A novel method to evaluate real-time path integral for scalar ϕ^4 theory

Shinji Takeda
（Kanazawa University）

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Study of real-time dynamics

- Complex Langevin
  - 3+1D ϕ^4 theory PRL95,202003(2005) Berges et al.
  - 3+1D SU(2), Schwinger-Keldysh (SK) PRD75,045007(2007) Berges et al.
- Lefschetz thimble
  - 1+1D ϕ^4 theory PRD95,114501(2017) Alexandru et al.
  - 0+1D ϕ^4 theory JHEP06(2019)094 Mou et al.
- Quantum computations
  - QED_2 arXiv2001.00485 Chakraborty et al., ….
  - Gauge Ising Yamamoto PTEP2021(2021)013B06
- Tensor networks
  - Hamiltonian approach PRD100,094504(2019) Lin et al., 1+1D Thirring model
  - Lagrangian approach Lattice 2019 ST 1+1D ϕ^4 theory using Feynman prescription

  ⇒ Today, I introduce different approach and show results of time-correlator

Closed time formalism

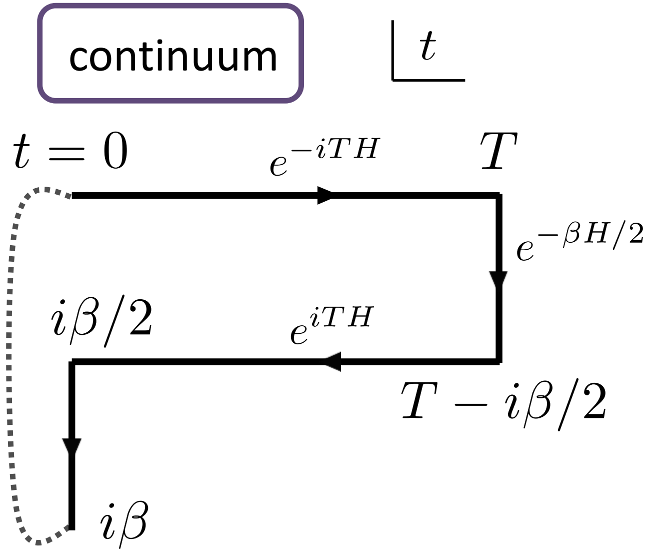
Schwinger 1961
Keldysh 1964

$$\langle \mathcal{O}(T) \rangle = \mathrm{Tr}[\mathcal{O}\rho(T)] = \mathrm{Tr}[\mathcal{O}(T)\rho(0)]$$

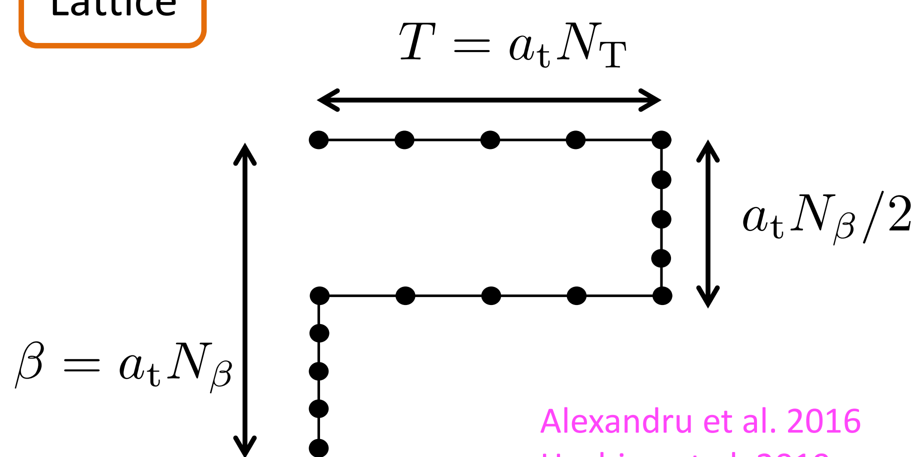
$$\mathcal{O}(T) = e^{iTH}\mathcal{O}e^{-iTH}$$

In the following $\rho(0) = e^{-\beta H}/\mathrm{Tr}e^{-\beta H},$

continuum

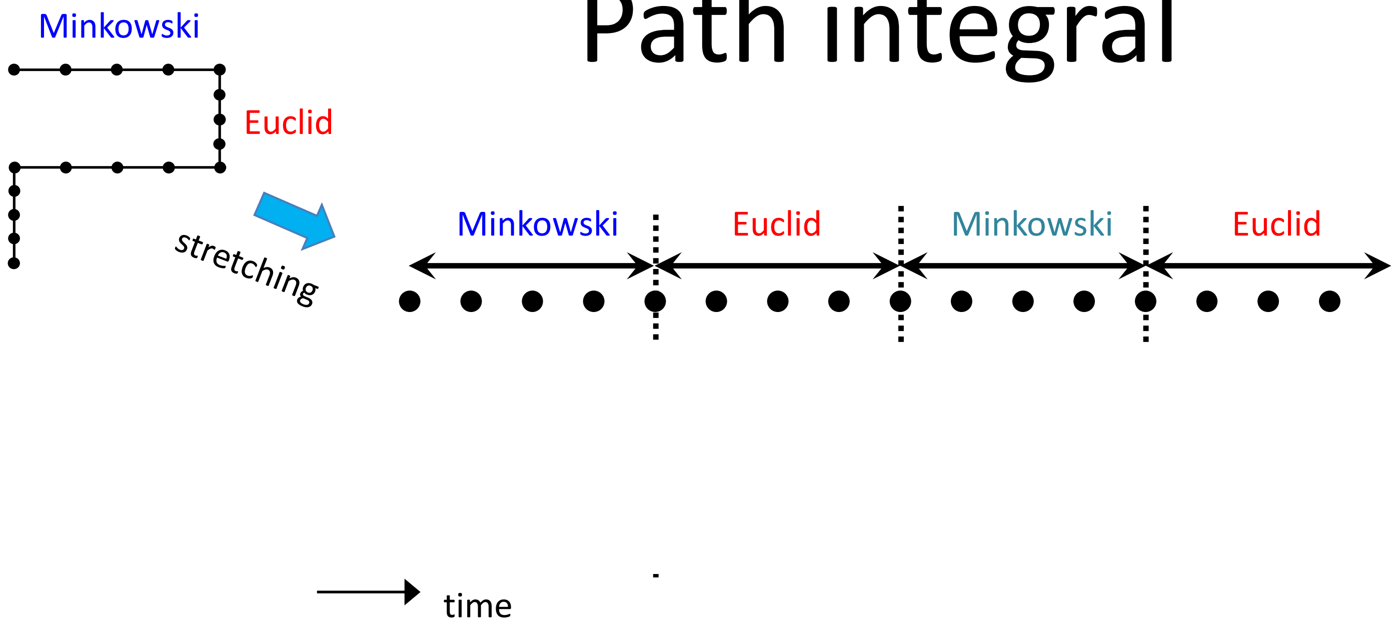


Lattice



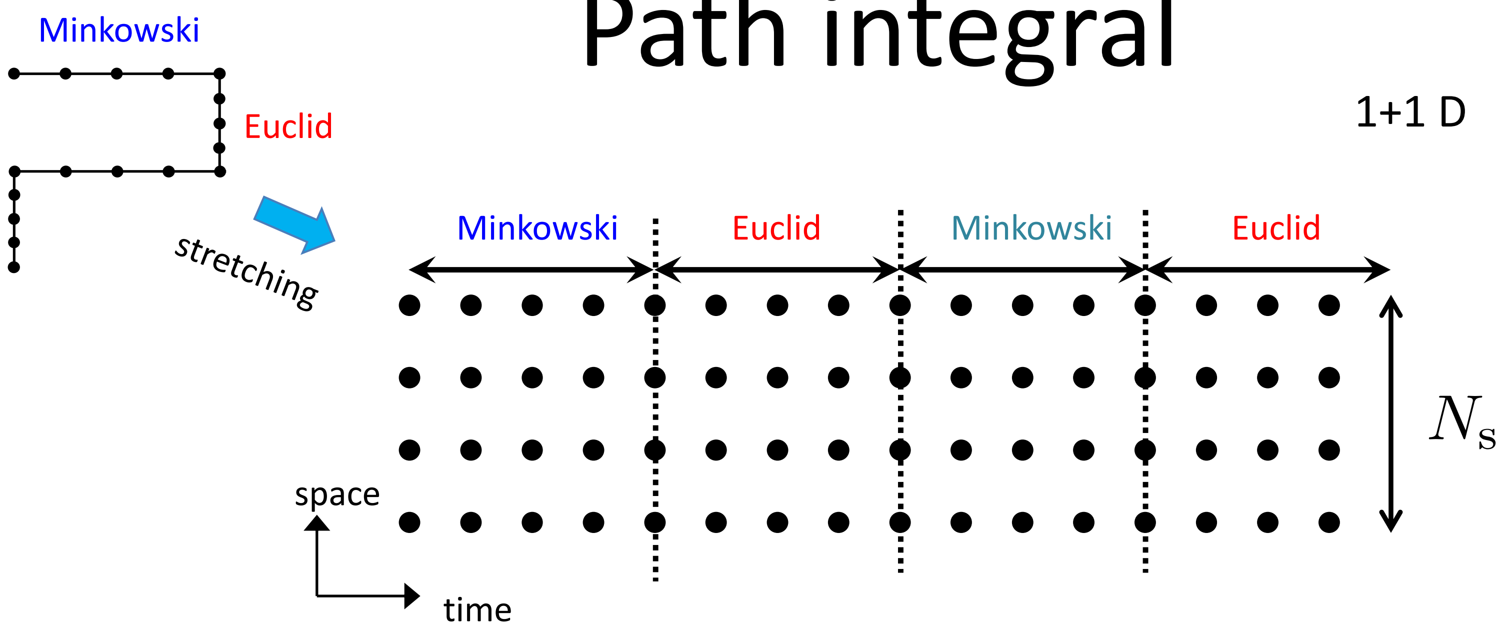
Alexandru et al. 2016
Hoshina et al. 2019

Path integral

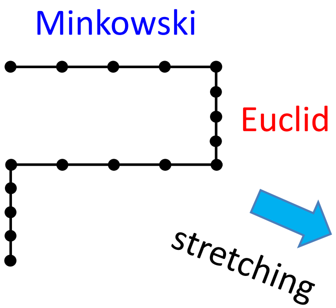


Path integral

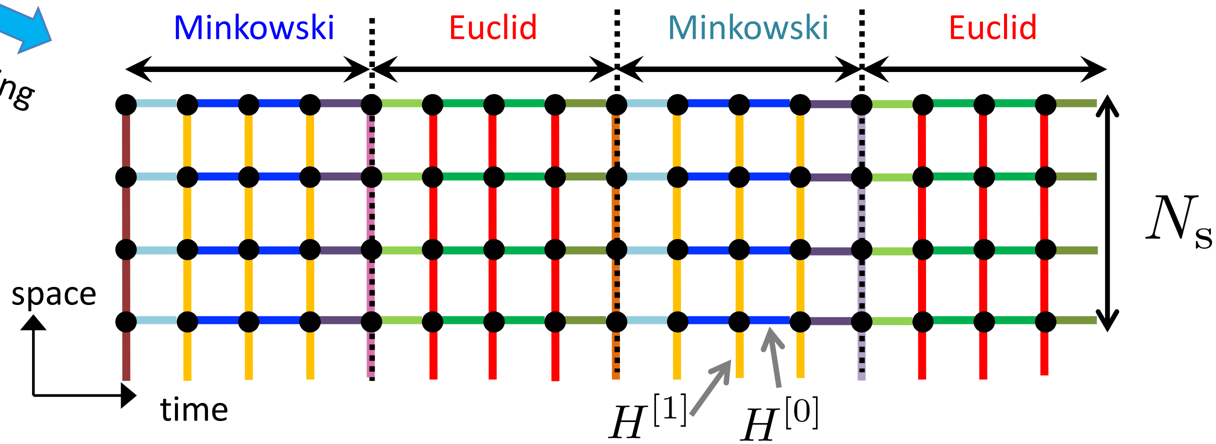
1+1 D



Path integral



1+1 D



$$\mathcal{Z}_{\rm lat} = \int [d\phi] e^{iS_{\rm lat}[\phi]} = \int [d\phi] \prod_x \prod_{\mu=0,1} H^{[\mu]}(x_0; \phi(x), \phi(x+\hat{\mu}))$$

e.g. In Minkowski space region

$$V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{4!}\lambda\phi^4$$

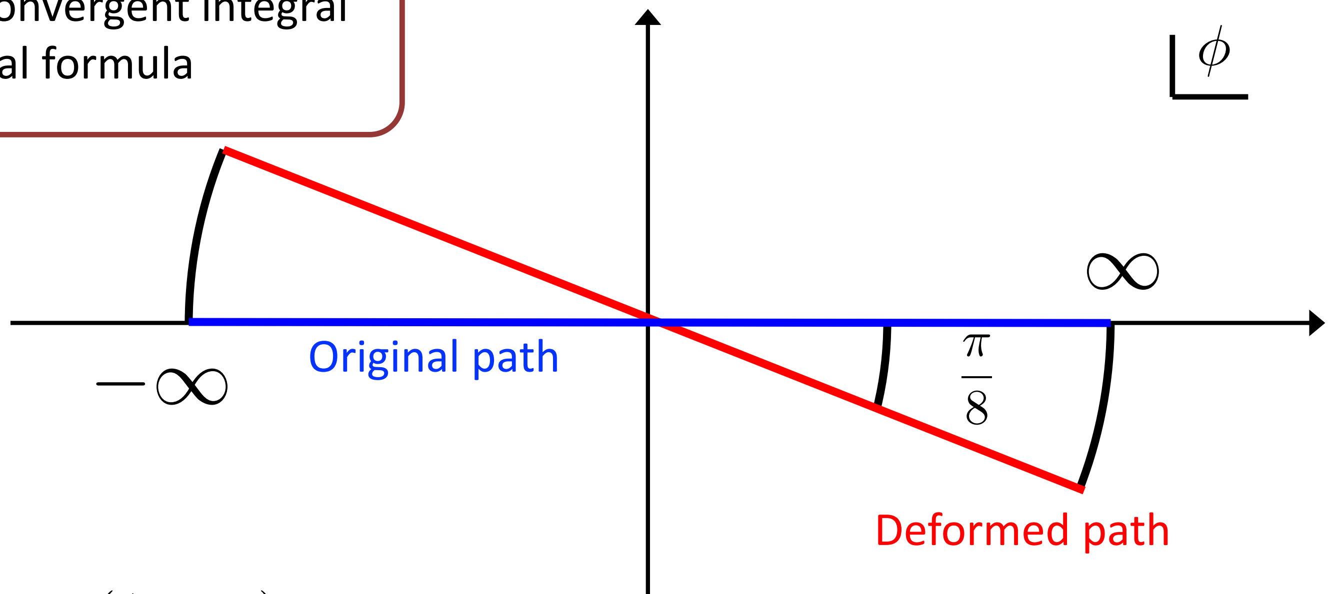
$$H^{[0]}(\phi,\phi') = \exp\left\{ +i\frac{(\phi'-\phi)^2}{2} - i\frac{1}{4}V(\phi) - i\frac{1}{4}V(\phi') \right\}$$

$$H^{[1]}(\phi,\phi') = \exp\left\{ -i\frac{(\phi'-\phi)^2}{2} - i\frac{1}{4}V(\phi) - i\frac{1}{4}V(\phi') \right\}$$

$\Rightarrow$ multiple oscillating integral

Deformation of integration path for scalar field

Rewrite in terms of convergent integral using Cauchy's integral formula



Basic idea

$$\phi \longrightarrow e^{-i\pi/8}\xi \qquad (\xi \in \mathbb{R})$$

$$V(\phi) = \cdots + \frac{\lambda}{4!}\phi^4 \longrightarrow -i\frac{\lambda}{4!}\xi^4$$

$$e^{iS_{\mathrm{lat}}[\phi]} = e^{\cdots - iV(\phi)} \longrightarrow e^{\cdots - \frac{\lambda}{4!}\xi^4}$$

$\Rightarrow$ integrand of path integral is localized

$\Rightarrow$ Path integral turns out to be convergent

But, the integrand is complex…

Gauss Hermite quadrature

2018 Sakai et al.

$$\int_{-\infty}^{\infty} d\xi f(\xi) \approx \sum_{a=1}^{N} w_a e^{\xi_a^2} f(\xi_a)$$

N: Truncation order

w_a: weight

ξ_a: Zeros of N-th Hermite polynomials

Replace integral to discrete sum for all scalar fields

$$\mathcal{Z}_{\rm lat} = \int_{-\infty}^{\infty} \prod_x d\xi(x) \ldots \approx \sum_{\{a\}} w_a e^{\xi_a^2} \ldots$$

$$H^{[\mu]}(\phi(\xi_a), \phi(\xi_b)) \longrightarrow A_{ab}^{[\mu]}$$: $N \times N$ matrix

$\Rightarrow$ Singular value decomposition (SVD)

Singular value of A

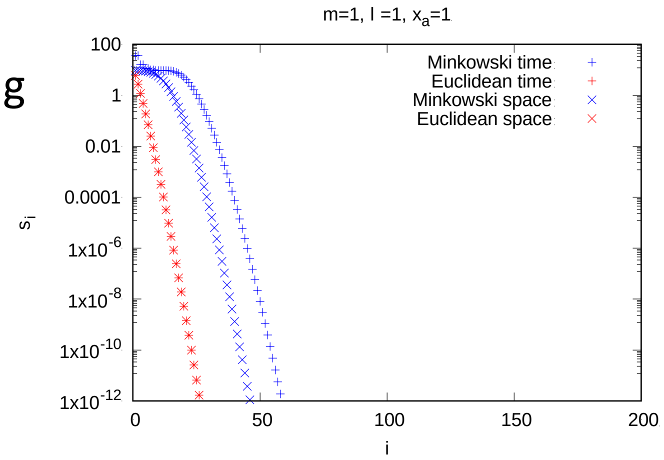
anisotropic parameter

$$\xi_a = \frac{a}{a_{\mathrm{t}}}$$

isotropic ($\xi_a = 1$)

Weak coupling

$\lambda = 1$



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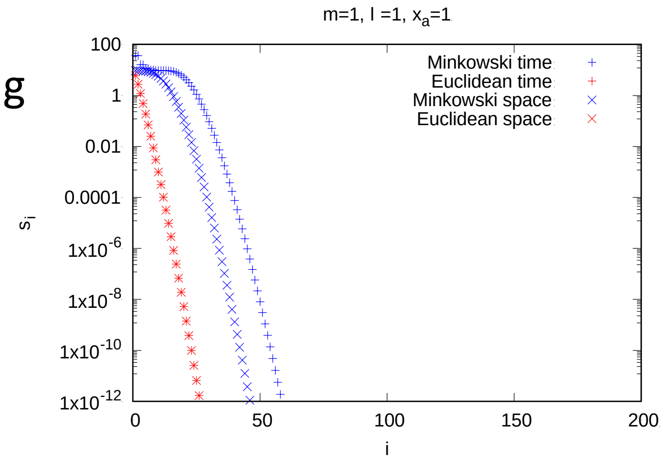
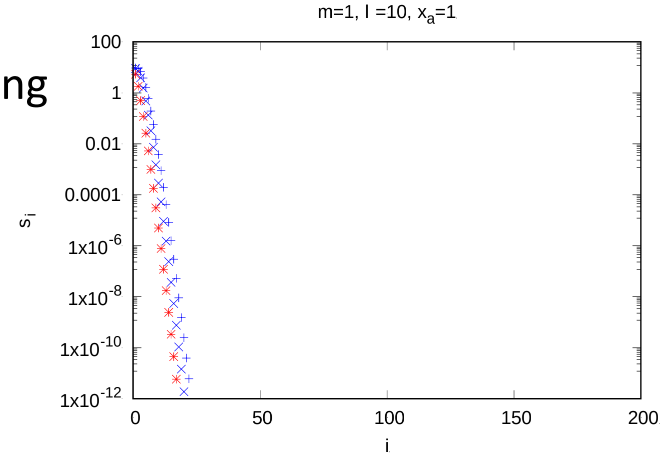


Strong coupling
$\lambda = 10$



Singular value of A

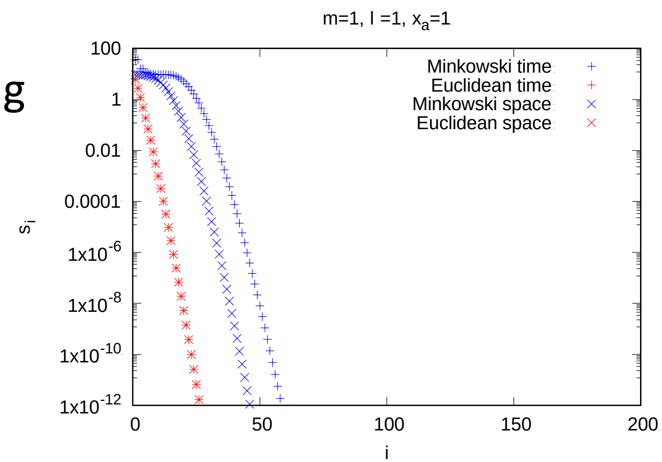
anisotropic parameter

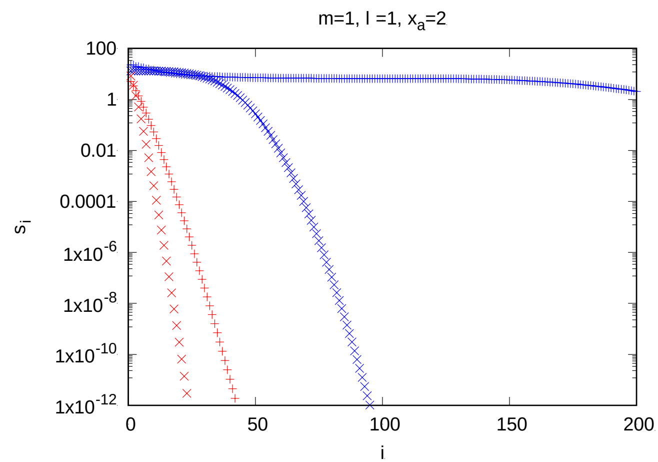
$$\xi_a = \frac{a}{a_t}$$

isotropic ($\xi_a = 1$)

anisotropic ($\xi_a = 2$)

Weak coupling
$\lambda = 1$





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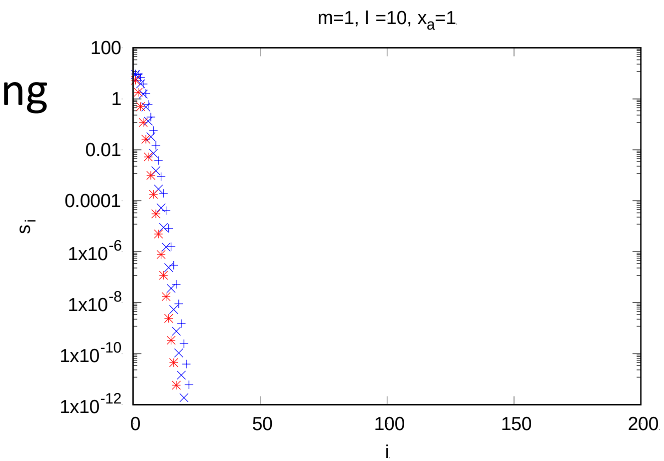
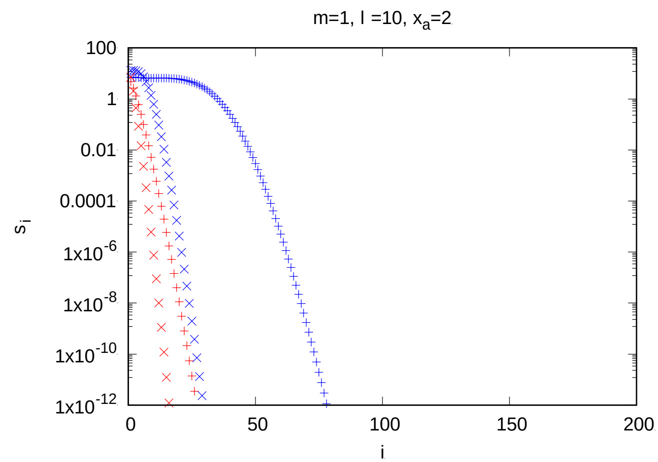




How to make tensor (1+1D)

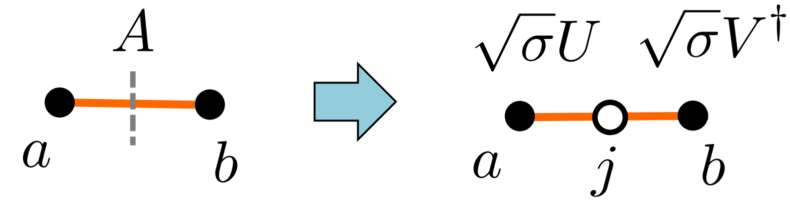
SVD

$$A^{[\mu]}_{ab} = \sum_j U^{[\mu]}_{aj} \sigma^{[\mu]}_j V^{[\mu]\dagger}_{jb}$$

unitary matrix

Singular value

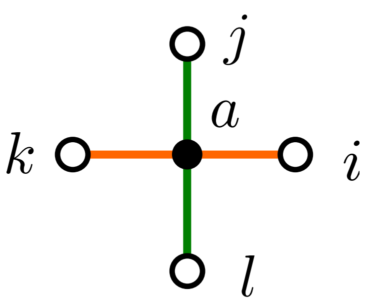
for $\mu = 0, 1$



Tensor consists of the unitary matrices and singular values

$$T_{ijkl} = \sqrt{\sigma^{[0]}_i \sigma^{[1]}_j \sigma^{[0]}_k \sigma^{[1]}_l} \sum_{a=1}^{N} J(\xi_a) U^{[0]}_{ai} U^{[1]}_{aj} V^{[0]*}_{ak} V^{[1]*}_{al}$$

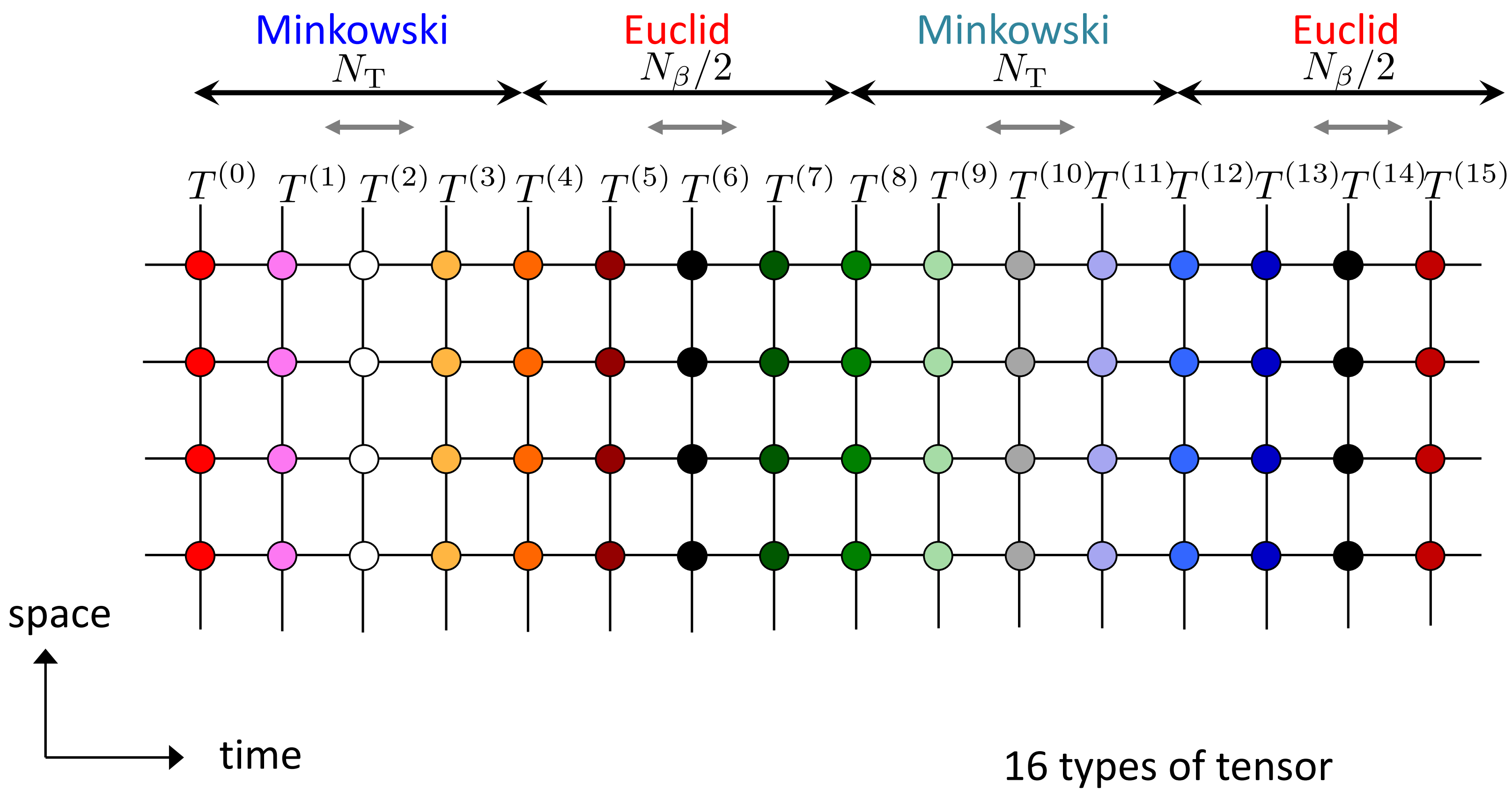
$$J = \frac{d\phi}{d\xi}$$



Bond dimension of i, j, k, l is determined adaptively according to a required precision

Tensor network of the system

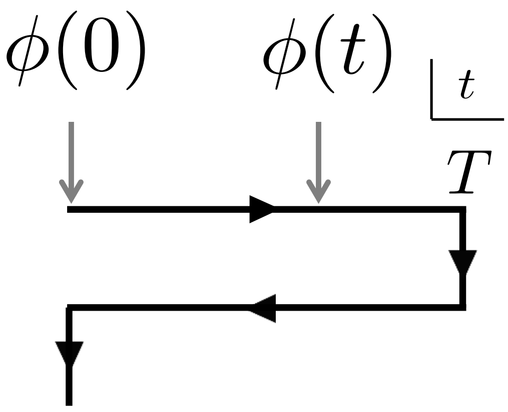
1+1D



16 types of tensor

Real-time correlator for $N_s = 2$

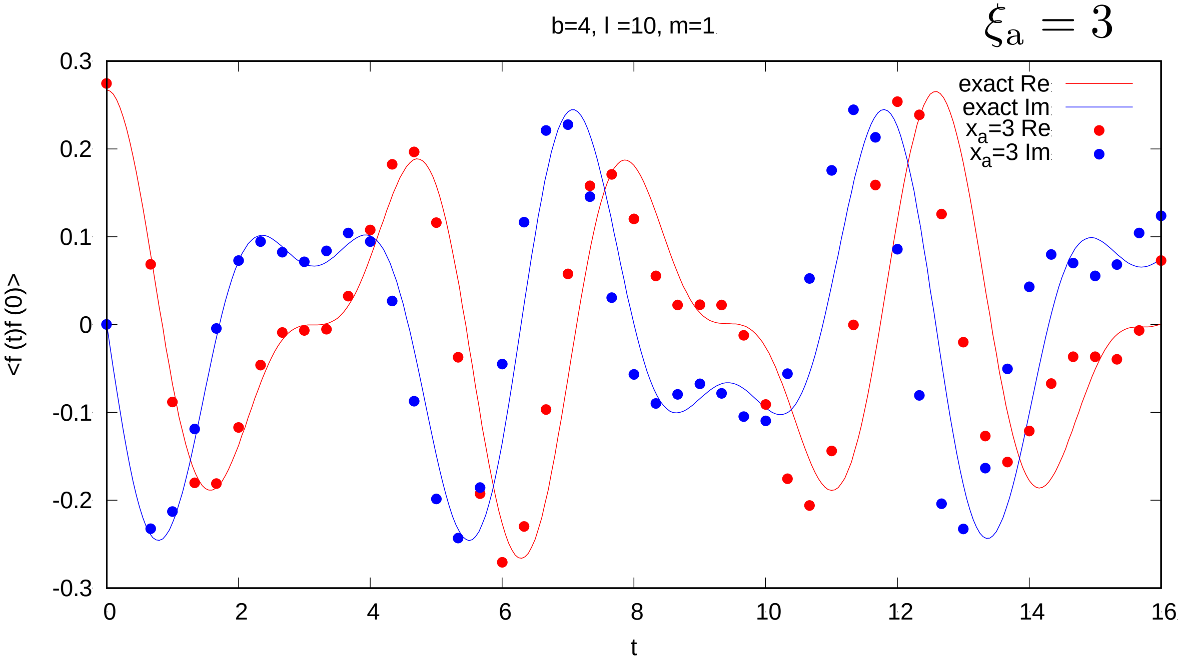
$$\langle \phi(t)\phi(0) \rangle$$



$$0 \leq t \leq T$$

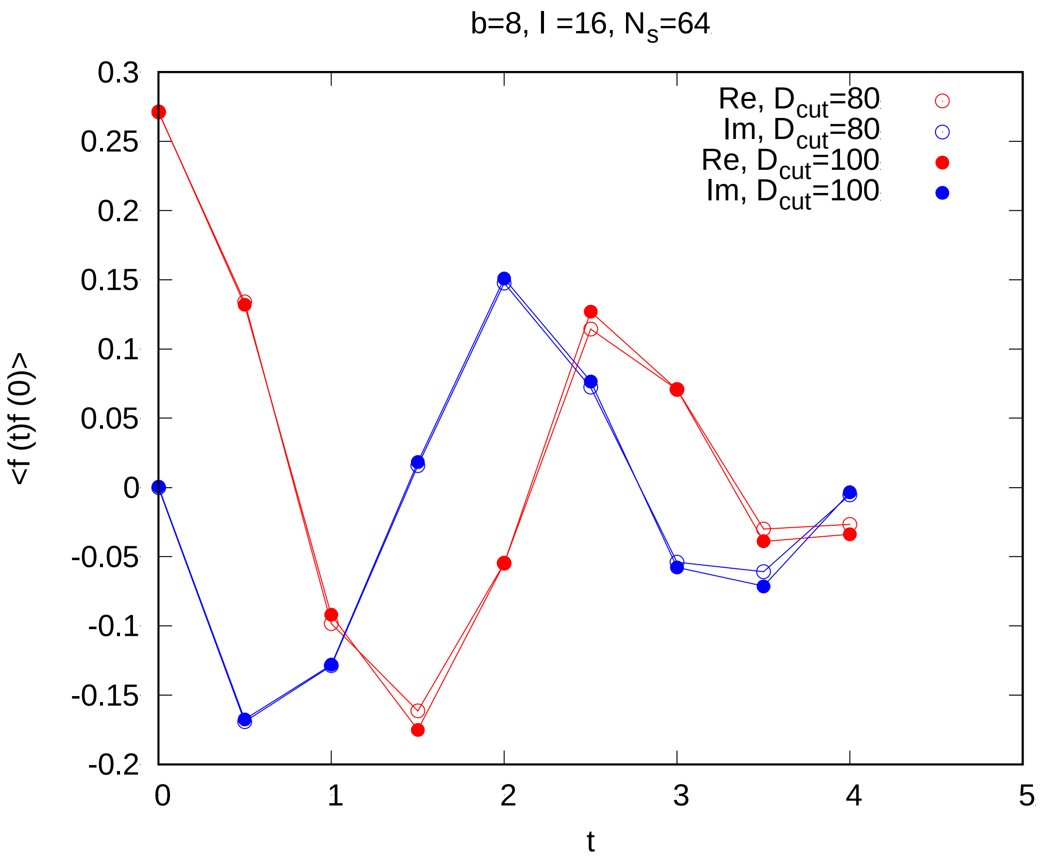
- coarse-graining is not applied
- for checking initial tensor

$N_{\mathrm{T}} = 48$
$N_{\beta} = 12$
$\xi_{\mathrm{a}} = 3$



- Required relative accuracy : 10^{-4}
- Max bond dimension ~ 200

Real-time correlator for $N_s = 64$



$N_{\mathrm{T}} = 8$

$N_\beta = 16$

$\xi_{\mathrm{a}} = 2$

- D_{cut}: truncation order for coarse-graining
- Required relative accuracy : 10^{-4}

Summary

- Derive tensor network (TN) representation of closed-time formalism for real scalar theory
- Demonstrate real-time correlator at $N_s = 2$ and 64

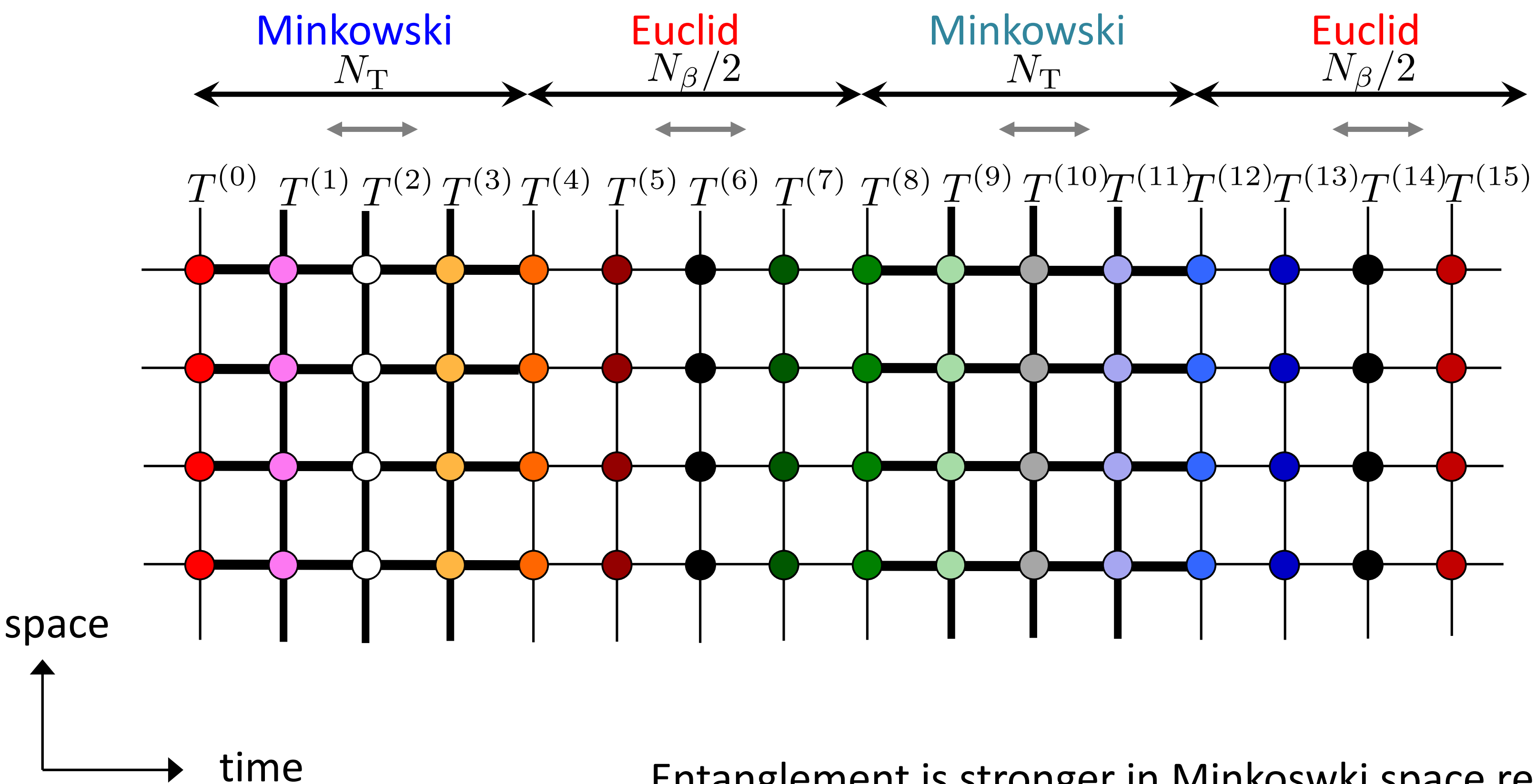
Future

- Entanglement tends to be stronger for Weak coupling / anisotropic case. How to improve in such case?
- Better coarse-graining scheme for $N_s > 2$?
- Computation of spectral function

Backup

Tensor network of the system

1+1D

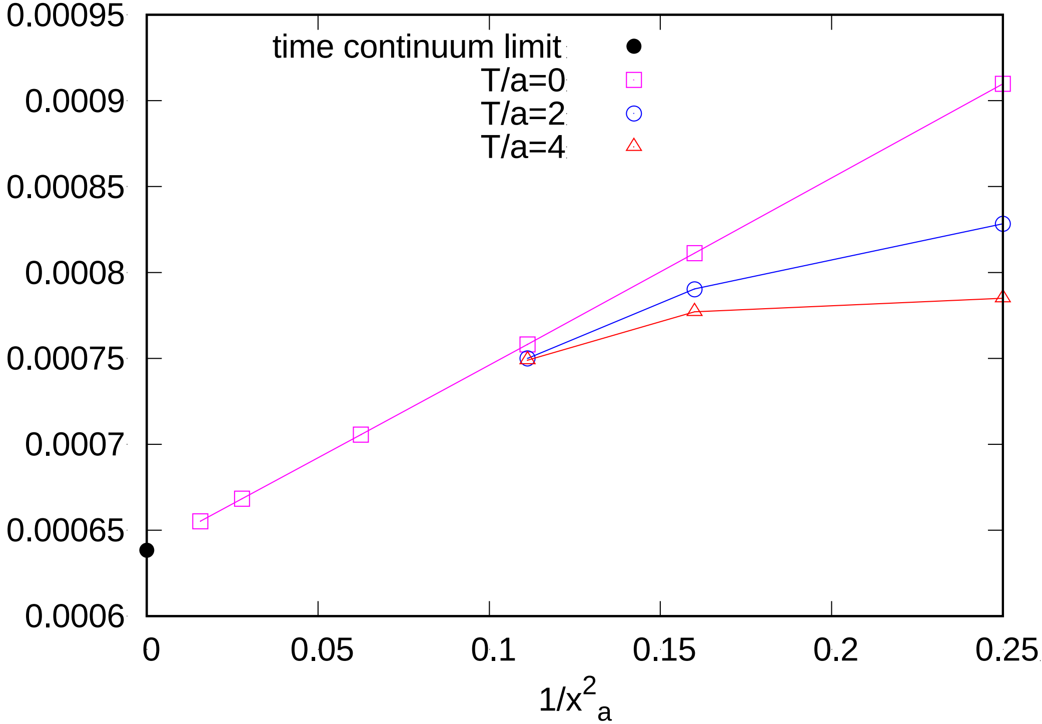


Entanglement is stronger in Minkoswki space region

Check of initial tensor

$$\mathrm{tr}\left[e^{-iTH}e^{-\beta H/2}e^{iTH}e^{-\beta H/2}\right] \begin{cases} = \mathrm{tr}[e^{-\beta H}] & \text{for continuum time} \\ \uparrow a_{\mathrm{t}} \to 0 & \\ \neq \mathrm{tr}[e^{-\beta H}] & \text{for lattice} \end{cases}$$

b=4, l =10 $m = 1$



On the lattice

$$[e^{-iTH}, e^{-\beta H}] \neq 0$$

$$1/\xi_a = \frac{a_{\mathrm{t}}}{a}$$

Time-correlator for $N_s = 2$