A variance reduction technique for hadronic correlators with partially twisted boundary conditions

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General idea

GOAL: we want to induce momentum into hadrons through **twisted boundary conditions**[a] with less computational effort but still with good statistical precision

[a] [Sachrajda, Villadoro, PLB 609 (2005) 73-85], [de Divitiis et al., PLB 595 (2004) 408-413]

IDEA: exploit the known **untwisted** correlators, which are in general computed with good precision

We construct the twisted correlator (2-point and 3-point) using

$$C_\theta^\alpha(t) = \tilde{C}_\theta(t) + \alpha(t)\underbrace{\left(C_0(t) - \tilde{C}_0(t)\right)}_{\text{noisy zero}}$$

$$\begin{cases} C = \frac{1}{N}\sum_{i=1}^{N} C^i \\ \tilde{C} = \frac{1}{n}\sum_{i=1}^{n} C^i \end{cases} \qquad N > n,$$

N=full number of source planes ($\mathbb{Z}_2$) per configuration

Computing the error

$$(\sigma_\theta^\alpha)^2 =< (\Delta C_\theta^\alpha)^2 >$$

we can optimize with respect to α imposing $\frac{\partial}{\partial\alpha}(\sigma_\theta^\alpha)^2 = 0$:

$$\alpha_{min} = -\frac{\mathrm{Cov}(C_0, \tilde{C}_\theta) - \mathrm{Cov}(\tilde{C}_0, \tilde{C}_\theta)}{\sigma_0^2 + \tilde{\sigma}_0^2 - 2\mathrm{Cov}(C_0, \tilde{C}_0)}$$

$$(\sigma_\theta^\alpha)^2 = \tilde{\sigma}_\theta^2 - \frac{\left[\mathrm{Cov}(C_0, \tilde{C}_\theta) - \mathrm{Cov}(\tilde{C}_0, \tilde{C}_\theta)\right]^2}{\sigma_0^2 + \tilde{\sigma}_0^2 - 2\mathrm{Cov}(C_0, \tilde{C}_0)}$$

Test on Kaon/Pion on $24^3 \times 64$ RBC/UKQCD DWF ensembles

[Allton et al, Phys. Rev. D (2008) 114509]

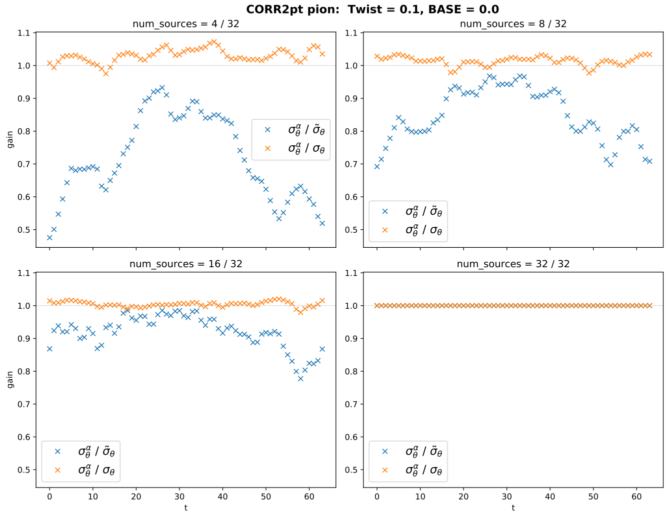
32 total sources / 39 configurations

Twist on light quark ($m_\pi \simeq 330\,\text{MeV}$) on 3 spatial directions

$$\begin{cases} \theta_1 = 0.1 \times (2\pi, 2\pi, 2\pi, 0) \\ \theta_2 = 0.2 \times (2\pi, 2\pi, 2\pi, 0) \\ \theta_3 = 0.3 \times (2\pi, 2\pi, 2\pi, 0) \\ \theta_4 = 0.4 \times (2\pi, 2\pi, 2\pi, 0) \end{cases} \qquad \begin{cases} p_1 = 0.1\sqrt{3} \cdot \frac{2\pi}{L} \simeq 0.17 \cdot \frac{2\pi}{L} \\ p_2 = 0.2\sqrt{3} \cdot \frac{2\pi}{L} \simeq 0.35 \cdot \frac{2\pi}{L} \\ p_3 = 0.3\sqrt{3} \cdot \frac{2\pi}{L} \simeq 0.52 \cdot \frac{2\pi}{L} \\ p_4 = 0.4\sqrt{3} \cdot \frac{2\pi}{L} \simeq 0.69 \cdot \frac{2\pi}{L} \end{cases}$$

Simulations carried out on the DiRAC Extreme Scaling service at the University of Edinburgh using the Hadrons and Grid software packages

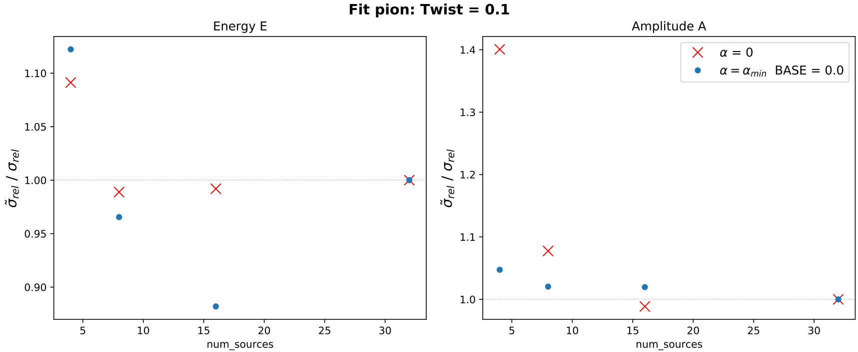
Results for 2pt $C_\pi(t, \boldsymbol{p}_\theta)$ with $\theta = 0.1$



With 8 sources (1/4 of the cost) we get a good error improvement, $\sigma^\alpha_\theta/\sigma_\theta \simeq 1$ (orange data).

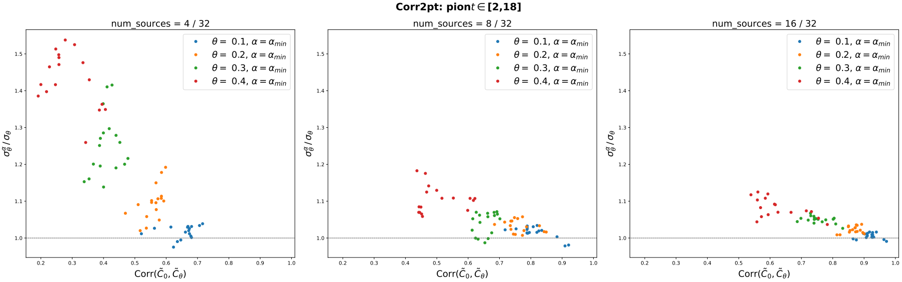
2pt Correlator Fit

$$C_\pi(t) = 2Ae^{-E\frac{T}{2}} \cosh\left[E\left(\frac{T}{2} - t\right)\right]$$



The blue points ($\alpha = \alpha_{min}$) show that the relative error with the new approach is smaller than the standard reduced case. In this case, 8 sources out of 32 gives a good improvement with only 1/4 of the total cost.

2pt improvement as function of correlation



The improvement $\sigma_\theta^\alpha/\sigma_\theta$ (y-axis) is controlled by the correlation $\mathrm{Corr}(\tilde{C}_0, \tilde{C}_\theta)$ between twisted and untwisted correlator. The smaller the twist, the higher the correlation and the better is the improvement $\sigma_\theta^\alpha/\sigma_\theta$.

$K \to \pi$ SEMI-LEPTONIC FORM FACTOR

The matrix element is

$$M_\mu = \langle \pi(p_\pi)| \, V_\mu \, |K(p_K)\rangle = f_+(q^2)(p_K + p_\pi)_\mu + f_-(q^2)(p_K - p_\pi)_\mu \, ,$$

$$M_\mu = 4\sqrt{E_K E_\pi}\sqrt{\frac{C^{\alpha_{min}}_{\mu,K\pi}(t)\, C^{\alpha_{min}}_{\mu,\pi K}(t)}{C_K(t)\, C^{\alpha_{min}}_{\pi}(t)}}$$

The form factors are

$$\begin{cases} f_+(q^2) = \frac{1}{2E_K}\left(M_0 + \frac{E_K - E_\pi}{p_\pi} M_s\right) \\ f_-(q^2) = \frac{1}{2E_K}\left(M_0 - \frac{E_K + E_\pi}{p_\pi} M_s\right) \end{cases}$$

and the twist is on the π.

So far we have tuned α in order to reduce the error on the **correlators** (α_{min}).

We could tune α in order to reduce the error on the **form factors** (α_{opt}).

We consider the matrix elements as functions of αs

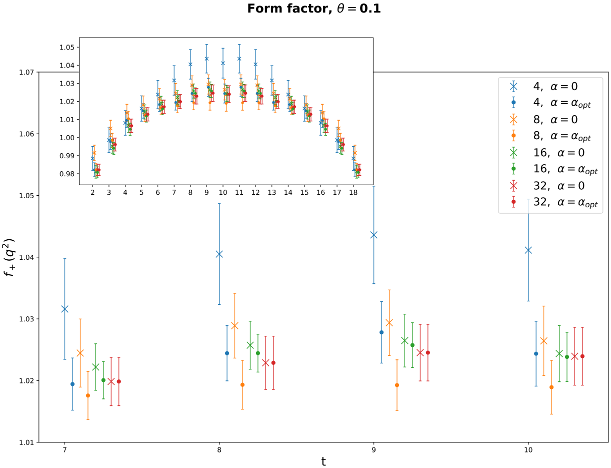
$$M_\mu(\alpha_1, \alpha_2, \alpha_3) \propto \sqrt{\frac{C^{\alpha_1}_{\mu,K\pi}(t)\, C^{\alpha_2}_{\mu,\pi K}(t)}{C_K(t)\, C^{\alpha_3}_{\pi}(t)}}$$

such that the error on the form factors is

$$\sigma_f \to \sigma_f(\boldsymbol{\alpha}) \qquad \text{with} \qquad \boldsymbol{\nabla}_{\boldsymbol{\alpha}} \sigma_f(\boldsymbol{\alpha}) = 0$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$, with n up to 9.

Form factor f_+ optimized



num_sources = **4** / 32		
t	$\tilde{\sigma}_\theta / \sigma_\theta$	$\sigma_\theta^{\alpha_{opt}} / \sigma_\theta$
7.0	2.0813	**1.0789**
8.0	1.8931	**1.0364**
9.0	1.7226	**1.0858**
10.0	1.7604	**1.1208**

num_sources = **8** / 32		
t	$\tilde{\sigma}_\theta / \sigma_\theta$	$\sigma_\theta^{\alpha_{opt}} / \sigma_\theta$
7.0	1.4084	**0.9922**
8.0	1.216	**0.9183**
9.0	1.1569	**0.8947**
10.0	1.201	**0.9299**

num_sources = **16** / 32		
t	$\tilde{\sigma}_\theta / \sigma_\theta$	$\sigma_\theta^{\alpha_{opt}} / \sigma_\theta$
7.0	0.9626	**0.774**
8.0	0.9015	**0.711**
9.0	0.9333	**0.7911**
10.0	0.9711	**0.8512**

The numerical optimization on the form factors $f_+(q^2)$ (in bold) improves significantly the error ($\sigma_\theta^\alpha \lesssim \sigma_\theta$) for at least $\sim 1/4$ of the total number of sources. The fits show a similar behaviour!

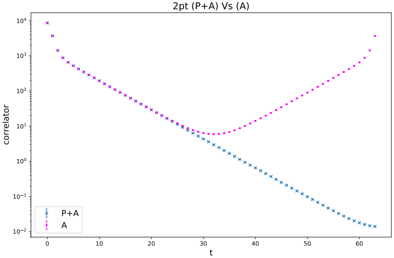
P±A Boundary Conditions

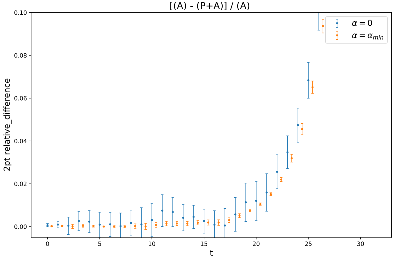
We could consider a similar procedure on a **pion** for the case

Periodic ± **Anti-periodic** Boundary Conditions in **time direction**

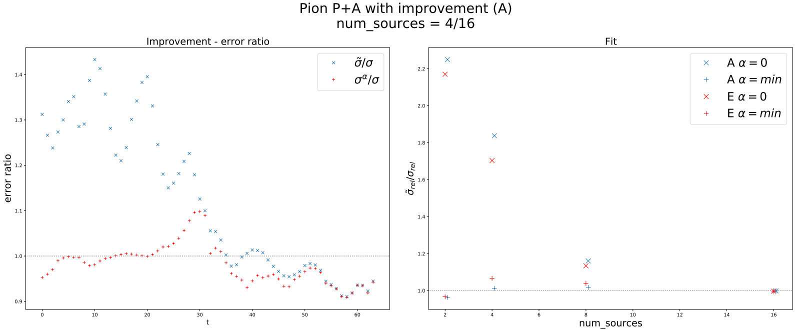
which have the advantage of **eliminating the around-the-world effects** related to the finite volume formulation. As before we can build

$$C^{\alpha}_{P\pm A} = \tilde{C}_{P\pm A} + \alpha(t)(C_A - \tilde{C}_A)$$





P±A Results



With 1/4 of the cost the improvement is already very good in the first half of the lattice, $\tilde{\sigma}^{\alpha}_{P+A} \simeq \sigma_{P+A}$. The fit parameters are basically always equivalent to the case with the full number of sources.

Outlook and future directions

Summary

On correlators (twisting/P±A):

- error is reduced, OK with $\sim \frac{1}{4}$ of total cost
- α_{min} has analytical expression
- central value (BIAS) must be taken into account

On generic observables (twisting):

- error is reduced, OK with $\sim \frac{1}{4}$ of total cost
- α_{opt} is optimized numerically

Next step

- BIAS correction
- Testing P±A to more complex analysis (*e.g.* $K\pi$ scattering)