Temporal Contact Terms in Lattice Feynman-Hellmann Methods

Kadir Utku Can **Alec Hannaford-Gunn** Roger Horsley Paul Rakow Edward Sankey Gerrit Schierholz Ross Young James Zanotti

UKQCD/QCDSF collaboration & CSSM, The University of Adelaide

Introduction

Aim: Investigate **short-distance artifacts** in lattice QCD calculations of the **Compton amplitude subtraction function**, $S_1(Q^2)$.

This quantity is of great physical interest as it is related to the proton-neutron mass difference and muonic hydrogen Lamb shift (discussed in the talk *Lattice QCD calculation of the subtraction function*, by Edward Sankey).

Using Feynman-Hellmann methods, we can calculate four-point functions from perturbed two-point functions. We start with the perturbed quark propagator:

$$C_\lambda = \left[M - \lambda \mathcal{J}_\mu(\vec{q})\right]^{-1} = \underbrace{M^{-1}}_{\text{unperturbed}} + \lambda \underbrace{M^{-1}\mathcal{J}_\mu(\vec{q})M^{-1}}_{\text{three-point}} + \lambda^2 \underbrace{M^{-1}\mathcal{J}_\mu(\vec{q})M^{-1}\mathcal{J}_\mu(\vec{q})M^{-1}}_{\text{four-point}} + \ldots \quad (1)$$

Putting these perturbed quark propagators into a nucleon propagator, $\mathcal{G}_\lambda(\vec{p}, t)$, we get

$$\left.\frac{\partial^2}{\partial\lambda^2}\frac{\mathcal{G}_\lambda(t,\vec{p})}{\mathcal{G}_0(t,\vec{p})}\right|_{\lambda=0} \propto \sum_{t_1,t_2}^{t}\sum_{\vec{z}} e^{-i\vec{q}\cdot\vec{z}}\langle N(\vec{p})|T\{J_\mu(\vec{z},t_1)J_\mu(0,t_2)\}|N(\vec{p})\rangle. \quad (2)$$

With a judicious choice of kinematics, the RHS of Eq. 2 is proportional to a discretisation of the subtraction function, $S_1(Q^2)$, where $Q^2 = \vec{q}^2$.

OPE Prediction

The operator product expansion (OPE) describes the short-distance (large Q^2) behaviour of the Compton amplitude. It predicts that the subtraction term behaves like

$$S_1(Q^2) \sim \frac{1}{Q^2}, \quad \text{for } Q^2 \text{ large.} \quad (3)$$

and hence S_1 **asymptotes to zero**.

Therefore, the subtraction function also tells us how well the OPE is obeyed on the lattice.

Prior Feynman-Hellmann Results

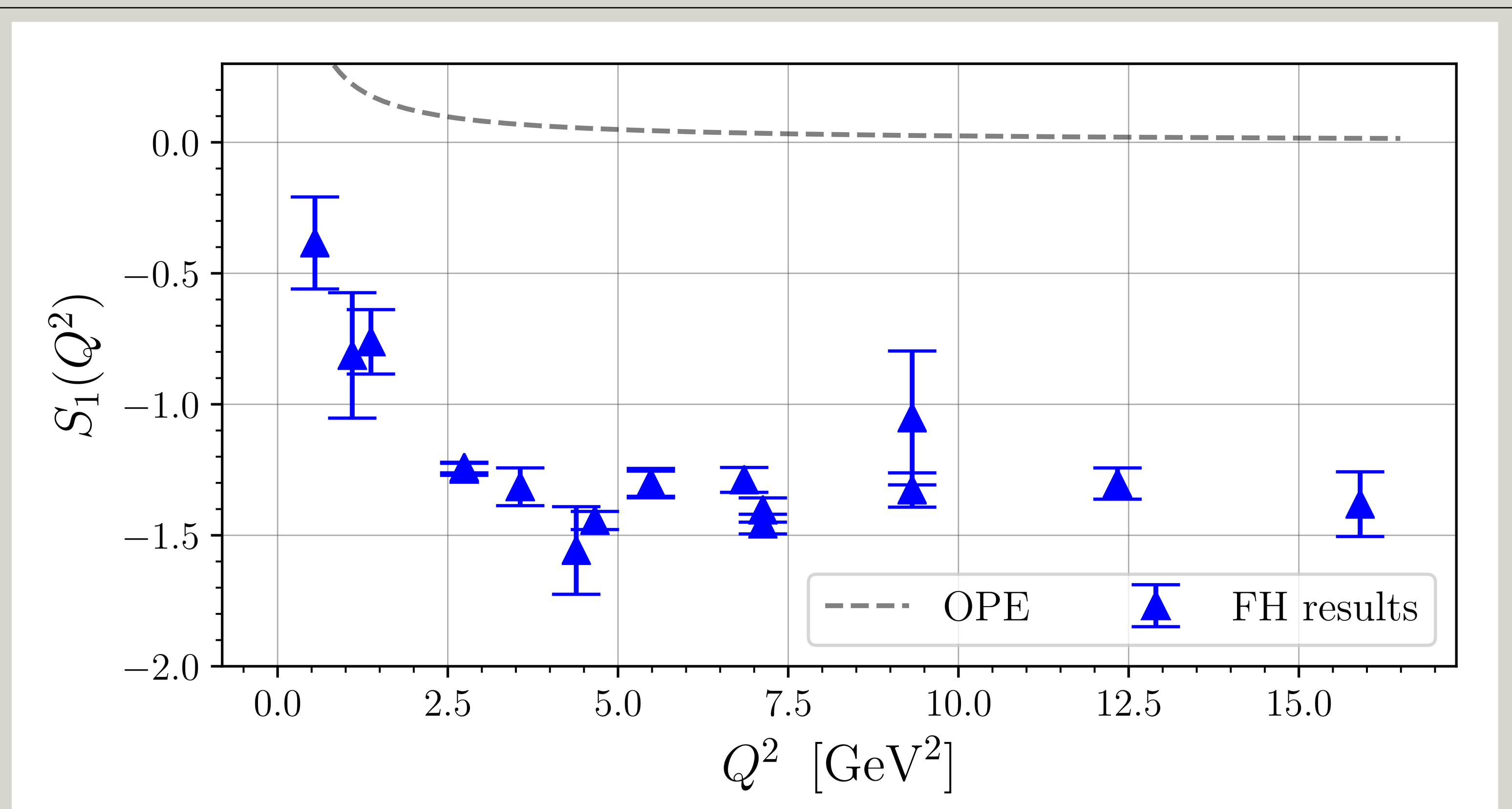


Figure 1. Existing results for the proton Compton amplitude subtraction function. Feynman-Hellmann results from Ref. [1]; OPE prediction from Ref. [2].

Instead of the predicted OPE behaviour (Eq. 3), our previous Feynman-Hellman results show that $S_1(Q^2)$ **asymptotes to a large non-zero value**.

Since the OPE is a very successful theoretical tool, **this is concerning!**

Lattice OPE-breaking

On the lattice, the continuum OPE receives **lattice spacing artifacts** [3]:

$$T\{J(z)J(0)\} = \underbrace{\sum_i \mathcal{C}_i(z^2)\sum_n z_{\mu_1}\ldots z_{\mu_n}\mathcal{O}^{\mu_1\ldots\mu_n}_{i,n}(0)}_{\text{continuum OPE}} + \underbrace{\mathcal{O}\left(\frac{a}{|z|}\right)}_{\text{lattice artifacts}}, \quad \text{for } |z| \approx 0, \quad (4)$$

where a is the lattice spacing. We will argue that these artifacts are responsible for the OPE-breaking seen in figure 1.

Temporal Contact Terms

In Eq. 2, notice that our Feynman-Hellmann calculation includes terms where currents lie on the same time slice: $t_1 = t_2$ (see figure 2).

Therefore, in coordinate space our calculation includes terms where $|z| = \sqrt{(t_2 - t_1)^2 + \vec{z}^2} = 0$ and hence the $a/|z|$ artifacts blow up.

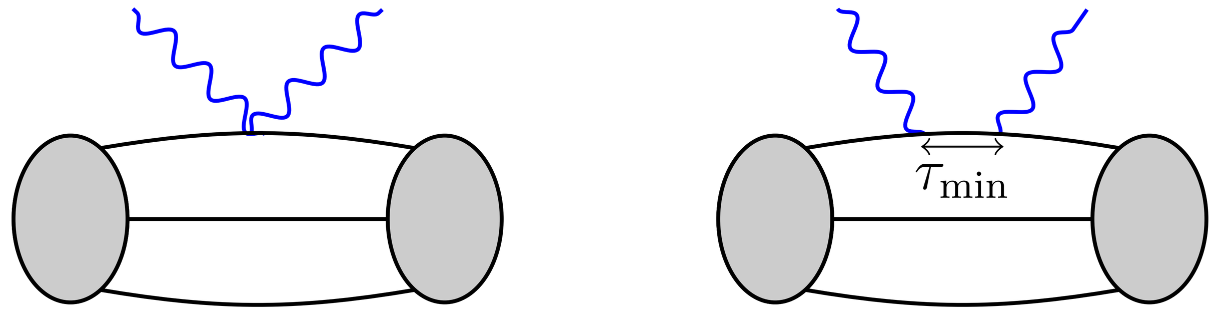


Figure 2. Left: temporal contact terms, $t_1 = t_2$. Right: a non-zero minimum temporal separation is introduced.

To suppress these artifacts, we sum over our time-slices such that there is a minimum separation: $|t_1 - t_2| \geq \tau_{\text{min}}$. Then, since $|z| \geq \tau_{\text{min}}$, the lattice spacing artifacts are suppressed by a/τ_{min}.

Temporal Interlacing

To implement this, we have two currents where we usually have one in Eq. 1:

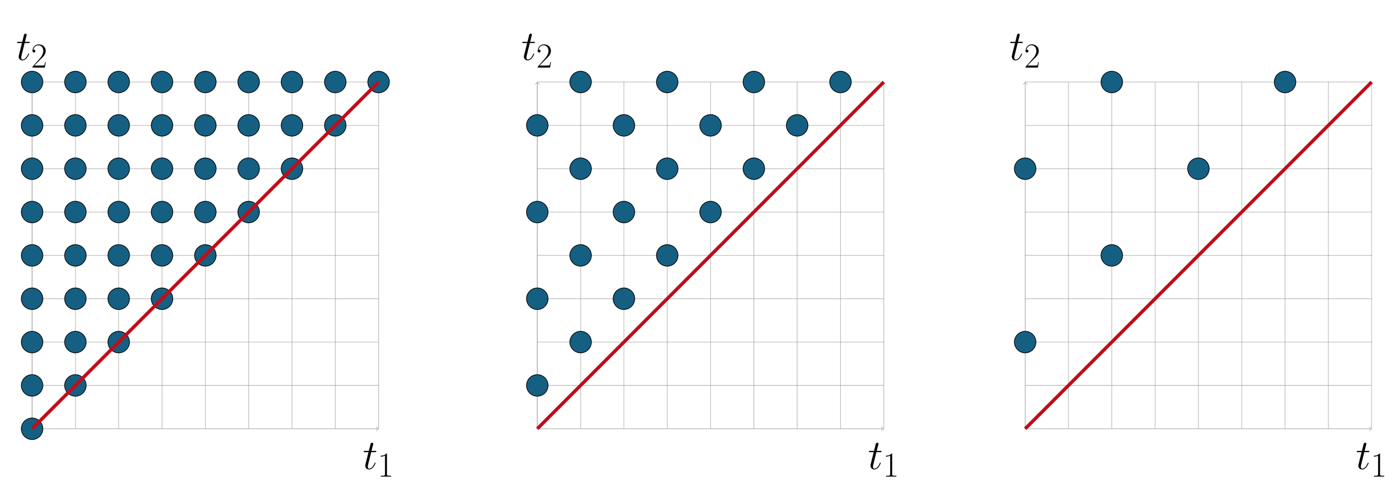
$$C_{\vec{\lambda}} = \left[M - \lambda_1\mathcal{J}_1 - \lambda_2\mathcal{J}_2\right]^{-1}, \quad (5)$$

where $\mathcal{J}_1$ is zero for time-slices $\tau_{\text{min}}, 3\tau_{\text{min}}, 5\tau_{\text{min}}\ldots$ and $\mathcal{J}_2$ is zero for $0, 2\tau_{\text{min}}, 4\tau_{\text{min}}\ldots$ We call this method **temporal interlacing**.

For instance, if $\tau_{\text{min}} = a$ (compare to figure 3(b)), then

$$\left.\frac{\partial^2}{\partial\lambda_1\lambda_2}\frac{\mathcal{G}_{\vec{\lambda}}(t,\vec{p})}{\mathcal{G}_0(t,\vec{p})}\right|_{\vec{\lambda}=0} \propto \left(\sum_{t_1 \text{ even}}\sum_{t_2 \text{ odd}} + \sum_{t_1 \text{ odd}}\sum_{t_2 \text{ even}}\right)\sum_{\vec{z}} e^{-i\vec{q}\cdot\vec{z}}\langle N(\vec{p})|T\{J_\mu(\vec{z},t_1)J_\mu(0,t_2)\}|N(\vec{p})\rangle. \quad (6)$$

With each new interlacing, we change the measure of our sum and hence the normalisation of the FH relation. This ensures that each interlacing has the same continuum limit.



(a) No interlacing. (b) Interlacing $\tau_{\text{min}} = a$. (c) Interlacing $\tau_{\text{min}} = 2a$.

Figure 3. Integration regions for different temporal interlacings, with the time-ordering $t_2 \gtrsim t_1$. The blue dots represent pairs of time slices with current insertions, while the red line is the $t_1 = t_2$ line.

Numerical Calculation

N_f	κ_l	κ_s	$L^3 \times T$	a [fm]	M_π [GeV]
$2+1$	0.1209	0.1209	$32^3 \times 64$	0.074(2)	0.467(12)

We implement three different interlacings: $\tau_{\text{min}} = 0, a, 2a$. Note that the $\tau_{\text{min}} = 0$ results are simply the 'old' results (figure 1).

The statistics are all relatively low, as this is a highly exploratory study.

Interlacing Results

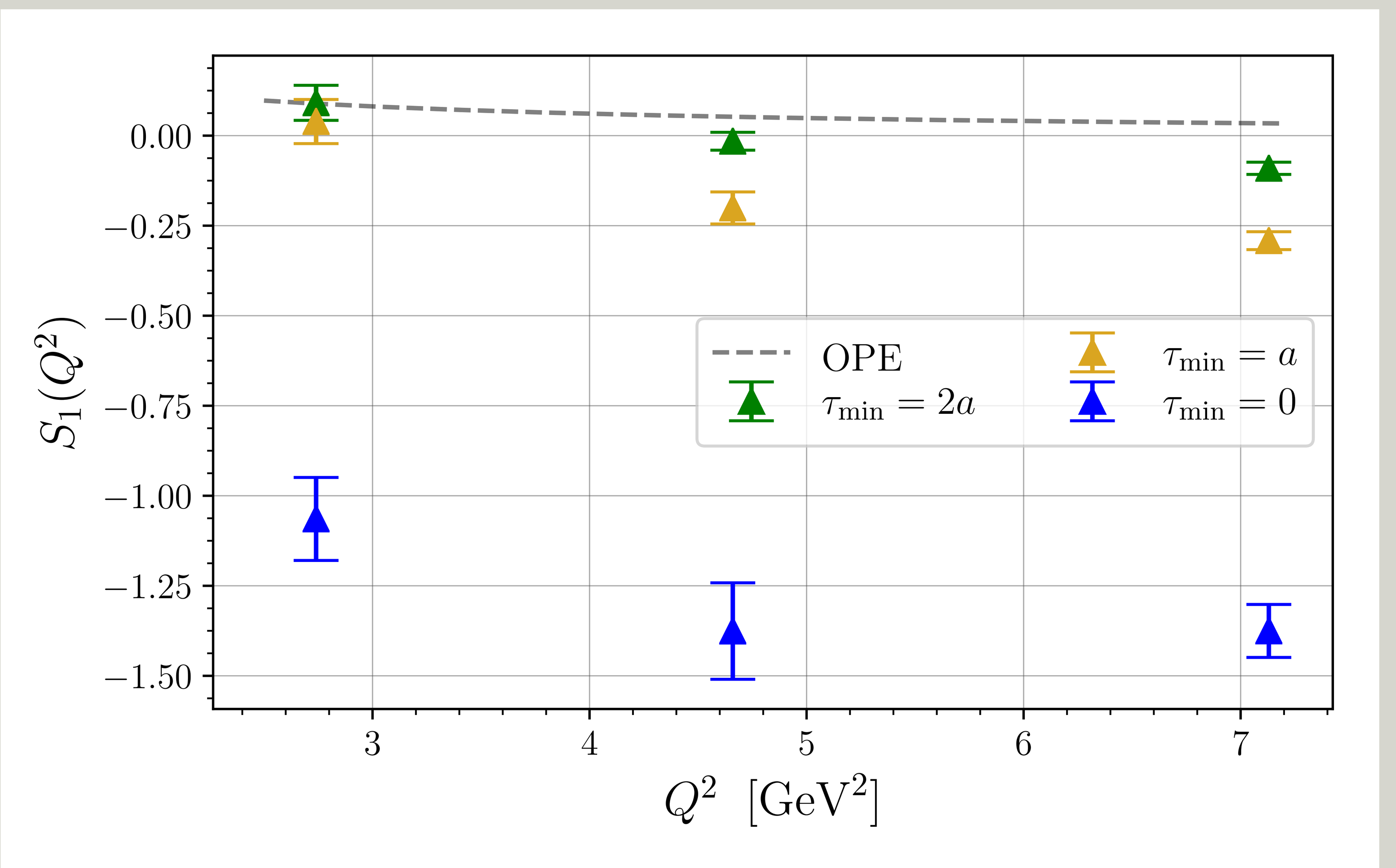


Figure 4. Proton subtraction function calculated with different interlacings. OPE prediction from Ref. [2].

Conclusions

From figure 4, we can see that, as τ_{min} increases, the subtraction term becomes **significantly more consistent with the OPE**.

This suggests that lattice spacing artifacts are responsible for the OPE-breaking (see Eq. 4). And hence we can suppress them by a/τ_{min}, using temporal interlacing.

Therefore, we can use this method to

1. calculate the physical subtraction term,
2. and investigate the short-distance lattice artifacts for the Compton amplitude.

References

1. Can KU, Hannaford-Gunn A, Horsley R, et al. Lattice QCD evaluation of the Compton amplitude employing the Feynman-Hellmann theorem. Physical Review D 2020;102.
2. Hill RJ and Paz G. Nucleon spin-averaged forward virtual Compton tensor at large Q^2. Physical Review D 2017;95.
3. Martinelli G. Hadronic weak interactions of light quarks. Nucl. Phys. B Proc. Suppl. 1999;73:58–71.

alec.hannafordgunn@adelaide.edu.au