Pion electric polarizabilities from lattice QCD

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Outline

- **Introduction**
- Obtaining the formula which relate the pion polarizability with the 4-point function
- Lattice calculation
- Conclusion and outlook

In Minkowski space-time, for *neutral pion* (charged pion discuss in later slides):

$$H_{\text{eff}} = -\frac{4\pi}{2}\alpha_\pi E^2 - \frac{4\pi}{2}\beta_\pi B^2 \quad \text{Minkowski} \tag{1}$$

In Euclidean space-time, the electric field absorbs an i coefficient. Therefore, we have

$$H_{\text{eff}} = \frac{4\pi}{2}\alpha_\pi E^2 - \frac{4\pi}{2}\beta_\pi B^2 \tag{2}$$

$$J_\mu(x) = J_\mu(t_x, \vec{x}) = e\Big(e_u \bar{u}(x)\gamma_\mu u(x) + e_d \bar{d}(x)\gamma_\mu d(x) + e_s \bar{s}(x)\gamma_\mu s(x)\Big) \tag{3}$$

where $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu,\nu}$, $e_u = 2/3$, $e_d = e_s = -1/3$, and $\alpha_{\text{QED}} = e^2/(4\pi) \approx 1/137$. We will use Euclidean space-time convention by default in the rest of the talk.

Introduction

There are Chiral Perturbation Theory calculations to two-loop order:

- Charged pion: U. Burgi (hep-ph/9602421, hep-ph/9602429), J. Gasser et al. (hep-ph/0602234)
- Neutral pion: J. Gasser et al. (hep-ph/9401206, hep-ph/0506265)

There are also some lattice calculations with the background field method. Recently, we have

- α_π: H. Niyazi (arXiv:2105.06906)
- β_π: R. Bignell et al (arXiv:2005.10453), H.T Ding et al (arXiv:2008.00493)

There are also attempts to use hadronic tensor (4-point function) to extract polarizablities: M. Burkardt et al (hep-lat/9406009), W. Wilcox (arXiv:2106.02557). Realistic lattice calculations along this direction difficult.

In this work, we derive different position space formulas with the hadronic tensor to obtain the pion electric polarizabilities. We demonstrate these formulas allow efficient lattice calculations and will show some numerical results.

Objective of the derivation

- We are looking for the second order effects on the pion energy due to the static E&M fields.

$$H_{\text{eff}} = \frac{4\pi}{2}\alpha_\pi E^2 - \frac{4\pi}{2}\beta_\pi B^2 \tag{4}$$

- E&M fields are described by the vector potential $A_\mu(x)$. It couples with pion through the vector current $J_\mu(x)$.

$$\alpha_\pi, \beta_\pi \sim \langle\pi|T J_\mu(x) J_\nu(0)|\pi\rangle \tag{5}$$

We start the derivation in a very large, finite volume, periodic box. We can imagine the finite volume box during the derivation to be much larger than the real lattice sizes so the finite volume effects can be neglected. After the final master formula is obtained, we can then analyze possible finite volume effects for realistic lattice calculations.

Outline

- Introduction
- **Obtaining the formula which relate the pion polarizability with the 4-point function**
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Consider the zero momentum neutral pion correlation function ($t_{\rm snk} \gg 0 \gg t_{\rm src}$) in the presence of very smooth and slow varying external vector potential $A_\mu(x) = A_\mu(t_x, \vec{x})$.

$$\begin{aligned} & \langle T\pi(t_{\rm snk})\pi(t_{\rm src})\rangle_{A_\mu} \\ = \quad & \langle T\pi(t_{\rm snk})\pi(t_{\rm src})\rangle \exp\Big(-\frac{1}{L^3}\int_x \frac{4\pi}{2}\big(\alpha_\pi E(x)^2 - \beta_\pi B(x)^2\big)\Big) \end{aligned} \tag{6}$$

That is:

$$1 - \frac{\langle T\pi(t_{\rm snk})\pi(t_{\rm src})\rangle_{A_\mu}}{\langle T\pi(t_{\rm snk})\pi(t_{\rm src})\rangle} = \frac{1}{L^3}\int_x \frac{4\pi}{2}\big(\alpha_\pi E(x)^2 - \beta_\pi B(x)^2\big) \tag{7}$$

Using perturbation theory, we have (notice the term proportion to A vanishes)

$$1 - \frac{\langle T\pi(t_{\rm snk})\pi(t_{\rm src})\rangle_{A_\mu}}{\langle T\pi(t_{\rm snk})\pi(t_{\rm src})\rangle} = \frac{1}{2}\int_x \left(\int_y A_\mu(x+y)A_\nu(y)\right)\frac{\langle T\pi(t_{\rm snk})J_\mu(x)J_\nu(0)\pi(t_{\rm src})\rangle}{\langle T\pi(t_{\rm snk})\pi(t_{\rm src})\rangle} \tag{8}$$

Combine the two results (and use infinite volume pion matrix elements):

$$\int_x \frac{4\pi}{2}\big(\alpha_\pi E(x)^2 - \beta_\pi B(x)^2\big) = \int_x \left(\int_y A_\mu(x+y)A_\nu(y)\right)\frac{1}{2M_\pi}\frac{1}{2}\langle\pi|TJ_\mu(x)J_\nu(0)|\pi\rangle \tag{9}$$

Spatial position independent field

Consider a spatial position independent vector potential $\vec{A}(t_x) = \vec{A}(t_x, \vec{x}) = \vec{A}(x)$. Only electric field is non-zero. Performed the Taylor expansion for A and only keep the leading non-zero term:

$$
\begin{aligned}
I_{\text{eff}} &= \int_{t_x} \frac{4\pi}{2} \alpha_\pi E(t_x)^2 && (10)\\
&= \int_{t_x,\vec{x},t_y} A_j(t_y) \frac{1}{2} t_x^2 \partial_t^2 A_i(t_y) \frac{1}{2M_\pi} \frac{1}{2} \langle \pi | T J_i(t_x, \vec{x}) J_j(0, \vec{0}) | \pi \rangle && (11)\\
&= -\int_{t_y} \frac{1}{2} \partial_t \vec{A}(t_y) \cdot \partial_t \vec{A}(t_y) \int_{t_x,\vec{x}} \frac{1}{3} t_x^2 \frac{1}{2M_\pi} \frac{1}{2} \langle \pi | T \vec{J}(t_x, \vec{x}) \cdot \vec{J}(0, \vec{0}) | \pi \rangle && (12)
\end{aligned}
$$

Therefore, we obtain the master formula for neutral pion.

$$
\alpha_\pi = -\frac{1}{4\pi} \int_{t_x} \int_{\vec{x}} \frac{1}{3} t_x^2 \frac{1}{2M_\pi} \frac{1}{2} \langle \pi | T \vec{J}(t_x, \vec{x}) \cdot \vec{J}(0, \vec{0}) | \pi \rangle \qquad (13)
$$

While the formula is derived assuming a extremely large volume, both in the time and the spatial direction. We can use this formula with a modest lattice size and source-sink time separation. Finite volume errors are exponentially suppressed.

Time independent field

Similarly, we can choose $A_\mu(t,\vec{x}) = A_\mu(\vec{x})$ for a very large range of t, and obtain some different formulas for neutral pion:

$$\alpha_\pi = -\frac{1}{4\pi}\int_{t_x}\int_{\vec{x}}\frac{1}{3}\vec{x}^2\frac{1}{2M_\pi}\frac{1}{2}\langle\pi|TJ_t(t_x,\vec{x})J_t(0,\vec{0})|\pi\rangle \tag{14}$$

$$\beta_\pi = \frac{1}{4\pi}\int_{t_x}\int_{\vec{x}}\frac{1}{6}\vec{x}^2\frac{1}{2M_\pi}\frac{1}{2}\langle\pi|T\vec{J}(t_x,\vec{x})\cdot\vec{J}(0,\vec{0})|\pi\rangle \tag{15}$$

$$= -\frac{1}{4\pi}\int_{t_x}\int_{\vec{x}}\frac{1}{3}\frac{1}{2M_\pi}\frac{1}{2}\langle\pi|T\vec{x}\cdot\vec{J}(t_x,\vec{x})\vec{x}\cdot\vec{J}(0,\vec{0})|\pi\rangle \tag{16}$$

Note that the four point function $\langle\pi|TJ_\mu(x)J_\nu(0)|\pi\rangle$ satisfies current conservation constraints. Different formulas for α_π and β_π can be obtained. They are equivalent due to the constraint. In practical lattice calculations, they may have different finite volume error, discretization error (if we use local current), and statistical error.

Defined through the subtracted Compton tensor:

$$
\begin{aligned}
&\langle\pi|TJ_\mu(t_x,\vec{x})J_\nu(0,\vec{0})|\pi\rangle_S \\
&\quad= \langle\pi|TJ_\mu(t_x,\vec{x})J_\nu(0,\vec{0})|\pi\rangle - \langle\pi|TJ_\mu(t_x,\vec{x})J_\nu(0,\vec{0})|\pi\rangle_{\mathrm{Born}} \qquad (17)
\end{aligned}
$$

See arXiv:1905.05640 sec 4.3, 4.4.

$$T^{\mu\nu} = T_A^{\mu\nu} + T_B^{\mu\nu}. \qquad (114)$$

Here $T_A^{\mu\nu}$ will contain all of the terms in the amplitude which are singular as either $q \to 0$ or $q' \to 0$, together, perhaps, with some additional non-singular terms. $T_B^{\mu\nu}$ will contain everything else. We stress that this separation is not unique in the sense that non-singular terms may be shifted from $T_A^{\mu\nu}$ to $T_B^{\mu\nu}$ and vice versa.

Using, e.g., the soft-photon technique of Ref. [36], one may *define* the generalized Born terms of the virtual Compton scattering amplitude as [37]

$$T_{\mathrm{Born}}^{\mu\nu} = e^2 F(q^2)F(q'^2)\left[2g^{\mu\nu} - \frac{(2p_i+q)^\mu(2p_f+q')^\nu}{(p_i+q)^2 - M_\pi^2} - \frac{(2p_i-q')^\nu(2p_f-q)^\mu}{(p_i-q')^2 - M_\pi^2}\right], \qquad (115)$$

where F denotes the on-shell electromagnetic form factor. The s- and u-channel terms provide the singular contributions proportional to $1/(p_i \cdot q)$ and $1/(p_i \cdot q')$, respectively, whereas the term proportional to the metric tensor $g^{\mu\nu}$ makes the generalized Born terms gauge invariant. Moreover, $T_{\mathrm{Born}}^{\mu\nu}$ is symmetric under photon

The subtraction of Born term for the charged pion can be evaluated for the master formula:

$$\alpha_{\pi^\pm} = -\frac{1}{4\pi}\int_{t_x}\int_{\vec{x}}\frac{1}{3}t_x^2\frac{1}{2M_\pi}\frac{1}{2}\langle\pi^\pm|T\vec{J}(t_x,\vec{x})\cdot\vec{J}(0,\vec{0})|\pi^\pm\rangle_S \quad (18)$$

$$= -\int_{t_x,\vec{x}}\frac{t_x^2}{24\pi}\frac{1}{2M_\pi}\langle\pi^\pm|T\vec{J}(t_x,\vec{x})\cdot\vec{J}(0,\vec{0})|\pi^\pm\rangle - \alpha_{\pi^\pm}^{\mathrm{Born}} \quad (19)$$

where $\alpha_{\pi^\pm}^{\mathrm{Born}} = -\alpha_{\mathrm{QED}}\frac{r_\pi^2}{3M_\pi}$, and $r_\pi = 0.659(4)$ fm (PDG) is the $\pi^\pm$ charge radius.

Note that the single pion intermediate states do not contribute in the above matrix elements. This is not true for the other three formulas, in which case we need to subtract the Born term matrix elements in the same finite volume lattice and then perform the coordinate integration to ensure exponentially suppressed finite volume effects.

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Lattice calculation

Use the RBC-UKQCD 48I and 64I physical pion ensemble.

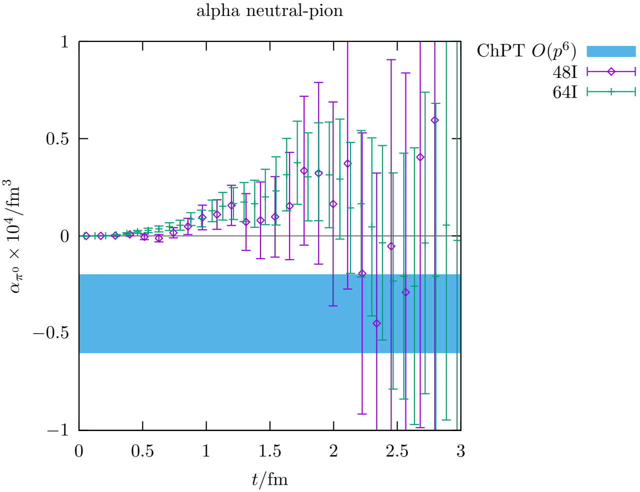
- $m_\pi = 0.139$ GeV.
  Calculation use partially quenched pion mass 0.135 GeV.
- 48I: $a^{-1} = 1.730$ GeV, 64I: $a^{-1} = 2.359$ GeV.
- 48I: $L = 5.48$ fm, 64I: $L = 5.35$ fm.

Polarization of the sea quark is not included in the calculation (most disconnected diagram is not included yet).

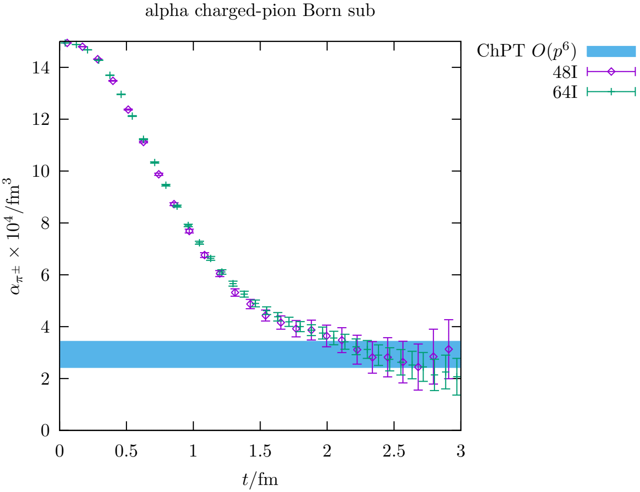
$$\alpha_\pi(t) = -\int_{-t<t_x<t} \int_{\vec{x}} \frac{t_x^2}{24\pi} \frac{1}{2M_\pi} \langle \pi | T \vec{J}(t_x, \vec{x}) \cdot \vec{J}(0, \vec{0}) | \pi \rangle - \alpha_\pi^{\text{Born}} \tag{20}$$

We will plot the results as a function of t, and $\alpha_\pi = \alpha_\pi(t \to +\infty)$.

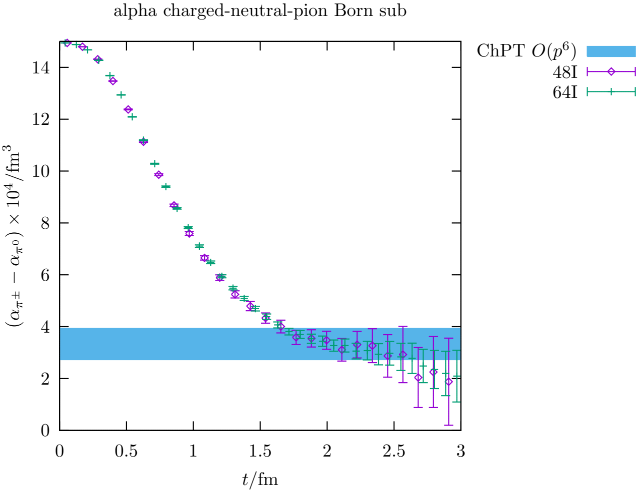
Preliminary results - α_{π^0}



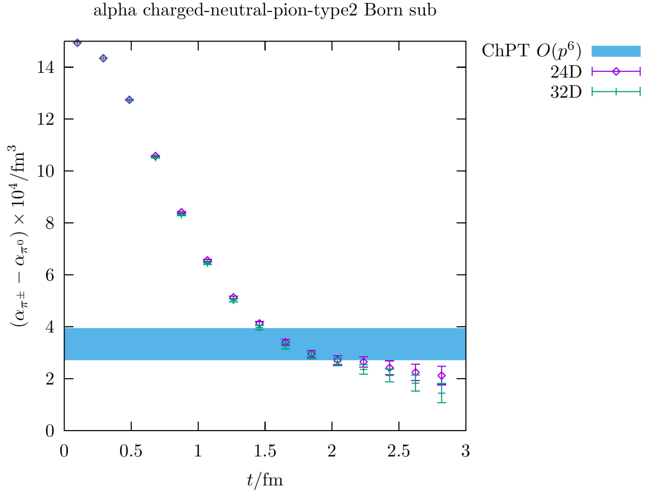
Some disconnected diagrams are not included yet.



Some disconnected diagrams are not included yet.



Only two types of diagrams remain in the difference. Both are included.



Finite volume effects study with 24D/32D (4.7 fm v.s. 6.3 fm) ensembles.

Only the connected diagram.

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Conclusion and outlook

- Derived the formula to obtain the pion electric and magnetic polarizabilities.
- Preliminary results for pion electric polarizabilities α_π is obtained at physical pion mass. The result is consistent with ChPT predictions with competitive and improvable accuracy. We can expect the precision of lattice calculation to improve in the future.
- We plan to calculate the missing disconnected diagrams and the kaon polarizabilities.

- Calculation performed by reusing propagators generated for the lattice HLbL calculation at MIRA.

Thank You!

Detailed derivations

Using perturbation theory, we have

$$\begin{aligned}
&\langle T\pi(t_{\text{snk}})\pi(t_{\text{src}})\rangle_{A_\mu} \\
&= \langle T\pi(t_{\text{snk}})\pi(t_{\text{src}})\rangle + \langle T\pi(t_{\text{snk}}) \int_x iA_\mu(x) J_\mu(x)\pi(t_{\text{src}})\rangle \\
&\quad + \frac{1}{2}\langle T\pi(t_{\text{snk}}) \int_x iA_\mu(x)J_\mu(x) \int_y iA_\nu(y)J_\nu(y)\pi(t_{\text{src}})\rangle
\end{aligned} \tag{21}$$

Recall for neutral pion, we have

$$\langle \pi(t_{\text{snk}})J_\mu(x)\pi(t_{\text{src}})\rangle = 0 \tag{22}$$

Also with the translation invariance of the matrix elements and then shift the integration for x,

$$\begin{aligned}
&\frac{1}{2}\langle T\pi(t_{\text{snk}}) \int_x iA_\mu(x)J_\mu(x) \int_y iA_\nu(y)J_\nu(x)\pi(t_{\text{src}})\rangle \\
&= -\int_x \left(\int_y A_\mu(x+y)A_\nu(y) \right) \frac{1}{2}\langle T\pi(t_{\text{snk}})J_\mu(x)J_\nu(0)\pi(t_{\text{src}})\rangle
\end{aligned} \tag{23}$$

Therefore, we have:

$$1 - \frac{\langle T\pi(t_{\mathrm{snk}})\pi(t_{\mathrm{src}})\rangle_{A_\mu}}{\langle T\pi(t_{\mathrm{snk}})\pi(t_{\mathrm{src}})\rangle} = \int_x \left(\int_y A_\mu(x+y)A_\nu(y) \right) \frac{1}{2} \frac{\langle T\pi(t_{\mathrm{snk}})J_\mu(x)J_\nu(0)\pi(t_{\mathrm{src}})\rangle}{\langle T\pi(t_{\mathrm{snk}})\pi(t_{\mathrm{src}})\rangle} \quad (24)$$

Combining with the definitions of the pion polarizabilities, we have:

$$\frac{1}{L^3}\int_x \frac{4\pi}{2}\alpha_\pi E(x)^2 - \frac{1}{L^3}\int_x \frac{4\pi}{2}\beta_\pi B(x)^2 = \int_x \left(\int_y A_\mu(x+y)A_\nu(y) \right) \frac{1}{2} \frac{\langle T\pi(t_{\mathrm{snk}})J_\mu(x)J_\nu(0)\pi(t_{\mathrm{src}})\rangle}{\langle T\pi(t_{\mathrm{snk}})\pi(t_{\mathrm{src}})\rangle} \quad (25)$$

To be precise, we actually need to subtract the vacuum contribution to remove the effect of the vacuum polarization.

$$\frac{\langle T\pi(t_{\mathrm{snk}})J_\mu(x)J_\nu(0)\pi(t_{\mathrm{src}})\rangle}{\langle T\pi(t_{\mathrm{snk}})\pi(t_{\mathrm{src}})\rangle} \to \frac{\langle T\pi(t_{\mathrm{snk}})J_\mu(x)J_\nu(0)\pi(t_{\mathrm{src}})\rangle}{\langle T\pi(t_{\mathrm{snk}})\pi(t_{\mathrm{src}})\rangle} - \langle T J_\mu(x)J_\nu(0)\rangle \quad (26)$$

We will assume this subtraction in later discussion without explicitly writing it down.

Note that, with infinite volume state normalization condition, we have:

$$\frac{1}{2M_\pi}\frac{1}{2}\langle\pi|TJ_\mu(x)J_\nu(0)|\pi\rangle = L^3\frac{1}{2}\frac{\langle T\pi(t_{\mathrm{snk}})J_\mu(x)J_\nu(0)\pi(t_{\mathrm{src}})\rangle}{\langle T\pi(t_{\mathrm{snk}})\pi(t_{\mathrm{src}})\rangle} \tag{27}$$

We can then rewrite the finite volume results in terms of the infinite volume convention expression:

$$\begin{aligned}\int_x \frac{4\pi}{2}\alpha_\pi E(x)^2 - \int_x \frac{4\pi}{2}\beta_\pi B(x)^2 \\ = \int_x \left(\int_y A_\mu(x+y)A_\nu(y)\right)\frac{1}{2M_\pi}\frac{1}{2}\langle\pi|TJ_\mu(x)J_\nu(0)|\pi\rangle\end{aligned} \tag{28}$$

where the normalization of the infinite volume state is

$$\langle\pi(\vec{p}')|\pi(\vec{p})\rangle = (2\pi)^3 2E_{\pi,\vec{p}}\delta^{(3)}(\vec{p}-\vec{p}') \tag{29}$$

Spatial position independent field

Consider a spatial position independent vector potential $\vec{A}(t_x) = \vec{A}(t_x, \vec{x}) = \vec{A}(x)$. Only electric field is non-zero, therefore we have:

$$
\begin{aligned}
I_{\text{eff}} &= \int_{t_x} \frac{4\pi}{2} \alpha_\pi E(t_x)^2 & (30) \\
&= \int_{t_x,\vec{x}} \left(\int_{t_y} A_i(t_x + t_y) A_j(t_y) \right) \frac{1}{2M_\pi} \frac{1}{2} \langle \pi | T J_i(t_x, \vec{x}) J_j(0, \vec{0}) | \pi \rangle & (31)
\end{aligned}
$$

Since the time dependence is very mild, we have:

$$
\begin{aligned}
I_{\text{eff}} &= \int_{t_x,\vec{x},t_y} A_j(t_y) \Big(A_i(t_y) + t_x \partial_t A_i(t_y) + \frac{1}{2} t_x^2 \partial_t^2 A_i(t_y) \Big) \\
&\qquad \times \frac{1}{2M_\pi} \frac{1}{2} \langle \pi | T J_i(t_x, \vec{x}) J_j(0, \vec{0}) | \pi \rangle & (32)
\end{aligned}
$$

The first and the second term vanishes. The third term is proportion to the E&M field strength square, which can be matched with the polarizability expression.

Spatial position independent field

$$I_{\text{eff}} = \int_{t_x} \frac{4\pi}{2}\alpha_\pi E(t_x)^2 \tag{33}$$

$$= \int_{t_x,\vec{x},t_y} A_j(t_y)\Big(A_i(t_y) + t_x\partial_t A_i(t_y) + \frac{1}{2}t_x^2\partial_t^2 A_i(t_y)\Big) \times \frac{1}{2M_\pi}\frac{1}{2}\langle\pi|TJ_i(t_x,\vec{x})J_j(0,\vec{0})|\pi\rangle \tag{34}$$

The first term vanishes due to current conservation and vanishing boundary terms.

$$\int_x \langle\pi|TJ_\mu(x)J_\nu(0,\vec{0})|\pi\rangle = \int_x \langle\pi|T\partial_\rho\big(x_\mu J_\rho(x)\big)J_\nu(0,\vec{0})|\pi\rangle = 0 \tag{35}$$

The second term vanishes due to spatial and time reflection symmetry.

$$\int_{\vec{x}} \langle\pi|TJ_i(t_x,\vec{x})J_j(0,\vec{0})|\pi\rangle = \frac{1}{3}\delta_{i,j}\int_{\vec{x}}\langle\pi|T\vec{J}(t_x,\vec{x})\cdot\vec{J}(0,\vec{0})|\pi\rangle \tag{36}$$

$$= \frac{1}{3}\delta_{i,j}\int_{\vec{x}}\langle\pi|T\vec{J}(-t_x,\vec{x})\cdot\vec{J}(0,\vec{0})|\pi\rangle \tag{37}$$

Therefore, only the third term remains.

Spatial position independent field

$$
\begin{aligned}
I_{\text{eff}} &= \int_{t_x} \frac{4\pi}{2} \alpha_\pi E(t_x)^2 & (38) \\
&= \int_{t_x,\vec{x},t_y} A_j(t_y) \frac{1}{2} t_x^2 \partial_t^2 A_i(t_y) \frac{1}{2M_\pi} \frac{1}{2} \langle \pi | T J_i(t_x,\vec{x}) J_j(0,\vec{0}) | \pi \rangle & (39) \\
&= -\int_{t_y} \frac{1}{2} \partial_t \vec{A}(t_y) \cdot \partial_t \vec{A}(t_y) \int_{t_x,\vec{x}} \frac{1}{3} t_x^2 \frac{1}{2M_\pi} \frac{1}{2} \langle \pi | T \vec{J}(t_x,\vec{x}) \cdot \vec{J}(0,\vec{0}) | \pi \rangle & (40)
\end{aligned}
$$

Therefore:

$$
\alpha_\pi = -\frac{1}{4\pi} \int_{t_x} \int_{\vec{x}} \frac{1}{3} t_x^2 \frac{1}{2M_\pi} \frac{1}{2} \langle \pi | T \vec{J}(t_x,\vec{x}) \cdot \vec{J}(0,\vec{0}) | \pi \rangle \qquad (41)
$$

While the formula is derived assuming a extremely large lattice, both in the time and the spatial direction. This final formula can be calculated using a modest lattice size and source-sink time separation with exponentially suppressed finite volume errors.

Charged pion polarizabilities Born term

Based on the Born term definition, we have:

$$T^{\text{Born}}_{\mu,\nu}(q_t,\vec{q}) = \int_{t_x,\vec{x}} e^{iq_t t_x - i\vec{q}\cdot\vec{x}} \langle \pi | T J_\mu(t_x,\vec{x}) J_\nu(0,\vec{0}) | \pi \rangle \tag{42}$$

$$= e^2 F^2(q_t^2+\vec{q}^2)\Big(2\delta_{\mu,\nu} - \frac{(2p+q)_\mu(2p+q)_\nu}{(p+q)^2+M_\pi^2} - \frac{(2p-q)_\mu(2p-q)_\nu}{(p-q)^2+M_\pi^2}\Big) \tag{43}$$

where $p=(iM_\pi,\vec{0})$. Therefore:

$$\frac{\partial^2}{\partial q_t^2} T^{\text{Born}}_{k,k}(q_t,\vec{0})\bigg|_{q_t=0} = -\int_{t_x,\vec{x}} t_x^2 \langle \pi | T J_k(t_x,\vec{x}) J_k(0,\vec{0}) | \pi \rangle_{\text{Born}} \tag{44}$$

$$= \frac{\partial^2}{\partial q_t^2}\Big(e^2 F^2(q_t^2) 2\delta_{k,k}\Big) \tag{45}$$

For charged pion, we have $F_{\pi^\pm}(q^2) \approx 1 - r_\pi^2 q^2/6$, where $r_\pi = 0.659(4)$ fm (PDG), is the $\pi^\pm$ charge radius. Combining the above equations, we obtain the expression for $\alpha_{\pi^\pm}$:

$$\alpha_{\pi^\pm} = -\int_{t_x,\vec{x}} \frac{t_x^2}{24\pi}\frac{1}{2M_\pi} \langle \pi^\pm | T \vec{J}(t_x,\vec{x}) \cdot \vec{J}(0,\vec{0}) | \pi^\pm \rangle - \alpha^{\text{Born}}_{\pi^\pm} \tag{46}$$

where $\alpha^{\text{Born}}_{\pi^\pm} = -\alpha_{\text{QED}}\frac{r_\pi^2}{3M_\pi} = -14.94(18)\times 10^{-4}\ \text{fm}^3$.

Diagrams for $\alpha_{\pi^\pm} - \alpha_{\pi^0}$