Lattice $\mathcal{N} = 4$ super Yang-Mills at Strong Coupling

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Overview

1. Supersymmetry on Lattice
   - Brief Review of Lattice $\mathcal{N} = 4$ super Yang-Mills Theory
   - Details of the $\mathcal{Q}$ Invariant Construction
2. New Action
   - Phase structure of the new action
   - Supersymmetric Wilson Loops
3. Conclusions

Brief Review of Lattice $\mathcal{N} = 4$ super Yang-Mills Theory

- We start with a twisted $4d$ $\mathcal{N} = 4$ supersymmetric theory has 16 supercharges. On lattice we can only preserve one of them exactly.
- The other 15 are broken by lattice artifacts and recovered only in the continuum limit.
- Bosons and fermions treated symmetrically meaning that they both live on links as required by the exact susy and lattice gauge invariance.

Brief Review of Lattice $\mathcal{N} = 4$ super Yang-Mills Theory

- A_4^* lattice is used as the underlying lattice structure which packs the 4 gauge fields and 6 scalars into 5 complex bosons each associated with one of the basis vectors of the lattice. They are also valued in the adjoint representation of the algebra not in the group.
- All fields transform under the twisted rotation group

$$\mathrm{diag}(SO(4)_L \times SO(4)_R) \tag{1}$$

Where L denotes the Lorentz Symmetry and R the R-Symmetry

$\mathcal{Q}$ Invariant Construction

Let's start with the supersymmetric lattice action

$$S = \frac{N}{4\lambda} \mathcal{Q} \sum_{x} \mathrm{Tr} \left(\chi_{ab} \mathcal{F}_{ab} + \eta \overline{\mathcal{D}}_a \mathcal{U}_a + \frac{1}{2} \eta d \right) + S_{\text{closed}} \tag{2}$$

- The second term in the action S_{closed} is given as.

$$S_{\text{closed}} = -\frac{N}{16\lambda} \sum_{x} \mathrm{Tr}\ \epsilon_{abcde} \chi_{ab} \overline{\mathcal{D}}_c \chi_{de} \tag{3}$$

$\mathcal{Q}$ Invariant Construction

Carrying out the $\mathcal{Q}$ variation and integrating out the auxiliary field d we obtain the supersymmetric lattice action $S = S_b + S_f$ where

$$S_b = \frac{N}{4\lambda} \sum_x \mathrm{Tr} \left(\mathcal{F}_{ab} \overline{\mathcal{F}}_{ab} + \frac{1}{2} \mathrm{Tr} \, (\overline{\mathcal{D}}_a \mathcal{U}_a)^2 \right) \tag{4}$$

and

$$S_f = -\frac{N}{4\lambda} \sum_x \left(\mathrm{Tr} \; \chi_{ab} \mathcal{D}_{[a} \psi_{b]} + \mathrm{Tr} \; \eta \overline{\mathcal{D}}_a \psi_a \right) \tag{5}$$

Fermionic part of this action is also known as Kähler-Dirac action.

- The continuum limit of this action corresponds to the Marcus or GL twist of $\mathcal{N} = 4$ Yang-Mills.

Regulating the flat directions

- There are flat directions that corresponds to the classical vacuum solutions of the bosonic action.
- To regulate these flat directions we add a term that gives a vacuum expectation value to the imaginary part of the trace mode of $\mathcal{U}$

$$S_{\rm mass} = \mu^2 \sum_x \mathrm{Tr}\ \left(\overline{\mathcal{U}}_a(x)\mathcal{U}_a(x) - I\right)^2 \tag{6}$$

- This term gives masses to the scalars.
- Lifts the degeneracy and provides a unique ground state.

Controlling the $U(1)$ modes

- Since each link is an element of the algebra $\mathfrak{gl}(N, C)$, this formulations naturally describes the gauge group $U(N) = SU(N) \times U(1)$.
- Even though the $U(1)$ gauge degrees of freedom decouple in the continuum limit they introduce lattice artifacts at strong coupling and need to be suppressed to access strong coupling regimes.

A Previous attempt to control this $U(1)$ modes include,

- A plaquette determinant term can go up to $\lambda \sim 10$
  arXiv: 1505.03135 by S. Catterall and D. Schaich

New Action

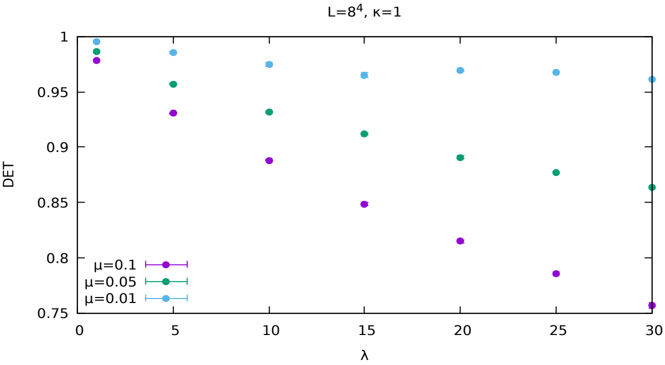
Now we include a new term to the action which drives the determinant of each individual gauge link to unity.

$$\frac{N}{4\lambda}\kappa\mathcal{Q}\sum_{x,a}\mathrm{Tr}\,(\eta)\left(\mathrm{Re}\det\left(U_a(x)-1\right)\right) \tag{7}$$

- This term breaks the $U(1)$ symmetry explicitly. But since $U(1)$ is a decoupled free theory in the continuum limit we are still preserving the $SU(N)$ gauge invariance.
- Most important result of this new term is that it allows simulations to access strong coupling regimes that was inaccessible with the original action up to arbitrarily large couplings.

Phase structure of the new action

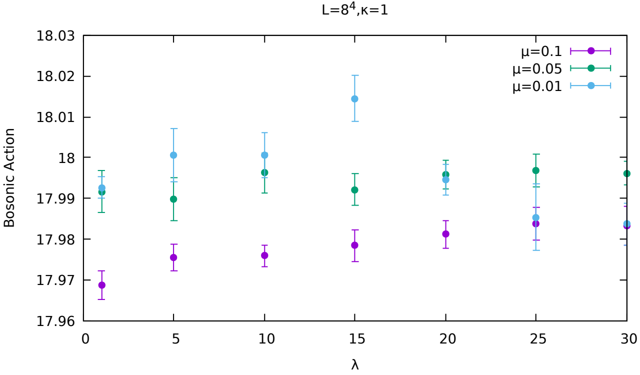
As a first test, We plot the Expectation value of the link determinant vs λ for 8^4 lattices at $\mu = 0.1, 0.05, 0.01$.



The expectation value is close to unity out to very large λ provided μ^2 is small enough confirming that we have effectively reduced the gauge fields to $SU(2)$.

Phase structure of the new action

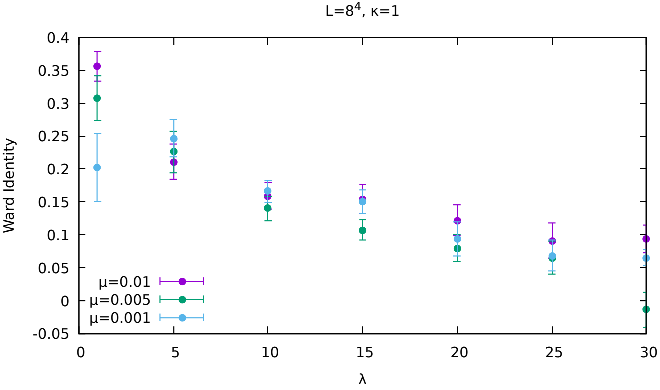
Next we look at the expectation value of the bosonic action $\frac{1}{V} < S_b >= \frac{9N^2}{2}$ for an N color theory on a system with (lattice) volume V *independent* of coupling λ.



- From these two plots we see that there is no phase transition as we vary λ as expected for a $\mathcal{N} = 4$ SYM.

Phase structure of the new action

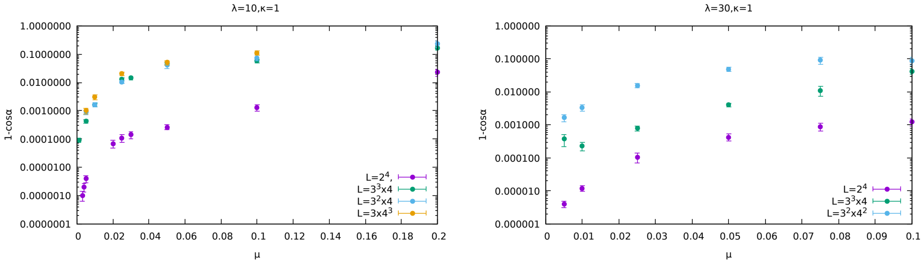
Finally we turn on to calculating a simple bilinear Ward identity given by $\left\langle \mathcal{Q}\mathrm{Tr}\left(\eta \mathcal{U}_a \overline{\mathcal{U}}_a\right)\right\rangle = 0$



Which is indeed small but not exactly 0 due to the thermal boundary condition.

Absence of a sign problem

- Writing the Pfaffian phase as $e^{i\alpha(\lambda,U)}$ we plot the quantity $1 - \cos\alpha$ as a function of μ.



- Pfaffian phase saturates as $L \to \infty$ and decreases as $\mu \to 0$.

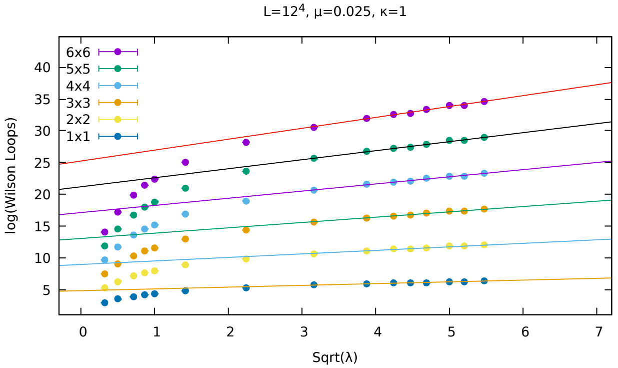
Supersymmetric Wilson Loops

We showed strong evidence that the lattice theory

- Exists in a single phase with unbroken supersymmetry out to very large coupling
- And it can be simulated with a Monte Carlo algorithm without encountering a sign problem.

We can turn on to confirming known results for $\mathcal{N} = 4$ Yang-Mills at strong coupling for Supersymmetric Wilson loops.

- Supersymmetric Wilson loops are generalization of regular Wilson loops by including contibutions from the scalars and are realized in the twisted construction by forming path ordered products of complexified lattice gauge fields $\mathcal{U}_a$
- The holographic prediction for the supersymmetric Wilson loops is that at strong coupling they depend on $W(R,T) = e^{(c\sqrt{\lambda}T/R)}$ not on λ as expected from perturbation theory.



Renormalized Supersymmetric Wilson Loops

Another interesting question is whether we can see evidence for a non-abelian Coulomb potential at small R.

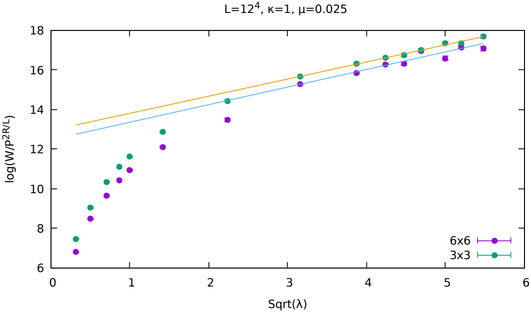
- To probe for this is we define renormalized Wilson loops by dividing the original Wilson loops by an appropriate power of the measured Polyakov line P.

Renormalized Wilson loop on a L^4 lattice:

$$W^R(R,R) = \frac{W(R,R)}{P^{\frac{2R}{L}}} \tag{8}$$

Renormalized Supersymmetric Wilson Loops

Here we show the renormalized supersymmetric Wilson loops for 12^4 lattices.



- Notice that the 3×3 and 6×6 loops now lie near to each other which is consistent with conformal invariance and the presence of a non-abelian Coulomb term while the strong coupling behavior still exhibits a dependence on $\sqrt{\lambda}$.

Conclusions

- The square root behavior at large λ is consistent with the result for circular Wilson loops in $\mathcal{N} = 4$ SYM derived by N. Drukker and D. Gross. arXiv:hep-th/0010274
- The strange $\sqrt{\lambda}$ dependence *cannot* be seen in perturbation theory and this preliminary result is a very non-trivial test of the correctness of the lattice approach in a non-perturbative regime.
- The current study has been limited to gauge group $SU(2)$. It is natural to inquire what occurs for this construction for other $SU(N)$.

Thanks for listening.

Loop Size	a$\sqrt{\lambda}$+b	Reduced-χ^2
4×4	$0.651978\sqrt{\lambda} + 8.04784$	8.1069
2×2	$0.590375\sqrt{\lambda} + 8.86867$	2.25436

Table: Normalized Supersymmetric Wilson loop fits on 8^4 lattice at $\mu = 0.025$ for $f(\lambda) = a\sqrt{\lambda} + b$

Loop Size	a$\sqrt{\lambda}$+b	Reduced-χ^2
6×6	$0.888503\sqrt{\lambda} + 12.4715$	6.5785
3×3	$0.86448\sqrt{\lambda} + 12.9472$	0.90153

Table: Normalized Supersymmetric Wilson loop fits on 12^4 lattice at $\mu = 0.025$ for $f(\lambda) = a\sqrt{\lambda} + b$

Renormalization of Wilson loops.

- Constant term in $V(R)$ receives a e^{-Cx2R} where the coefficient 2 comes from taking only the temporal like which mattter for the propagation.
- So we want to divide the loops by this coefficient. Polyakov loop itself goes like e^{-TC} so we need to divide by $(P^{(1/T)})^{2R}$ Letting $T = L$ we get $P^{2R/L}$

Volume dependence of Ward Identities

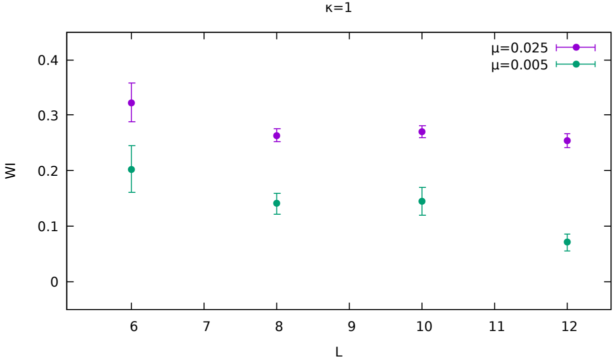


Figure: Ward Identity vs L at λ=10