Non-Perturbative Renormalisation with Interpolating Momentum schemes

Nicolas Garron

Liverpool Hope University, University of Liverpool

30^{th} July 2021

In collaboration with

- Caroline Cahill, Martin Gorbahn, John Gracey, Paul Rakow (Liverpool)
- Thanks to RBC-UKQCD collaborations, in particular Tobias Tsang (Southern Danemark)
- Thanks to Holger Perlt (Leipzig) and QCDSF
- Thanks to CalLat (Henry Jose Monge Camacho, Amy Nicholson, André Walker-Loud)

The RBC & UKQCD collaborations

UC Berkeley/LBNL
Aaron Meyer

BNL and BNL/RBRC
Yasumichi Aoki (KEK)
Peter Boyle (Edinburgh)
Taku Izubuchi
Yong-Chull Jang
Chulwoo Jung
Christopher Kelly
Meifeng Lin
Hiroshi Ohki
Shigemi Ohta (KEK)
Amarjit Soni

CERN
Andreas Jüttner (Southampton)

Columbia University
Norman Christ
Duo Guo
Yikai Huo
Yong-Chull Jang
Joseph Karpie
Bob Mawhinney
Ahmed Sheta
Bigeng Wang
Tianle Wang
Yidi Zhao

University of Connecticut
Tom Blum
Luchang Jin (RBRC)
Michael Riberdy
Masaaki Tomii

Edinburgh University
Matteo Di Carlo
Luigi Del Debbio
Felix Erben
Vera Gülpers
Tim Harris
Raoul Hodgson
Nelson Lachini
Michael Marshall
Fionn Ó hÓgáin
Antonin Portelli
James Richings
Azusa Yamaguchi
Andrew Z.N. Yong

KEK
Julien Frison

Liverpool Hope / UoL
Nicolas Garron

Michigan State University
Dan Hoying

Milano Bicocca
Mattia Bruno

Peking University
Xu Feng

University of Regensburg
Davide Giusti
Christoph Lehner (BNL)

University of Siegen
Matthew Black
Oliver Witzel

University of Southampton
Nils Asmussen
Alessandro Barone
Jonathan Flynn
Ryan Hill
Rajnandini Mukherjee
Chris Sachrajda

University of Southern Denmark
Tobias Tsang

Stony Brook University
Jun-Sik Yoo
Sergey Syritsyn (RBRC)

Non Perturbative Renormalisation (NPR)

Reminders - general strategy

1. First step: remove the divergences

   For a generic composite operator $Q^{bare}(a)$ which renormalises multiplicatively, determine the Z-factor such that

   $$Q^{scheme}(\mu, a) = Z^{scheme}(\mu, a) Q^{bare}(a)$$

   has well-defined continuum limit.

   This step can be done non-perturbatively.

2. Second step: match to phenomenology (e.g. $\overline{\mathrm{MS}}$), This step has to be done in (continuum) perturbation theory .

   $$Q_i^{scheme}(\mu, 0) \longrightarrow Q^{\overline{\mathrm{MS}}}(\mu) = (1 + r_1 \alpha_S(\mu) + r_2 \alpha_S^2(\mu) + \ldots) Q^{scheme}(\mu, 0)$$

Non-Perturbative Renormalisation (NPR)

There are two popular methods for the non-perturbative determination of the Z-factors

- Schrödinger Functional (SF)
- Rome-Southampton: RI/MOM, RI/MOM', RI/SMOM, RI/mSMOM, ...

Here I am talking about extensions of the latter.

The Rome Southampon method [Martinelli et al '95]

Original setup [Martinelli et al '95]

- Chiral extrapolation, scale given by some momentum (off-shell)
- Gauge fixed (Landau).

Prescription: one requires some amputated Green function(s) to be finite.

The Rome Southampon method [Martinelli et al '95]

Original setup [Martinelli et al '95]

- Chiral extrapolation, scale given by some momentum (off-shell)
- Gauge fixed (Landau).

Prescription: one requires some amputated Green function(s) to be finite.

Several extensions and improvement, most notably

- Non-exceptional kinematics (SMOM) [RBC, RBC-UKQCD, Sturm et al., Lehner and Sturm, Almeida and Sturm, Gorbahn and Jäger, Gracey, . . .]
- Momentum sources (QCDSF)
- Twisted boundary conditions, eg RBC-UKQCD, [Arthur and Boyle, 2010]
- Massive momentum scheme [Boyle, Del Debbio and Khamseh, 2016]

Example: quark bilinear

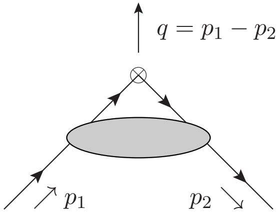
Consider a quark bilinear $O_\Gamma = \bar{\psi}_2 \Gamma \psi_1$, where $\Gamma = \mathbb{1}, \gamma_\mu, \sigma_{\mu\nu}, \gamma_\mu\gamma_5, \gamma_5$

Define

$$\Pi(x_2, x_1) = \langle \psi_2(x_2) O_\Gamma(0) \bar{\psi}_1(x_1) \rangle = \langle G_2(x_2, 0) \Gamma G_1(0, x_1) \rangle$$

In Fourier space $G(p) = \sum_x G(x, 0) e^{ip.x}$ and $G(-p) = \gamma_5 G(p)^\dagger \gamma_5$

$$V(p_2, p_1) = \langle G_2(-p_2) \Gamma G_1(p_1)^\dagger) \rangle$$



Example: quark bilinear

Consider a quark bilinear $O_\Gamma = \bar{\psi}_2 \Gamma \psi_1$, where $\Gamma = \mathbb{1}, \gamma_\mu, \sigma_{\mu\nu}, \gamma_\mu\gamma_5, \gamma_5$

Define

$$\Pi(x_2, x_1) = \langle \psi_2(x_2) O_\Gamma(0) \bar{\psi}_1(x_1) \rangle = \langle G_2(x_2, 0) \Gamma G_1(0, x_1) \rangle$$

In Fourier space $G(p) = \sum_x G(x, 0) e^{ip.x}$ and $G(-p) = \gamma_5 G(p)^\dagger \gamma_5$

$$V(p_2, p_1) = \langle G_2(-p_2) \Gamma G_1(p_1)^\dagger) \rangle$$

Amputated Green function

$$\Pi(p_2, p_1) = \langle G_2(p_2)^{-1} \rangle \langle G_2(p_2) \Gamma G_1(p_1)^\dagger) \rangle \langle (G_2(p_1)^{\dagger^{-1}}) \rangle$$

Example: quark bilinear

Consider a quark bilinear $O_\Gamma = \bar{\psi}_2 \Gamma \psi_1$, where $\Gamma = \mathbb{1}, \gamma_\mu, \sigma_{\mu\nu}, \gamma_\mu\gamma_5, \gamma_5$

Define

$$\Pi(x_2, x_1) = \langle \psi_2(x_2) O_\Gamma(0) \bar{\psi}_1(x_1) \rangle = \langle G_2(x_2, 0) \Gamma G_1(0, x_1) \rangle$$

In Fourier space $G(p) = \sum_x G(x,0) e^{ip.x}$ and $G(-p) = \gamma_5 G(p)^\dagger \gamma_5$

$$V(p_2, p_1) = \langle G_2(-p_2) \Gamma G_1(p_1)^\dagger) \rangle$$

Amputated Green function

$$\Pi(p_2, p_1) = \langle G_2(p_2)^{-1} \rangle \langle G_2(p_2) \Gamma G_1(p_1)^\dagger) \rangle \langle (G_2(p_1)^{\dagger^{-1}}) \rangle$$

Rome Southampton original scheme (RI/MOM), $p_1 = p_2 = p$ and $\mu = \sqrt{p^2}$

$$\frac{Z}{Z_q}(\mu, a) \times \lim_{m \to 0} \mathrm{Tr}(\Gamma\, \Pi(p,p))_{\mu^2 = p^2} = \mathrm{Tree}$$

SMOM and IMOM

More MOM schemes

Renormalisation scale is μ, given by the choice of kinematics

- Original RI-MOM scheme

$$p_1 = p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2$$

But this lead to "exceptional kinematics' and bad IR poles

More MOM schemes

Renormalisation scale is μ, given by the choice of kinematics

- Original RI-MOM scheme

$$p_1 = p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2$$

But this lead to “exceptional kinematics' and bad IR poles

- then RI-SMOM scheme

$$p_1 \neq p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2 = (p_1 - p_2)^2$$

Much better IR behaviour [Sturm et al., Lehner and Sturm, Gorbahn and Jäger, Gracey, . . .]

More MOM schemes

Renormalisation scale is μ, given by the choice of kinematics

- Original RI-MOM scheme

$$p_1 = p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2$$

But this lead to "exceptional kinematics' and bad IR poles

- then RI-SMOM scheme

$$p_1 \neq p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2 = (p_1 - p_2)^2$$

Much better IR behaviour [Sturm et al., Lehner and Sturm, Gorbahn and Jäger, Gracey, ...]

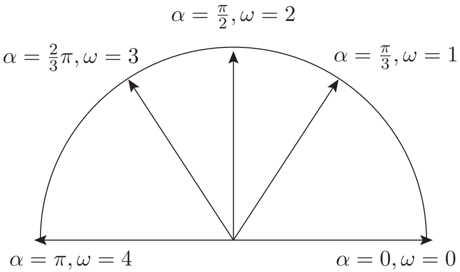
- We are now studying a generalisation (see also [Bell and Gracey])

$$p_1 \neq p_2 \text{ and } \mu^2 \equiv p_1^2 = p_2^2, \quad (p_1 - p_2)^2 = \omega\mu^2 \text{ where } \omega \in [0, 4]$$

Note that $\omega = 0 \leftrightarrow RI/MOM$ and $\omega = 1 \leftrightarrow RI/SMOM$

IMOM schemes

α is the angle between p_1 and p_2



$$\omega = 2(1 - \cos\alpha)$$

Implementation (1)

We want to achieve $p_1^2 = p_2^2 \equiv \mu^2\,, \quad q^2 = (p_1 - p_2)^2 = \omega\mu^2\;,$

Implementation (1)

We want to achieve $p_1^2 = p_2^2 \equiv \mu^2\,, \quad q^2 = (p_1 - p_2)^2 = \omega\mu^2\;,$

One possibility, for example [QCDSF'17]

$$p_1 = \frac{2\pi}{L}\,(m, m, m, m)\;, \quad p_2 = \frac{2\pi}{L}\,(-m, -m, -m, m)$$

$$\Rightarrow q = \frac{2\pi}{L}(2m, 2m, 2m, 0)$$

gives

$$\mu^2 = \left(\frac{2\pi}{L}\right)^2 4m^2\;, \text{and} \quad q^2 = 3\mu^2$$

Implementation (1)

We want to achieve $p_1^2 = p_2^2 \equiv \mu^2\,, \quad q^2 = (p_1 - p_2)^2 = \omega\mu^2\,,$

One possibility, for example [QCDSF'17]

$$p_1 = \frac{2\pi}{L}\,(m, m, m, m)\;, \quad p_2 = \frac{2\pi}{L}\,(-m, -m, -m, m)$$

$$\Rightarrow q = \frac{2\pi}{L}(2m, 2m, 2m, 0)$$

gives

$$\mu^2 = \left(\frac{2\pi}{L}\right)^2 4m^2\;,\text{and} \quad q^2 = 3\mu^2$$

The number of $-$ signs in p_2 gives the value of $\omega = 0, 1, \ldots, 4$.

Implementation (2)

Another possibility is to take advantage of twisted boundary conditions, say take

$$p_1 = \frac{2\pi}{L}\,(l,0,0,0) \qquad p_2 = \frac{2\pi}{L}\,(m,n,0,0)$$

$$\Rightarrow q = \frac{2\pi}{L}(l-m,-n,0,0)$$

And for each pair of desired (μ,ω), just need to solve

$$\begin{aligned} \mu &= 2\pi/L \\ l^2 &= m^2+n^2 \\ \omega l^2 &= (l-m)^2+n^2 \end{aligned}$$

Definitions

We call Λ_Γ the projected-amputated Green function, normalised by its tree value

For example $$\Lambda_S = \frac{1}{12}\mathrm{Tr}\left(\Pi_S\right) \; .$$

Definitions

We call Λ_Γ the projected-amputated Green function, normalised by its tree value

For example $$\Lambda_S = \frac{1}{12}\mathrm{Tr}\left(\Pi_S\right) \ .$$

We define $Z_m = 1/Z_S$ and compute the Z-factors for the scalar density

$$\left(\frac{Z_S}{Z_q}(\mu,\omega)\right)^{\mathrm{IMOM}} \times \left(\Lambda_S\right)_{q^2=\omega\mu^2} = 1$$

For Z_q we use the vector current

$$\left(\frac{Z_V}{Z_q}(\mu,\omega)\right)^{\mathrm{IMOM}-\gamma_\mu} \times \left(\Lambda_V^{(\gamma_\mu)}\right)_{q^2=\omega\mu^2} = 1$$

and

$$\left(\frac{Z_V}{Z_q}(\mu,\omega)\right)^{\mathrm{IMOM}-\not{q}} \times \left(\Lambda_V^{(\not{q})}\right)_{q^2=\omega\mu^2} = 1$$

Projectors

The difference between $\mathrm{IMOM} - \gamma_\mu$ and $\mathrm{IMOM} - \not{q}$ lies in the projector

$$\Lambda_V^{(\gamma_\mu)} = \frac{1}{48}\mathrm{Tr}\left(\gamma_\mu \Pi_{V^\mu}\right)$$

$$\Lambda_V^{(\not{q})} = \frac{q^\mu}{12q^2}\mathrm{Tr}\left(\not{q}\Pi_{V^\mu}\right)$$

Simulation

We use RBC-UKQCD ensembes, IW, 2+1 Domain-Wall fermions

We have two lattice spacings:

$$a^{-1} = 1.785(5)\ \text{GeV} \quad (24^3) \qquad (1)$$
$$a^{-1} = 2.383(9)\ \text{GeV} \quad (32^3), \qquad (2)$$

sea quark masses, $am = 0.005,\ 0.010,\ 0.020$ for the 24^3 x 64 x 16 lattice

and $am = 0.004,\ 0.006,\ 0.008$ for the 32^3 x 64 x 16 lattice.

We take the chiral limit on each lattice spacing using the values

$$am_{res} = 0.003152(43) \quad (24^3), \qquad (3)$$
$$am_{res} = 0.0006664(76) \quad (32^3). \qquad (4)$$

Our values for Z_V are

$$Z_V = Z_A = 0.71651(46) \quad (24^3), \qquad (5)$$
$$Z_V = Z_A = 0.74475(12) \quad (32^2). \qquad (6)$$

Results

Non-perturbative scale evolution (running), taking the continuum limit

$$\sigma(\mu,\omega,\mu_0,\omega_0) = \lim_{a^2\to 0} \frac{Z(\mu,\omega)}{Z(\mu_0,\omega_0)}$$

We have computed the perturbative prediction at NNLO, $U(\mu,\omega,\mu_0,\omega_0)$

In the next slides, I show some plots for the ratios

$$\frac{\sigma(\mu,\omega,\mu_0,\omega_0)}{U(\mu,\omega,\mu_0,\omega_0)}$$

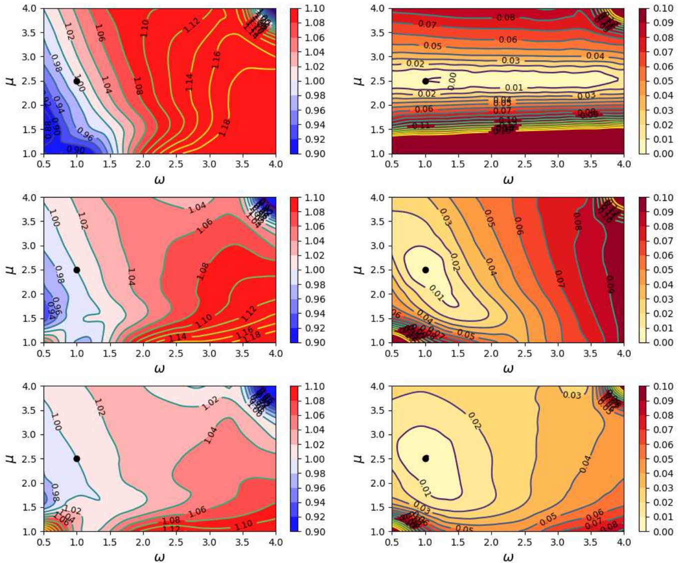
for fixed μ_0, ω_0 and various order in PT

Results for Z_m

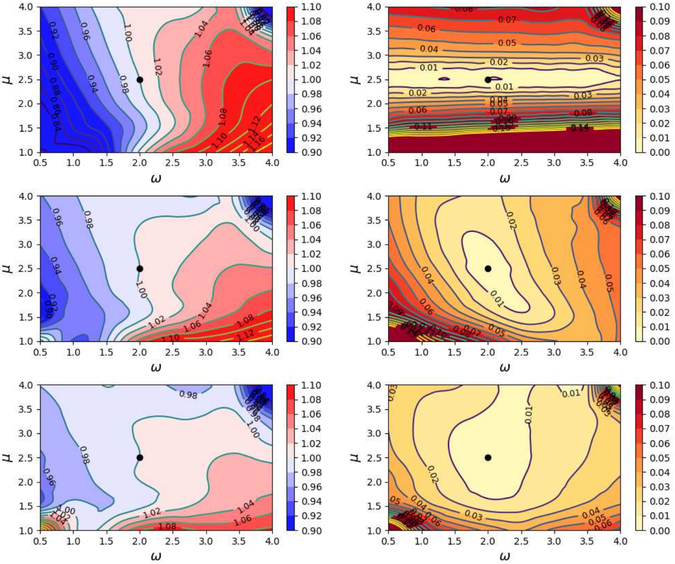
Running between $\mu_0 = 2$ GeV and $\mu = 3$ GeV, for various values of $\omega = \omega_0$

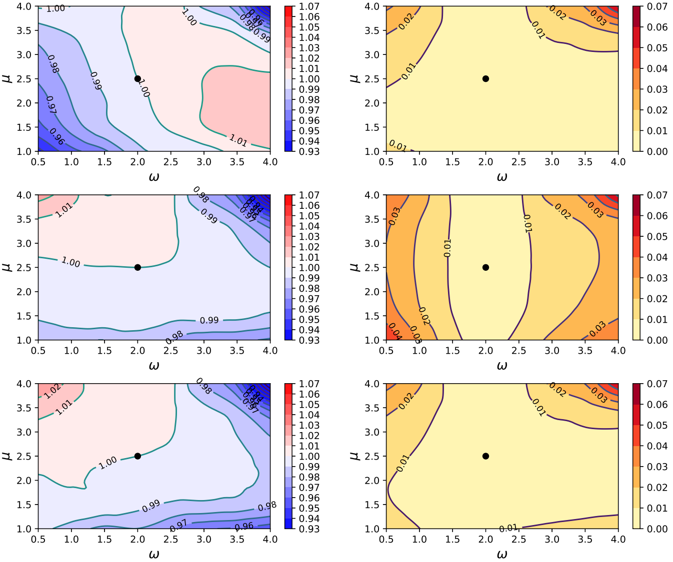
$(\mu, \mu_0) = (3.0, 2.0)$ $\omega = \omega_0 \downarrow$	$\sigma_\omega^{(\gamma_\mu)}$	σ_ω / LO	σ_ω / NLO	$\sigma_\omega / NNLO$
0.5	1.153(17)	1.062(15)	1.029(15)	1.014(15)
1.0	1.133(13)	1.044(12)	1.018(12)	1.010(12)
1.5	1.118(13)	1.030(12)	1.010(11)	1.006(11)
2.0	1.106(12)	1.019(11)	1.003(11)	1.002(11)
2.5	1.091(10)	1.005(9)	0.992(9)	0.995(9)
3.0	1.081(9)	0.996(9)	0.986(9)	0.991(9)
3.5	1.067(7)	0.983(6)	0.976(6)	0.982(6)
4.0	1.033(11)	0.951(10)	0.947(10)	0.955(10)

Results for Z_m



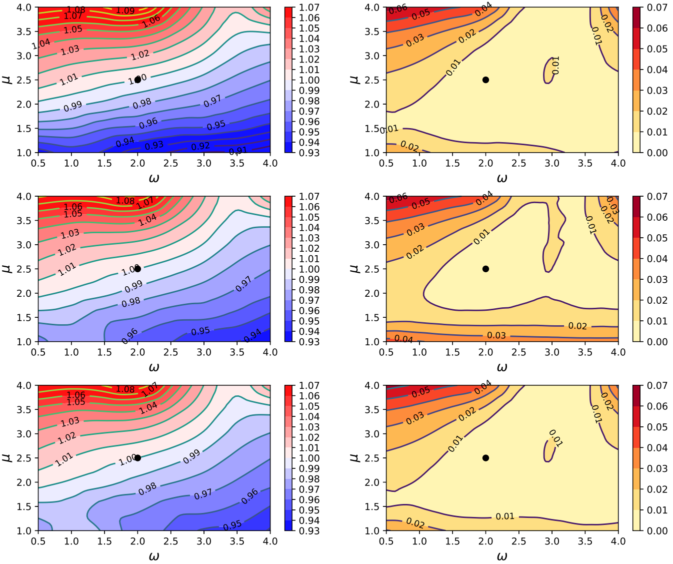
Results for Z_m





ω
μ

Results $Z_q^{(\not q)}$



Conclusions and outlook

- Proof of concept, first simulation of $\omega \neq 0, 1$
- Computation of Z_m, Z_q
- Perturbative matching factor to $\overline{\mathrm{MS}}$ at NNLO
- Non-perturbative and perturbative running (NNLO)
- Some work still to be done, chiral symmetry breaking effects
- Need a third lattice spacing
- Four-quark operators
- ...

Backup

Results for $Z_q^{(\gamma_\mu)}$

$\omega/\mu =$	1.0	1.5	2.5	3.0	3.5	4.0
0.5	0.972(8)	0.993(4)	1.008(4)	1.014(8)	1.023(14)	1.040(26)
1.0	0.976(8)	0.994(3)	1.004(2)	1.007(5)	1.012(8)	1.021(15)
1.5	0.978(4)	0.998(2)	1.003(1)	1.005(2)	1.006(3)	1.004(3)
2.0	0.990(7)	0.998(2)	1.003(0)	1.005(1)	1.007(1)	1.008(1)
2.5	0.987(5)	0.997(2)	1.001(1)	1.002(2)	1.002(3)	1.003(4)
3.0	0.985(4)	0.999(2)	1.000(2)	0.998(4)	0.993(9)	0.978(18)
3.5	0.989(5)	1.001(2)	0.997(2)	0.993(6)	0.982(13)	0.959(27)
4.0	0.990(5)	0.999(1)	0.994(3)	0.983(8)	0.957(22)	0.887(60)

Results for $Z_q^{(\not{q})}$

$\omega/\mu =$	1.0	1.5	2.5	3.0	3.5	4.0
0.5	0.935(27)	0.974(7)	1.018(7)	1.035(16)	1.059(31)	1.096(56)
1.0	0.951(19)	0.978(6)	1.017(6)	1.035(14)	1.058(28)	1.096(51)
1.5	0.950(15)	0.973(4)	1.017(4)	1.037(11)	1.064(25)	1.106(49)
2.0	0.942(15)	0.976(3)	1.020(3)	1.040(8)	1.069(20)	1.118(45)
2.5	0.942(14)	0.974(3)	1.019(1)	1.039(3)	1.060(9)	1.087(20)
3.0	0.937(12)	0.978(4)	1.017(2)	1.033(2)	1.048(2)	1.060(2)
3.5	0.943(10)	0.979(3)	1.014(2)	1.027(3)	1.039(3)	1.050(2)
4.0	0.940(10)	0.975(4)	1.013(3)	1.027(8)	1.046(18)	1.084(40)