Tensor network study of the 3d $O(2)$ model at nonzero chemical potential and temperature

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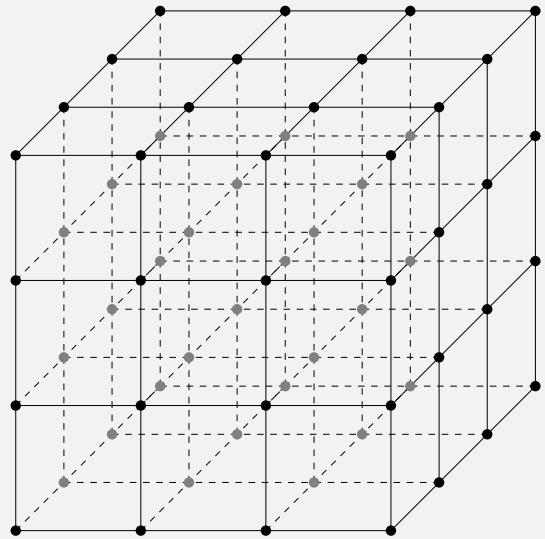
Monte Carlo versus Tensor Networks

Consider classical or quantum spin system in thermal equilibrium on d-dimensional lattice described by partition function Z

- Standard method: stochastic sampling using importance sampling Monte Carlo methods
- New alternative: reformulate Z as a fully contracted tensor network and use HOTRG to compute observables



Aim

Apply tensor method when action is complex and MC simulations fail

Partition function in tensor formulation similar to dual variables formulation of worm algorithm. However, computation of observables is different: stochastic for worm algorithm, deterministic for tensor networks, based on SVD approximations.

3d $O(2)$ model – action and partition function

$O(2)$ action

Three-dimensional quantum model with $O(2)$ action with chemical potential

$$S = -\beta \sum_{k=1}^{V} \sum_{\nu=1}^{3} \cos(\theta_k - \theta_{k+\hat{\nu}} - i\mu\delta_{\nu 3})$$

with coupling β and chemical potential μ in time direction.

Partition function

$$Z = \int_0^{2\pi} \frac{d\theta_1}{2\pi} \cdots \int_0^{2\pi} \frac{d\theta_V}{2\pi} e^{-S(\theta_1, \cdots, \theta_V)}$$

Complex action and sign problem for $\mu \neq 0$

Change to dual variables

Jacobi-Anger expansion

Boltzmann probability factorizes over links, and can be expanded using:

$$e^{\beta \cos\theta} = \sum_{n=-\infty}^{\infty} I_n(\beta)\, e^{in\theta}$$

$I_n(\beta)$: modified Bessel functions of the first kind

$\longrightarrow$ Introduces new link variables n_{kv}

Spin integration

Integration over spins factorizes:

$$\int_0^{2\pi} \frac{d\theta}{2\pi} e^{in\theta} = \delta_{n,0} \qquad n \in \mathbb{Z}$$

Tensor formulation of Z

Tensor formulation of Z

$$Z = \sum_{\{n\}} \prod_{k=1}^{V} T_{n_{k-\hat{1},1} n_{k,1} n_{k-\hat{2},2} n_{k,2} n_{k-\hat{3},3} n_{k,3}}$$

with local tensor T:

$$T_{n_{k-\hat{1},1} n_{k,1} n_{k-\hat{2},2} n_{k,2} n_{k-\hat{3},3} n_{k,3}} = \sqrt{e^{\mu(n_{k3}+n_{k-\hat{3},3})} \prod_{\nu=1}^{3} I_{n_{k\nu}}(\beta) I_{n_{k-\hat{\nu}}}(\beta)} \times \underbrace{\delta_{\Delta n_k,0}}_{\text{current conservation}}$$

and current

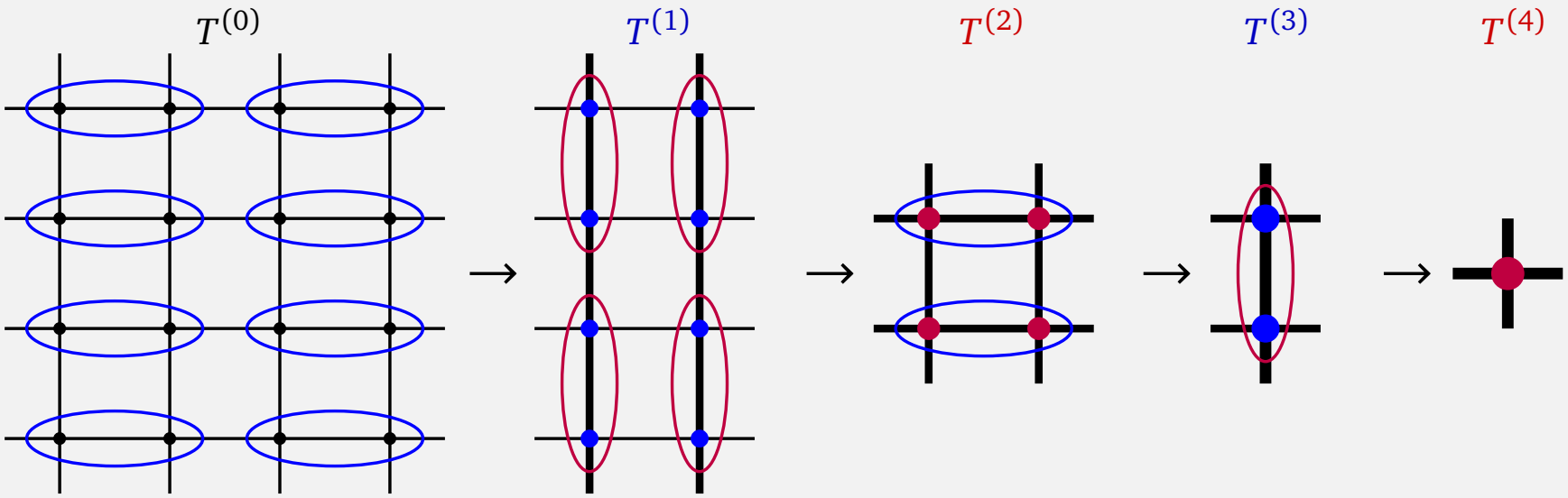
$$\Delta n_k = \sum_{\nu=1}^{3} (n_{k\nu} - n_{k-\hat{\nu},\nu})$$

$n \in \mathbb{Z} \rightarrow$ truncate sum to keep the D largest weights $I_n(\beta)\, e^{\mu n \delta_{\nu,3}}$

Higher order tensor renormalization group (HOTRG)

Typical blocking procedure in 2d HOTRG:

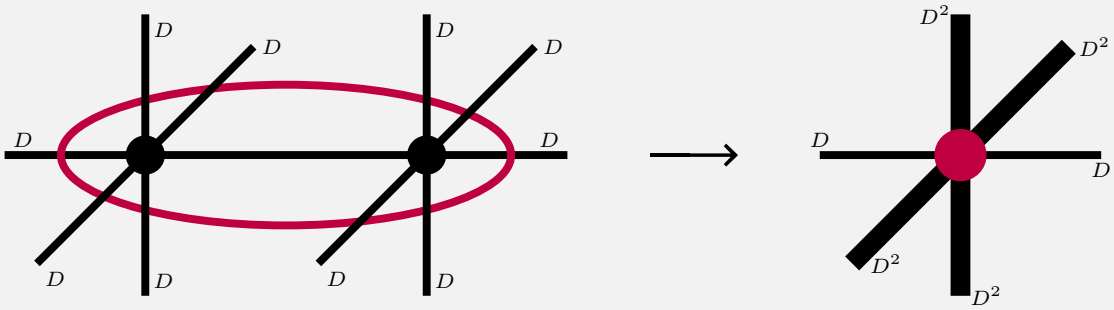
Alternating contraction directions



$$T^{(i)} \odot_{\mu} T^{(i)} \rightarrow T^{(i+1)}$$

3d HOTRG: contraction

Contraction



$$T^{(i)} \odot_x T^{(i)} \longrightarrow M$$

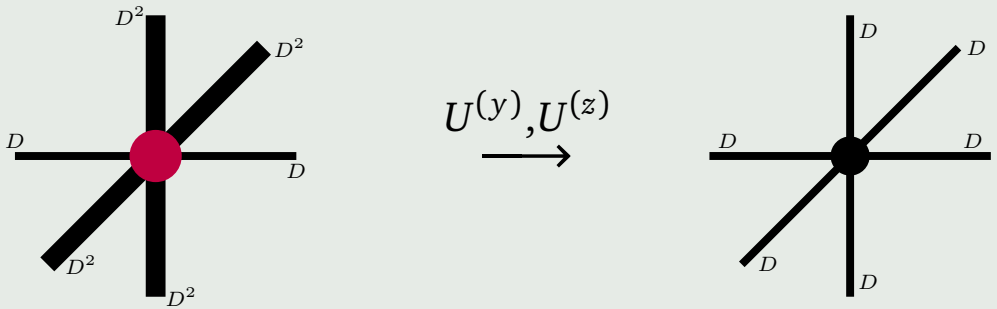
3d HOTRG: HOSVD truncation

Higher order singular value decomposition (HOSVD) truncation

- Compute SVD for unfoldings of M wrt all fat modes

  $\longrightarrow$ orthogonal $D^2 \times D^2$ frames: $\boxed{U^{(y^-)}, U^{(y^+)}, U^{(z^-)}, U^{(z^+)}}$

- Truncate frames $U^{(\nu)}$ from $D^2 \times D^2 \longrightarrow D^2 \times D$ by keeping singular vectors corresponding to the D largest singular values

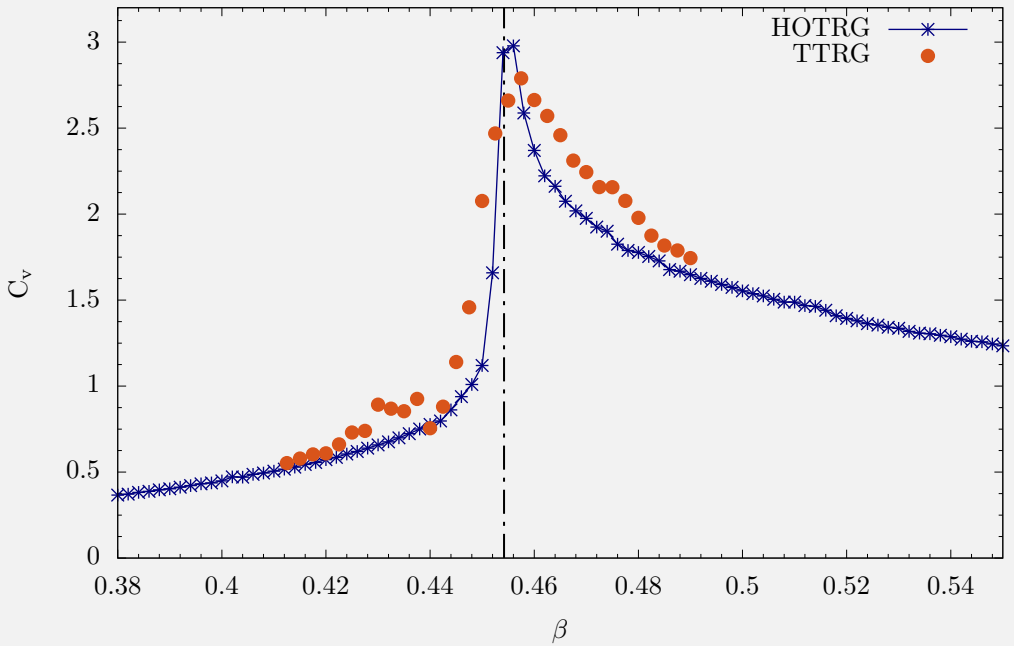


$$\boxed{M \xrightarrow{U^{(y)}, U^{(z)}} T^{(i+1)}}$$

Results: 3d $O(2)$ model – specific heat

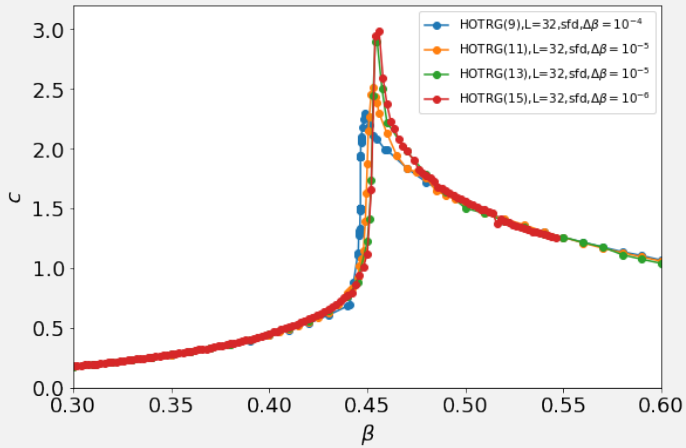
Specific heat – 32^3 lattice

Triad TRG with D=72 versus HOTRG with D=15



$$C_v = \frac{\beta^2}{V} \frac{\partial^2 \ln Z}{\partial \beta^2}$$

Convergence



- Triad TRG with $D = 72$ using 2nd order finite difference of $\ln Z$ with step size $\Delta\beta = 0.01$.
- HOTRG with $D = 15$ using a *stabilized* 2nd order finite-difference scheme with $\Delta\beta = 10^{-6}$.

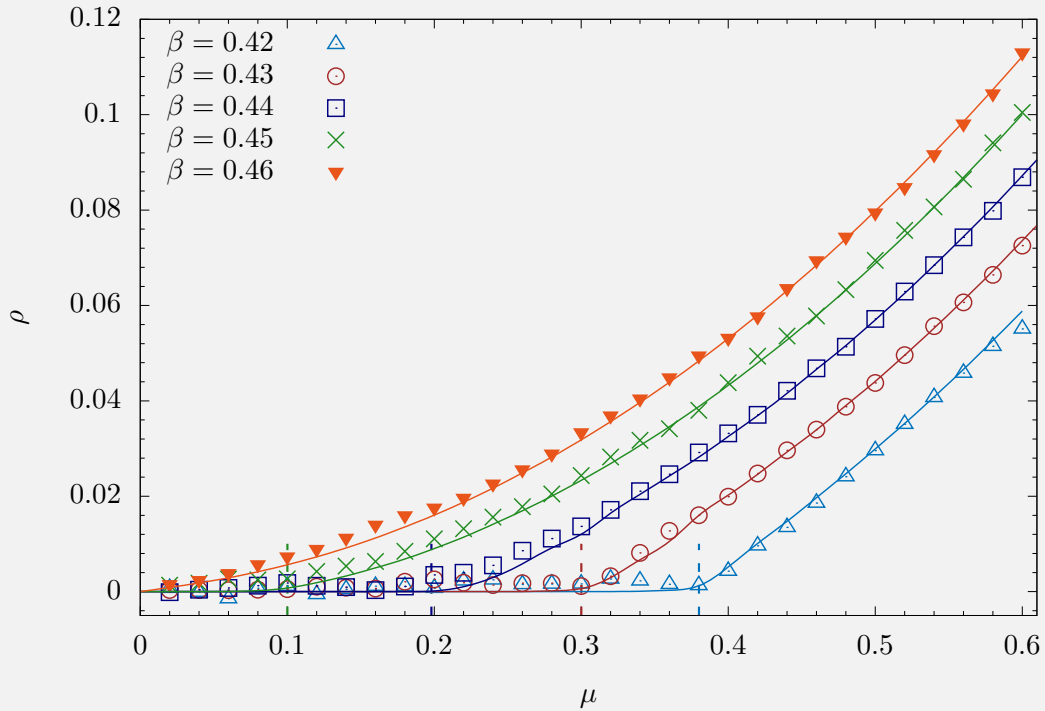
Number density ρ and Silver Blaze

Silver Blaze: at $T = 0$, particle density remains zero until μ reaches mass of lowest excitation

Property is very hard to reproduce numerically! Intrinsically coupled to sign problem in original formulation

Silver Blaze: ρ versus μ for $T = 0$ on 64^3 lattice

Triad TRG with $D = 50$ versus worm algorithm



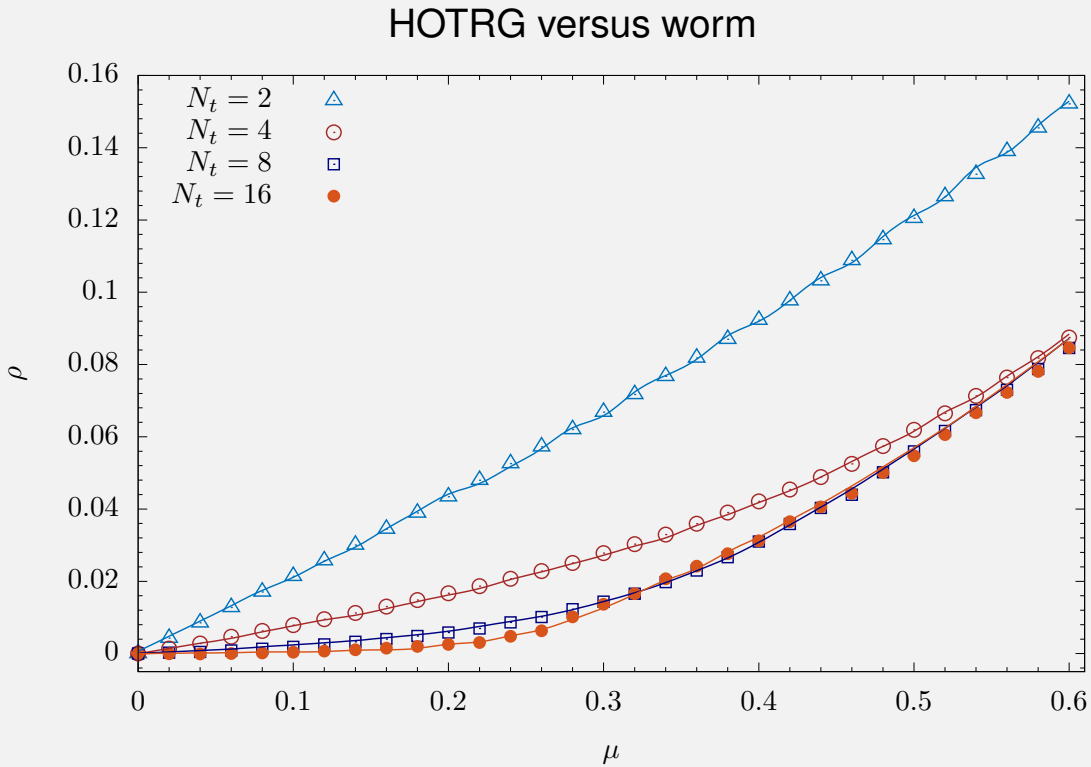
$$\rho = \frac{1}{V}\frac{\partial \ln Z}{\partial \mu}$$

$$ma(\beta) = \xi^{-1}(\beta)$$

$$\frac{\mu_{\text{phys}}}{m} = \xi(\beta)\mu$$

Non-zero temperature and chemical potential

Temperature dependence: ρ versus μ for $N_t < N_s$ ($N_s = 64$)



- HOTRG with $D = 13$, improved contraction order and stabilized finite differences on a $64^2 \times N_t$ lattice with $N_t = 2, 4, 8, 16$ at $\beta = 0.44$ versus worm algorithm

Improvements to standard HOTRG

Improved contraction order

Search contraction direction μ causing the smallest (approximate) truncation error in next coarsening step

- only sub-leading additional cost
- especially useful in case of anisotropic tensor
- naturally suits HOTRG on asymmetric lattices

Stabilized finite differences for observables

Problem: finite-differences suffer from jumps in $\ln Z$ due to level crossings or (almost) degenerate singular values

Solution: stabilized finite differences

- At each blocking step, analyze the spaces of singular vectors for the nearby parameter values used in finite difference formula
- Perform orthogonal transformations on singular vectors to maximize the overlap between these adjacent vector subspaces

Outlook

Improvements and extensions

- HT-HOTRG factorization: improves efficiency in 3d and 4d
- SuperQ method: improved construction of truncation frames
- Extend stabilized finite differences to factorization schemes like Triad-TRG and HT-HOTRG
- Current project: strong coupling $SU(3)$ → use Grassmann-HOTRG to handle baryon-meson system
- Future work: use environment information to improve local truncation iteratively (à la HOSRG)

More contributions

- Strong coupling $U(n)$ in 3d and 4d → talk Pascal Milde on Friday
- Effective Z_3 model for QCD at nonzero μ → poster Judah Unmuth-Yockey