Effective $\mathbb{Z}_3$ model for finite-density QCD with tensor networks

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Abstract

The tensor renormalization group is a promising numerical method used to study lattice statistical field theories. However, this approach is computationally expensive in 2+1 and 3+1 dimensions. Here we use tensor renormalization group methods to study an effective three-dimensional Z_3 model for the heavy-quark, high-temperature, strong-coupling limit of single-flavor 3+1 dimensional quantum chromodynamics. Our results are cross-checked using the worm Monte Carlo algorithm. We present the phase diagram of the model through the measurement of the Polyakov loop, the nearest-neighbor Polyakov loop correlator, and their susceptibilities. The tensor renormalization group results are in good agreement with the literature.

Single-flavor, finite-density QCD

The action has two parts, $S = S_g + S_f$, with **gauge action**

$$S_g = -\frac{\beta}{3}\sum_{x=1}^{N}\left[\frac{a}{a_t}\sum_{i=1}^{3}\Re\mathrm{Tr}[U_{x,i4}] + \frac{a_t}{a}\sum_{i<j=1}^{3}\Re\mathrm{Tr}[U_{x,ij}]\right] \tag{1}$$

and **fermion action**

$$S_f = \sum_{x,y}\bar\psi_x M_{xy}\psi_y, \quad M = \mathbf{1} - \kappa\frac{a_t}{a}H_s - \kappa H_t \tag{2}$$

and spatial and temporal hoppings

$$H_s = \sum_{i=1}^{3}\left(T_i^+ + T_i^-\right), \quad H_t = e^{\tilde\mu a_t}T_4^+ + e^{-\tilde\mu a_t}T_4^- \tag{3}$$

where

$$(T_\nu^\pm)_{xy} = (1 \pm \gamma_\nu)U_{x,\pm\nu}\delta_{y,x\pm\hat\nu}. \tag{4}$$

$\beta \sim 1/g^2$ and κ are couplings, $\tilde\mu$ is the **chemical potential**, a and a_t are the spatial and temporal lattice spacings. The **temperature** is given by the inverse, physical extent of the lattice,

$$T = \frac{1}{N_t a_t}, \tag{5}$$

with **(anti-)periodic boundary conditions** in (time) space.

The effective action

Take the **strong-coupling**, **high-temperature**, **large chemical potential**, **large quark mass** limit,

$$\beta \ll 1, \quad N_t = 1, \quad a_t \ll 1, \quad \tilde\mu \gg 1, \quad \kappa \ll 1. \tag{6}$$

In S_g spatial plaquettes are suppressed, and in S_f spatial hopping is suppressed. The $SU(3)$ group elements are replaced by elements of the **center**, $\mathbb{Z}_3$.

$$S_g \to -\frac{\beta}{2a_t}\sum_x\sum_{\nu=1}^{3}P_x^* P_{x+\hat\nu} + \text{c.c.} \tag{7}$$

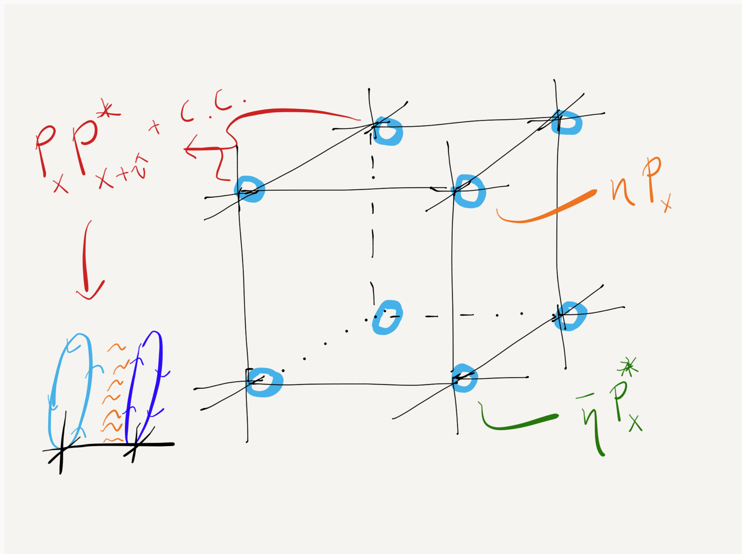
$$S_f \to -\kappa\sum_x\left[e^{\mu}P_x + e^{-\mu}P_x^*\right]. \tag{8}$$

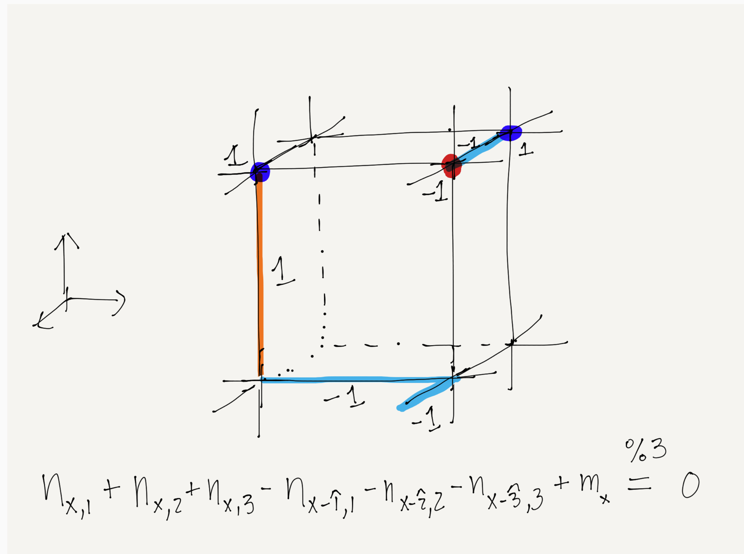
$P_x \in \mathbb{Z}_3$, which can be interpreted as a **Polyakov loop**. This results in an effective action

$$S_{\text{eff}} = -\sum_x\left[\tau\sum_{\nu=1}^{3}(P_x^* P_{x+\hat\nu} + \text{c.c.}) + (\eta P_x + \bar\eta P_x^*)\right] \tag{9}$$

with $\tau \equiv \beta/2a_t$ and $\eta \equiv \kappa e^{\mu}$ and $\bar\eta = \kappa e^{-\mu}$.

The tensor network





The partition function for the effective action describing this theory of interacting "spins" can be recast to a theory of "currents" and "charges" by **Fourier expansion**,

$$Z = \sum_{\{n\}}\sum_{\{m\}}\left(\prod_{x,\nu}B_{n_{x,\nu}}(\tau)\right)\left(\prod_x V_{m_x}(\eta,\bar\eta)\Delta^{(3)}(j_x)\right), \tag{10}$$

with $j_x = \sum_{\nu=1}^{3} n_{x,\nu} - n_{x-\hat\nu,\nu} + m_x$, and $\Delta^{(3)}(j_x) = \delta_{j_x \bmod 3, 0}$. A **tensor network** can be built from the tensor,

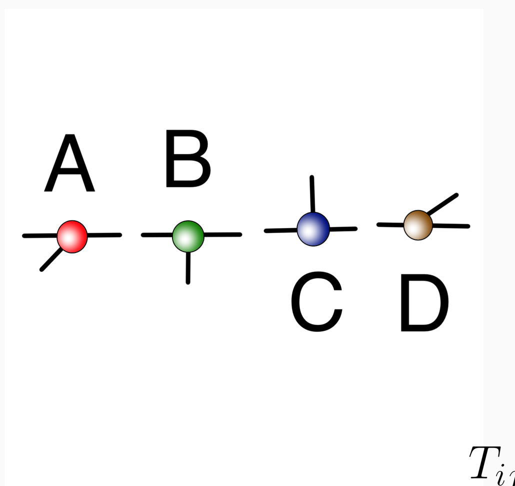
$$T^{(x)}_{n_{x-\hat1,1}n_{x,1}n_{x-\hat2,2}n_{x,2}n_{x-\hat3,3}n_{x,3}} = \sqrt{B_{n_{x-\hat1,1}}B_{n_{x,1}}B_{n_{x-\hat2,2}}B_{n_{x,2}}B_{n_{x-\hat3,3}}B_{n_{x,3}}(\tau)}\,V_{m_x}(\eta,\bar\eta)\Delta^{(3)}(j_x).$$

We calculate the average **Polyakov loop** and average **nearest-neighbor correlator**, and their susceptibilities,

$$\langle P\rangle = \frac{1}{V}\frac{\partial\ln Z}{\partial\eta}, \quad \langle E\rangle = -\frac{1}{V}\frac{\partial\ln Z}{\partial\tau} \tag{11}$$

$$\chi_P = V(\langle P^2\rangle - \langle P\rangle^2), \quad \chi_E = V(\langle E^2\rangle - \langle E\rangle^2). \tag{12}$$

Tensor methods

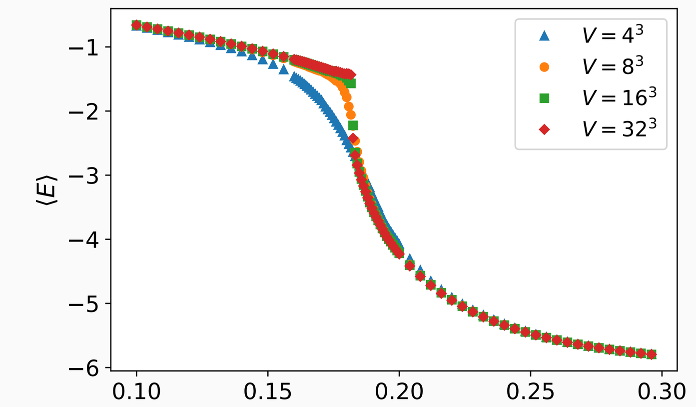


Using T we contract the tensor network using the **higher-order tensor renormalization group (HOTRG)**. This can be done in the original HOTRG prescription, or with additional approximations using the **triad** tensor renormalization group. The T tensor can be decomposed into four, three-indexed tensors called triads,

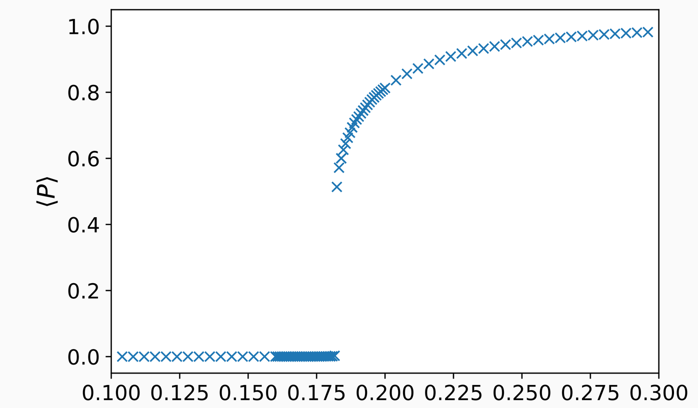
$$T_{ijklmn} = \sum_{\alpha,\beta,\gamma}A_{ij\alpha}B_{\alpha k\beta}C_{\beta l\gamma}D_{\gamma mn}. \tag{13}$$

The tensor network contraction with these can be done efficiently. We report results using original HOTRG, triad HOTRG, and **worm Monte Carlo**.

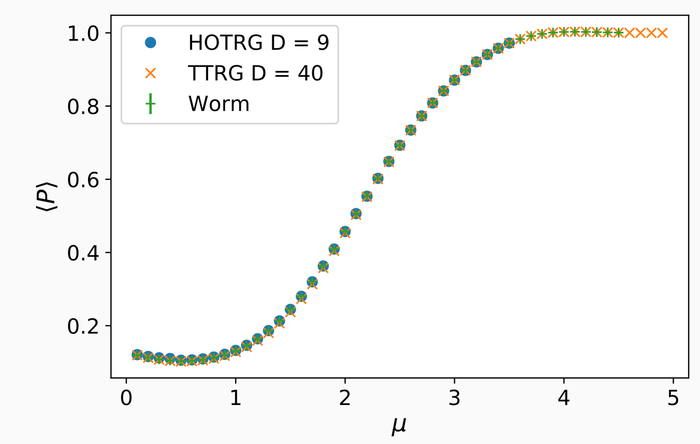
Results



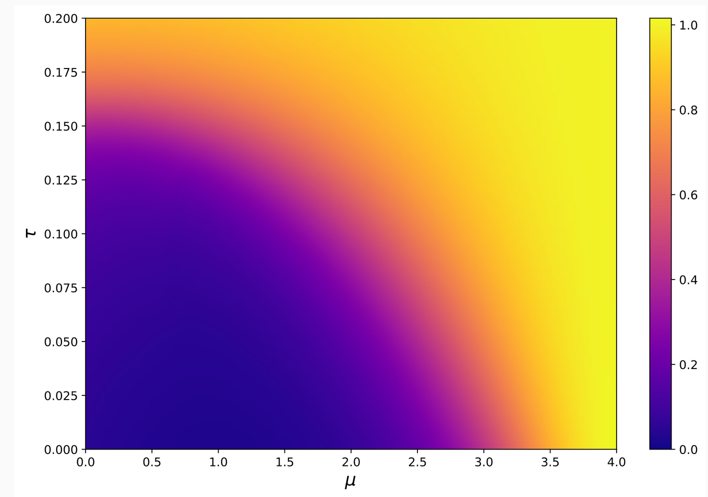
The average nearest-neighbor Polyakov loop correlator at $\kappa, \mu = 0$.



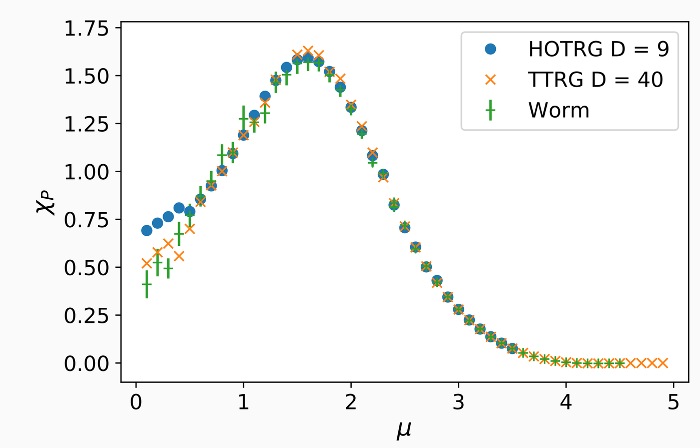
The average Polyakov loop at $\mu = 0$, $\kappa = 2\times10^{-5}$ at $V = 1024^3$.



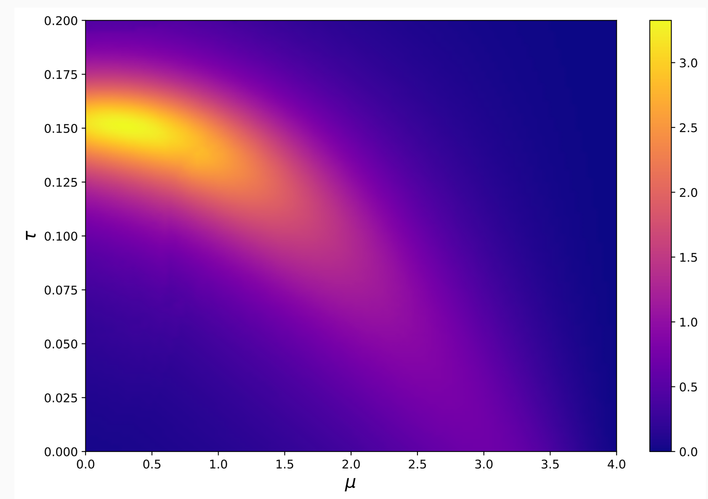
The Polyakov loop at $\kappa = 0.05$, and $\tau = 0.1$, $V = 32^3$.



Heat map of the Polyakov loop at $\kappa = 0.05$ using HOTRG with seven states.



The Polyakov loop susceptibility at $\kappa = 0.05$, and $\tau = 0.1$, $V = 32^3$.



Heat map of the Polyakov loop susceptibility at $\kappa = 0.05$ using HOTRG with seven states.