Lattice QCD calculation of the two-photon exchange contribution to the muonic-hydrogen Lamb shift

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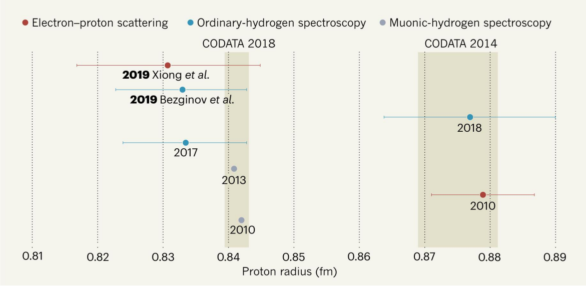
Outline

1. Background
   - Proton radius puzzle
   - Theory for μH Lamb shift & two-photon exchange
2. Lattice QCD approach
   - Master formula
   - Removal of the IR divergence
   - Preliminary results
3. Summary & outlook



JP Karr, Nature, 2019, 575, 61

- Puzzle raised in 2010 (R. Pohl et al. Nature 466 (2010) 213)
  - over 5σ discrepancy between muon and electron based measurements.
- After more than 10 years, the origin is still not very clear
  - is it a problem with the electron based measurements?
  - is it a hadronic uncertainty in μH spectroscopy?
  - or, is it new physics?

- Theory for μH Lamb shift splitting (Science 339 (2013) 417. Ann. of Phy. 331 (2013), 127)

$$\Delta E_{\text{2S-2P}} = \Delta E_{\text{QED}} + \Delta E_{\text{Proton size}} + \Delta E_{\text{TPE}}$$
$$= 206.0336(15) - 5.2275(10) r_p^2 + 0.0332(20)$$

(units are in meV and fm)

- Discrepancy is ~ 0.3meV.
- Two-photon exchange: biggest source of theoretical uncertainty.

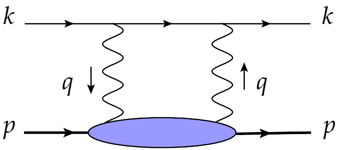
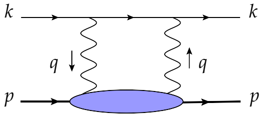


figure from PRA 84, 020102 (2011)

- How well do we know about the TPE contribution?

- All external lines have zero three-momentum.
- Kinematics: 2 variables (inside the loop)
  $q^2 = -Q^2$, $\nu = p \cdot q / M = q_0$.



Bottom part of the diagram: Compton tensor

$$T^{\mu\nu} = \frac{i}{8\pi M} \int \mathrm{d}^4 x e^{iqx} \langle p | \mathcal{T}[j^\mu(x) j^\nu(0)] | p \rangle$$

$$= \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2} \right) T_1(\nu, Q^2) + \left(p^\mu - \frac{p \cdot q}{q^2} q^\mu \right) \left(p^\nu - \frac{p \cdot q}{q^2} q^\nu \right) \frac{T_2(\nu, Q^2)}{M^2}$$

TPE contribution in terms of T_1 and T_2 is given by

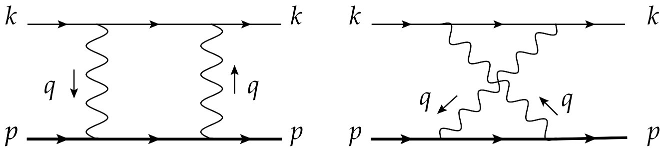
$$\Delta E = \frac{8\alpha^2 m}{\pi} |\phi_n(0)|^2 \int \mathrm{d}^4 Q \frac{(Q^2 + 2Q_0^2) T_1(iQ_0, Q^2) - (Q^2 - Q_0^2) T_2(iQ_0, Q^2)}{Q^4 (Q^4 + 4m^2 Q_0^2)}$$

This box diagram is essentially IR divergent.

$\Rightarrow$ terms from iterations of lower order contributions need to be subtracted.

IR subtraction

- IR divergence occurs in elastic contribution.



- Terms need to be subtracted: point particle with charge radius

1) point-like proton contribution (form factor $F_D = 1, \; F_P = 0$)

$$T_1 = \frac{M}{\pi} \frac{\nu^2}{Q^4 - 4M^2\nu^2}, \qquad T_2 = \frac{M}{\pi} \frac{Q^2}{Q^4 - 4M^2\nu^2}$$

2) charge radius term from third Zemach moment contribution

$$\Delta E_{\text{3rd Zemach moment}} = 16 m_r \alpha^2 |\phi_n(0)|^2 \int \frac{\mathrm{d}Q}{Q^4} \left[G_E^2(Q^2) - 1 - 2Q^2 G_E'(0) \right]$$

Carlson & Vanderhaeghen. PRA 84, 020102 (2011)

Conventional method: dispersive analysis

- T_i can be reconstructed using dispersion relation.
- Im T_i is related to data: form factors and structure functions

$$T_1(\nu, Q^2) = T_1(0, Q^2) + \frac{\nu^2}{\pi} \int_{\nu_{\text{el}}^2}^{\infty} \mathrm{d}\nu'^2 \frac{\text{Im } T_1(\nu, Q^2)}{\nu'^2(\nu'^2 - \nu^2)}$$

$$T_2(\nu, Q^2) = \frac{1}{\pi} \int_{\nu_{\text{el}}^2}^{\infty} \mathrm{d}\nu'^2 \frac{\text{Im } T_2(\nu, Q^2)}{\nu'^2 - \nu^2}$$

But T_1 requires a once subtracted dispersion relation (usually at $\nu = 0$).

- To date, the subtraction function $T_1(0, Q^2)$ is not well-constrained by scattering data and must be modeled.

$\Rightarrow$ potentially underestimate the hadronic uncertainty.

- Lattice QCD can determine the full TPE contribution.

Lattice QCD approach: master formula (naive)

- On lattice, we prefer to rewrite the ΔE in terms of T_{00} and $\sum_i T_{ii}$

$$\Delta E = \frac{8m\alpha^2}{\pi}|\phi_n(0)|^2 \int \mathrm{d}^4 Q \frac{-(Q^2+Q_0^2)T_{00} - Q_0^2 \sum_i T_{ii}}{Q^4(Q^4+4m^2Q_0^2)}$$

- Compton tensor in Euclidean space

$$T_{\mu\nu} = \frac{1}{8\pi M}\int \mathrm{d}^4 x e^{iQx} \underbrace{\langle p|\mathcal{T}[j_\mu(x)j_\nu(0)]|p\rangle}_{H_{\mu\nu}(\vec{x},t)}$$

- We get

$$\Delta E = \frac{2m\alpha^2}{\pi M}|\phi_n(0)|^2 \int \mathrm{d}^4 x\ \omega_1(\vec{x},t)H_1(\vec{x},t) + \omega_2(\vec{x},t)H_2(\vec{x},t)$$

with $H_1(\vec{x},t) = H_{00}(\vec{x},t)$, $H_2(\vec{x},t) = \sum_i H_{ii}(\vec{x},t)$, and

$$\omega_i(\vec{x},t) = -\int \frac{\mathrm{d}Q^2}{Q^2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \frac{\cos(Qt\sin\theta)j_0(Q|\vec{x}|\cos\theta)}{Q^2+4m^2\sin^2\theta} \times \begin{cases} 1-\sin^4\theta, & i=1 \\ \sin^2\theta(1-\sin^2\theta), & i=2 \end{cases}$$

- These weight functions are IR divergent.

Divergence from weight functions

- IR divergence only occurs in elastic (ground-state) contribution.

$\Rightarrow$ ω_i should be IR finite.

- E.g, for T_{00}, we have

$$T_{00} = \frac{1}{8\pi M} \int \mathrm{d}^4 x e^{iQ_0 t} e^{i\vec{Q}\cdot\vec{x}} H_{00}(\vec{x}, t)$$

But Lorentz structure shows

$$T_{00} = -\frac{Q^2 - Q_0^2}{Q^2} T_1 + \left(\frac{Q^2 - Q_0^2}{Q^2}\right)^2 T_2$$

therefore T_{00} vanishes at $Q = (Q_0, \vec{0})$, we can modify it to

$$T_{00} = \frac{1}{8\pi M} \int \mathrm{d}^4 x e^{iQ_0 t} (e^{i\vec{Q}\cdot\vec{x}} - 1) H_{00}(\vec{x}, t)$$

- the weight function is then given by

$$\omega_1(\vec{x}, t) = -\int \frac{\mathrm{d}Q^2}{Q^2} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta \frac{1 - \sin^4\theta}{Q^2 + 4m^2 \sin^2\theta} \cos(Qt\sin\theta) \left[j_0(Q|\vec{x}|\cos\theta) - 1\right]$$

Now it is IR finite.

Divergence from coordinate-space integral

- ΔE is still IR divergent since we haven't subtract anything non-trivial.

$\Rightarrow$ The coordinate-space integral must be divergent.

- Infinite-volume reconstruction (IVR) method. Feng, Jin, PRD 100, 094509 (2019)

idea: divide the time integral into two regions (assuming t_s is large enough)

$$\Delta E = \int_{|t|<t_s} \mathrm{d}^4 x \; \omega(\vec{x},t) H(\vec{x},t) + \underbrace{\int_{|t|>t_s} \mathrm{d}^4 x \; \omega(\vec{x},t) H(\vec{x},t)}_{\text{Ground-state dominance}}$$

$H(|t|>t_s)$ can be reconstructed by $H(t=t_s)$, leads to

$$\Delta E(t_s) = \int_{|t|<t_s} \mathrm{d}^4 x \; \omega(\vec{x},t) H(\vec{x},t) + \int \mathrm{d}^3 \vec{x} \; L(\vec{x},t_s) H(\vec{x},t_s)$$

We find $L(\vec{x},t_s)$ is still IR divergent.

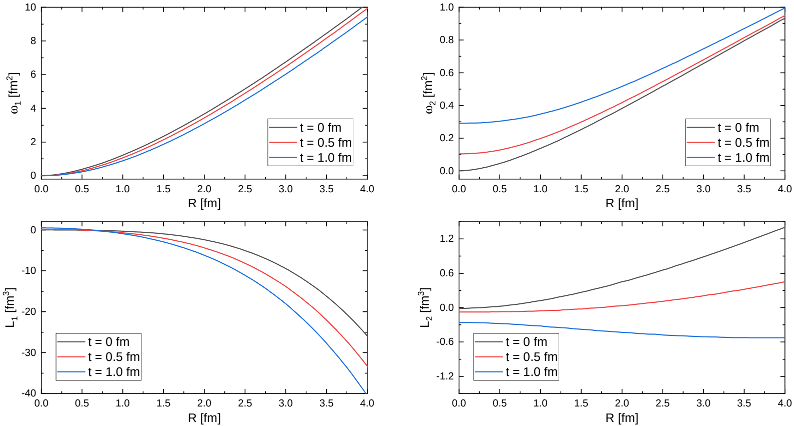
- The terms to be subtracted can also be represented by $H(\vec{x},t_s)$.

$\Rightarrow$ IR divergence can be removed in a straightforward way.

Master formula & Weight functions

- Master formula

$$\Delta E = \frac{2m\alpha^2}{\pi M}|\phi_n(0)|^2 \sum_{i=1,2} \Big[\underbrace{\int_{|t|<t_s} \mathrm{d}^4x\ \omega_i(\vec{x},t) H_i(\vec{x},t)}_{\text{short distance}} + \underbrace{\int \mathrm{d}^3\vec{x} L_i(\vec{x},t_s) H_i(\vec{x},t_s)}_{\text{long distance}} \Big]$$



- Long-distance (L_1) dominated → Signal-to-noise problem.

Ensemble & correlation function

Ensemble	m_π[MeV]	m_p[MeV]	L/a	T/a	a[fm]	N_{conf}
24D	141.7(2)	935(5)	24	64	0.1944	124

- Domain wall fermion ensemble generated by RBC/UKQCD.

  T. Blum et al., PRD 93, 074505 (2016)

- Correlation function constructed with random field selection method

  Li, Xia, et al., PRD 103, 014514 (2021)

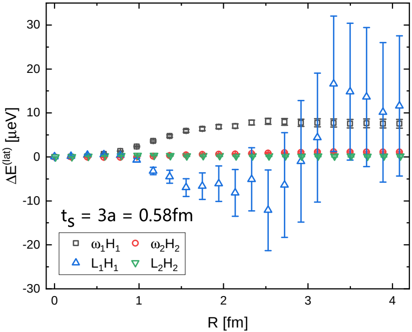
$$C^{4\mathrm{pt}}_{\mu\nu}(t,\vec{x}) = \sum_{\vec{x}_s,\vec{x}_0} \langle \mathcal{P}[O_p(t+\Delta t,\vec{x}_s) j_\mu(t,\vec{x}) j_\nu(0) O_p^\dagger(-\Delta t,\vec{x}_0)]\rangle$$

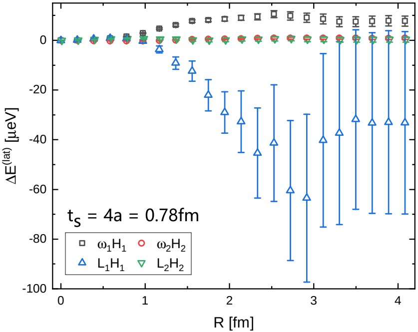
currently we choose $\Delta t = 2a = 0.39\mathrm{fm}$.

- Connected diagrams only.
- Same set up for proton EM polarizability — X-H Wang @ Tues 9:30 UTC.

Preliminary results

- ΔE as a function of spatial integral range R





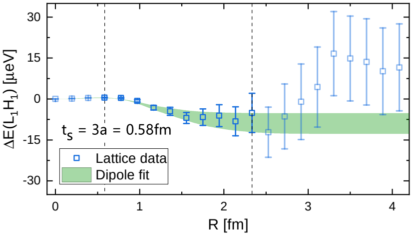
- ΔE dominated by $L_1(\vec{x}, t_s)H_1(\vec{x}, t_s)$ term — fluctuates at large $|\vec{x}|$.
- Long-distance → ground-state → proton form factors.

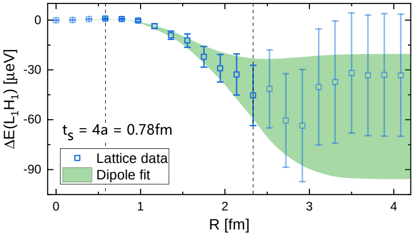
Preliminary results

- Fit $\Delta E[L_1 H_1] = \sum_{|\vec{x}|<R} L_1(\vec{x}, t_s) H_1(\vec{x}, t_s)$ with dipole form factor

$$G_E(Q^2) = \frac{G_E(0)}{(1 + Q^2 r_E^2/12)^2}$$

fitting range $R = 3a - 12a$.





	$G_E(0)$	r_E[fm]	Ground-state dominance?
$t_s = 3a$	0.755(68)	0.439(95)	×
$t_s = 4a$	0.95(12)	0.93(27)	should check $t_s = 5a$, but too noisy
Physical	1	0.8409(4) μH / 0.875(6) ep	

Preliminary results

- At $t_s = 4a$, we get $\Delta E_{\text{TPE}} = -50(37)\mu\text{eV}$

More specifically, $\Delta E^{(\text{SD})}_{\text{TPE}} = 8.6(2.1)\mu\text{eV}$, $\Delta E^{(\text{LD})}_{\text{TPE}} = -58.3(37.5)\mu\text{eV}$

- Possible improvement:

Long-distance 4pt correlation function $\rightarrow$ 3pt correlation function

- E.g, still using the dipole form as a simple estimate

$$G_E(Q^2) = \frac{1}{(1 + Q^2 r_E^2/12)^2}$$

with $r_E = 0.85(5)\text{fm}$ (achieveable with 3pt function), we can get

$$\Delta E^{(\text{LD})}_{\text{TPE}} = -55.2(5.4)\mu\text{eV}$$

- Comparison: μH experiment uses $\Delta E_{\text{TPE}} = -33.2(2.0)\mu\text{eV}$

Discrepancy is $\sim -300\mu\text{eV}$.

Summary & outlook

What we have done:

- The master formula for LQCD calculation is derived;
- In the framework of IVR, IR divergence can be removed naturally;
- Preliminary result is encouraging, but also suffers from signal-to-noise problem.

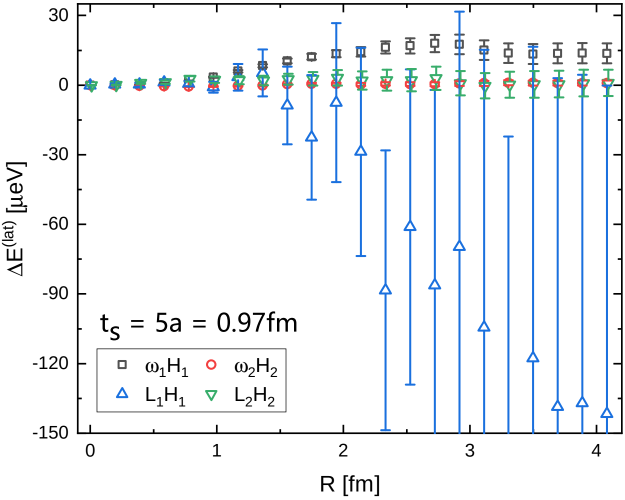
Future work:

- handle long-distance contribution with 3pt correlation function;
- Use ensembles with finer lattice spacing;
- Control excited-state contamination and lattice systematic uncertainties;

Thank you!

Backup Slides

$t_s = 5a$



30
0
-30
-60
-90
-120
-150
$\Delta E^{(lat)}$ [μeV]
0
1
2
3
4
R [fm]
t_S = 5a = 0.97fm
$\omega_1 H_1$
$\omega_2 H_2$
$L_1 H_1$
$L_2 H_2$