Z_V)_{\mathrm{RI'}}$

can be reduced using tm/OS method
& increasing statistics

$$Z_S = \left(\frac{Z_S}{Z_P}\right)_{\mathrm{tm/OS}} \cdot (Z_P)_{\mathrm{RI'}}$$

$\boldsymbol{O}$**(1%)** dominated by $(Z_P)_{\mathrm{RI'}}$

can be reduced improving
perturbative results

# ... and $Z_T$ ?

**In progress:** accuracy to be established, depending on hadronic contaminations and PT-evolution known only at 2-loops

expected to be $\boldsymbol{O}$**(1%)**

# Conclusions

- Reaching **few per mille precision** on RCs in the RI'-MOM scheme is possible!

- **Subtraction of $O(g^2a^\infty)$** artefacts & careful **control of hadronic contaminations** (less suppressed wrt RI-SMOM: *but please check there too!*) is crucial

- Combination with **alternative methods** for RGI RCs ($Z_V$, $Z_A/Z_V$, $Z_S/Z_P$) allows to reach even higher precision (using WTI & pushing correlator statistics - no hadron contam.)

- **Two independent RI'-MOM analyses** in the ETMC to better control systematics + cross-checks of raw data & intermediate steps *[publication in preparation]*

## Thanks for the attention!

& thanks to all the ETM collaborators

in particular, for the material presented here:
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