Structure-Dependent Electromagnetic Finite-Size Effects

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QED in a finite volume

- Difficult to define charged states in finite volume with periodic boundary conditions (Gauss' law)
- $\mathrm{QED_L}$ [Hayakawa,Uno 2008]: Photon zero-mode subtracted on every time slice

$$\sum_{\mathbf{k}} \longrightarrow \sum_{\mathbf{k}}{}' = \sum_{\mathbf{k}\neq 0}$$

- Massless photon $\implies$ Finite-size effects (FSEs) in observable $\mathcal{O}(L)$:

$$\Delta\mathcal{O}(L) = \mathcal{O}(L) - \mathcal{O}_{\mathrm{IV}} = C_0 + C_{\log}\log m_P L + C_1 \frac{1}{m_P L} + C_2 \frac{1}{(m_P L)^2} + \ldots$$

- Scaling in L is observable-dependent: e.g. self-energy $C_0 = C_{\log} = 0$
- Coefficients depend on physical particle properties: masses, charges, structure (form-factors): Point-like + structure-dependent

What we do

- We want analytical control of FSEs in a model-independent, relativistic set-up including structure-dependence
- QFT Correlators: Vertex functions from Ward identities contain unphysical terms that have to cancel at all orders:
  z_n, f_n [BMW 2015; RM-123 2017]
- What we do:
  1. Derive leading structure-dependence in self-energy $(1/L^3)$ and leptonic decays $(1/L^2)$ $\longrightarrow$ Only physical quantities appear
  2. z_n, f_n always cancel $\longrightarrow$ Skip them from the start?

Pseudoscalar self-energy

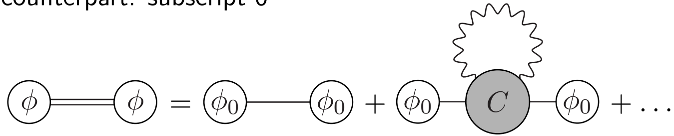
- Study Euclidean 2-point functions (QCD+QED), $p^2 = p_4^2 + \mathbf{p}^2$

$$C_2^{\infty}(p) = \int d^4x \, \langle 0|\mathrm{T}[\phi(x)\phi^\dagger(0)]|0\rangle e^{-ip\cdot x}$$

$$Z_P \equiv \langle 0|\phi(0)|P,\mathbf{p}\rangle$$

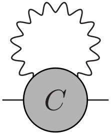
- QCD-only counterpart: subscript 0

$C_2^{\infty}(p) =$



$$= \frac{Z_{P,0}^2}{p^2 + m_{P,0}^2 - \Sigma_0(p^2) - \Sigma(p^2)}$$

$\Sigma(p^2) =$

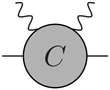


Pseudoscalar self-energy

- Power-law FSEs given by

$$\Delta m_P^2(L) = m_P(L)^2 - m_P^2 = -\left[\Sigma^L(-m_{P,0}^2, \mathbf{0}) - \Sigma(-m_{P,0}^2)\right]$$
$$= -\frac{e^2}{2} \lim_{p_4^2 \to -m_{P,0}^2} \left(\frac{1}{L^3} \sum_{\mathbf{k}}{}' - \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \right) \int \frac{\mathrm{d}k_4}{2\pi} \frac{C_{\mu\mu}(p,k,-k)}{k^2} \bigg|_{\mathbf{p}=\mathbf{0}}$$

- Full FSEs can be related to the *physical* Compton scattering amplitude

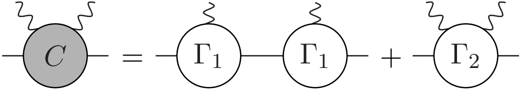


$$= C_{\mu\nu}(p,k,-k)$$

$$\lim_{p^2 \to -m_{P,0}^2} C_{\mu\nu}(p,k,-k) = e^2 \int \mathrm{d}^4x \, e^{-ik\cdot x} \, \langle P, \mathbf{p}| \, T\left\{J_\mu(x) J_\nu(0)\right\} |P, \mathbf{p}\rangle$$

The Compton scattering amplitude

- Decompose into irreducible vertex functions Γ_1, Γ_2



$$C_{\mu\mu}(p,k,-k) = \Gamma_\mu(p+k,-k)D_0(p+k)\Gamma_\mu(p,k) + \Gamma_\mu(p-k,k)D_0(p-k)\Gamma_\mu(p,-k) + \Gamma_{\mu\mu}(p,k,-k)$$

- Amplitude $C_{\mu\nu}(p,k,-k)$ satisfies Ward identities:
  - Γ_μ and $\Gamma_{\mu\nu}$ must satisfy these, but arbitrary separation!
- Can constrain the form of vertex functions from Ward identities, e.g.

$$k_\mu \Gamma^\mu(p,k) = D(p+k)^{-1} - D(p)^{-1}$$

- Full propagator ($Z(p^2)$: z_n)

$$D(p) = \frac{Z(p^2)}{p^2 + m_P^2}$$

Decomposing vertex functions

- Form-factor decomposition (structure-dependence!)

$$\Gamma_\mu(p,k) = (2p+k)_\mu\, F(k^2,(p+k)^2,p^2) + k_\mu\, G(k^2,(p+k)^2,p^2)$$

$$F^{(1,0,0)}(0,-m_P^2,-m_P^2) \equiv F'(0) = -\left\langle r_P^2 \right\rangle/6$$

- Ward identity yields G as a function of F and

$$F(0,p^2,-m^2) = F(0,-m^2,p^2) = Z(p^2)^{-1}$$

- We see e.g. $z_1 = F^{(0,0,1)}(0,-m_{P,0}^2,-m_{P,0}^2)$
  Unphysical derivative! $\longrightarrow$ Must always cancel in the end!

- General method: Expand order by order in $k \to$ arbitrary order in $1/L$

- Next step: Plug into Compton amplitude to calculate $\Delta m_P^2(L)$

Getting the FSEs

- Analytic structure: Photon pole, Pion pole, branch-cut

$$\Delta m_P^2(L) = \Delta m_{\text{pp}}^2(L) + \Delta m_{\text{psp}}^2(L) + \Delta m_{\text{cut}}^2(L)$$

- Cut-term: New, specific to QED_L (not in $\text{QED}_\mathcal{C}$ [Lucini, Patella, Ramos, Tantalo 2016])

$$\Delta m_P^2(L) = e^2 m_{P,0}^2 \left\{ \frac{c_2}{4\pi^2 m_{P,0} L} + \frac{c_1}{2\pi (m_{P,0}L)^2} - \frac{\langle r_P^2 \rangle c_0}{3 m_{P,0} L^3} + \frac{\mathcal{C}}{(m_{P,0}L)^3} + \mathcal{O}\left[\frac{1}{(m_{P,0}L)^4}\right] \right\}$$

- All unphysical terms z_n have cancelled! Result the same if one uses $F(k^2)$ instead of $F(k^2,(p+k)^2,p^2)$.
- Structure-dependence the same as in NRsQED! [Davoudi, Savage 2014]
- Branch-cut term: Will be studied in the future

Leptonic decays

- Leptonic decays $P^- \to \ell^- \nu_\ell[\gamma]$ give information on the ratio $|V_{us}/V_{ud}|$
  (see A. Yong's talk)
- Infrared-divergent process:

$$\Gamma\left(P^- \to \ell^- \nu_\ell[\gamma]\right) = \Gamma_0 + \Gamma_1(\Delta E_\gamma)$$

- RM-123 strategy 2015: Add and subtract point-like Γ_0^{pt}

$$\Gamma_0 + \Gamma_1(\Delta E_\gamma) = \lim_{L\to\infty}[\Gamma_0(L) - \Gamma_0^{\mathrm{pt}}(L)] + \lim_{m_\gamma\to 0}[\Gamma_0^{\mathrm{pt}}(m_\gamma) + \Gamma_1(m_\gamma, \Delta E_\gamma)]$$

- RM-123 2017: $\Gamma_0^{\mathrm{pt}}(L)$ calculated to give

$$\Gamma_0(L) - \Gamma_0^{\mathrm{pt}}(L) \sim \mathcal{O}\left(\frac{1}{L^2}\right)$$

- Our proposal: Replace $\Gamma_0^{\mathrm{pt}}(L)$ by

$$\Gamma_0^{(n)}(L) = \Gamma_0^{\mathrm{pt}}(L) + \sum_{j=2}^{n} \Delta\Gamma_0^{(j)}(L)$$

- $\Delta\Gamma_0^{(j)}(L)$ are here the FSEs of order $1/L^j$, containing both point-like and structure terms

Leptonic decays

- The residual volume-scaling is thus

$$\Gamma_0(L) - \Gamma_0^{(n)}(L) \sim \mathcal{O}\left(\frac{1}{L^{n+1}}\right)$$

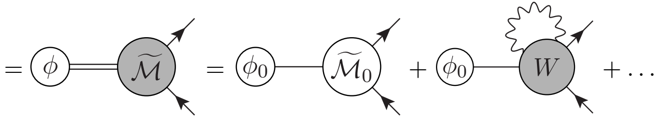
- Define the dimensionless FV function $Y^{(n)}(L)$ as

$$\Gamma_0^{(n)}(L) = \Gamma_0^{\mathrm{tree}}\left[1 + 2\frac{\alpha}{4\pi}\, Y^{(n)}(L)\right] + \mathcal{O}\left(\frac{1}{L^{n+1}}\right)$$

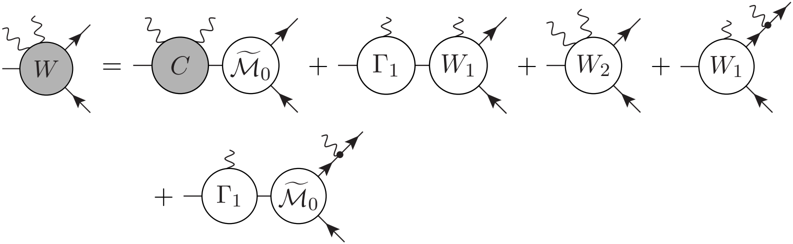
- NB: $Y^{(1)}(L) = Y(L)$ of [RM-123, 2017]

- Euclidean correlator for the decay $P^- \to \ell^- \nu_\ell$

$$C_W^{rs}(p, p_\ell) = \int \mathrm{d}^4 z\, e^{-ipz}\, \langle \ell^-, \mathbf{p}_\ell, r; \nu_\ell, \mathbf{p}_{\nu_\ell}, s|\, \mathrm{T}[\mathcal{O}_W(z)\phi^\dagger(0)]\, |0\rangle$$



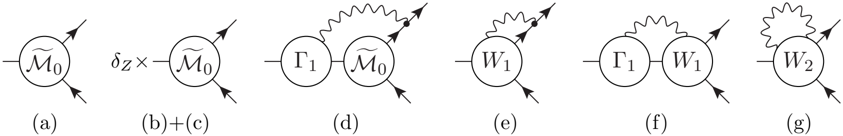
- Decompose into irreducible kernel functions as (Only two new: W_1 and W_2)



- Matrix element from reduction formula (be consistent with orders in e!)

$$\mathcal{M}^{rs} = \lim_{p^2 \to -m_P^2} Z_P^{-1} D(p)^{-1} C_W^{rs}(p, p_\ell)$$

- Contributions to $\mathcal{M}^{rs}$:



Vertex functions

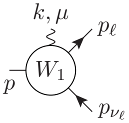
Weak vertex without photons:

$$C_W^\rho(p) = \int \mathrm{d}^4 z \, e^{-ipz} \, \langle 0 | \, \mathrm{T}[J_W^\rho(z)\phi^\dagger(0)] \, |0\rangle$$

$$W^\rho(p) = Z_P^{-1} D(p)^{-1} \, C_W^\rho(p) = -p^\rho F(p^2)$$

$$F(p^2) \equiv f_P \left\{ 1 + \sum_{n=1}^{\infty} (p^2 + m_{P,0}^2)^n f_n \right\}$$

Weak vertex with one photon:



$$= W^{\rho\mu}(p,k)\gamma_\rho(1-\gamma_5)$$

- Ward identity and solve order by order in k (for $1/L^2$):

$$k_\mu W^{\rho\mu}(p,k) = W^\rho(p) - W^\rho(p+k)$$

Vertex functions

$$W^{\rho\mu}(p,k) = \delta^{\rho\mu}\, F(p^2) + [2\,\delta^{\mu\rho}(p\cdot k) + p^\rho k^\mu + 2\,p^\rho p^\mu]\; F'(p^2)$$
$$+\,2\,(p\cdot k)\,p^\rho p^\mu\, F''(p^2) - \frac{V_1(k^2,(p+k)^2)}{m_P^2}\,\varepsilon_{\mu\rho\alpha\beta}\,k^\alpha p^\beta$$
$$+\,\frac{A_1(k^2,(p+k)^2)}{m_P^2}\,[\delta^{\rho\mu}\,(p\cdot k) - k^\rho p^\mu] + \mathcal{O}(k^2)$$

- $A_1(k^2,(p+k)^2)$, $V_1(k^2,(p+k)^2)$: appear in $P^- \to \ell^- \nu_\ell \gamma$
- On-shell: $F_A^P = A_1(0,-m_P^2)$ and $F_V^P = V_1(0,-m_P^2)$
- Known from chiral perturbation theory [Bijnens, Ecker, Gasser 1992], lattice [RM-123 2020], experiment [...] (Discrepancies [RM-123 2020])

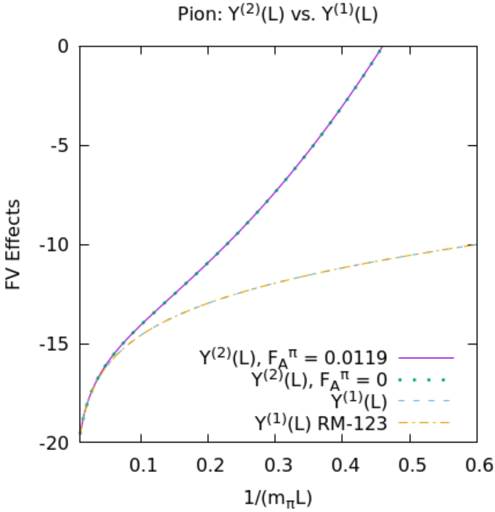
Finite-size effects

- Diagrams give $Y^{(n)}(L)$ for $n = 2$ as

$$Y^{(2)}(L) = \frac{3}{4} + 4\,\log\left(\frac{m_\ell}{m_W}\right) + \frac{c_3 - 2\,c_3(\mathbf{v}_\ell)}{2\pi} - 2\,A_1(\mathbf{v}_\ell) + 2\,\log\left(\frac{m_W L}{4\pi}\right)$$
$$- 2\,A_1(\mathbf{v}_\ell)\left[\log\left(\frac{m_P L}{4\pi}\right) + \log\left(\frac{m_\ell L}{4\pi}\right)\right] - \frac{1}{m_P L}\left[\frac{(1+r_\ell^2)^2\,c_2 - 4\,r_\ell^2\,c_2(\mathbf{v}_\ell)}{1-r_\ell^4}\right]$$
$$+ \frac{1}{(m_P L)^2}\left[-\frac{F_A^P}{f_P}\,\frac{4\pi\,m_P\,[(1+r_\ell^2)^2\,c_1 - 4\,r_\ell^2\,c_1(\mathbf{v}_\ell)]}{1-r_\ell^4} + \frac{8\pi\,[(1+r_\ell^2)\,c_1 - 2\,c_1(\mathbf{v}_\ell)]}{(1-r_\ell^4)}\right]$$

- All unphysical quantities vanish, i.e. we could put $f_n = z_n = 0$ from the start (as they must at all orders in $1/L$)
- Only F_A^P appears
- Charge radii $\langle r_P^2 \rangle$ cancel between diagrams due to charge conservation
- $c_j(\mathbf{v}_\ell)$ FS coefficients previously only known for $j < 3$, now for all $j \geq 3$ too

Numerical results: Physical Pion



- Perfect agreement with RM-123 for $Y^{(1)}(L)$
- The $1/L^2$-correction is sizeable
- Point-like $1/L^2$ completely dominates

Conclusions

- With model-independent principles it is indeed possible to predict FSEs beyond the point-like approximation (only physical form-factors and derivatives appear)
- Self-energy $(1/L^3)$:
  - Charge radii $\langle r_P^2 \rangle$
  - Non-locality of $\mathrm{QED_L}$: Branch-cut
- Leptonic decays $(1/L^2)$:
  - Radiative leptonic decay axial form-factor F_A^P
  - Charge radii cancel because of charge conservation
- Our method is general, and new software released
  - Infrared divergent FS coefficients
- Future: Semi-leptonic decays, ...