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Investigating quark confinement from the viewpoint of lattice gauge scalar models

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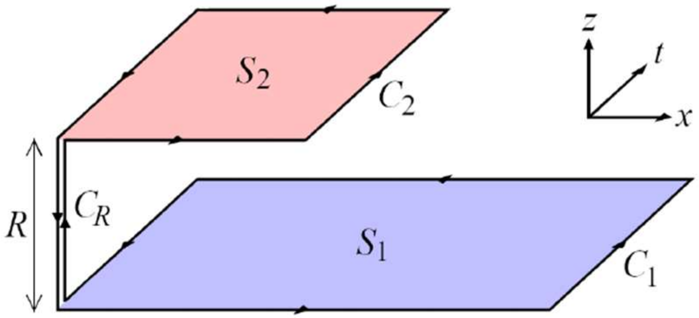
1. Introduction

■ About the double-winding Wilson loop operator

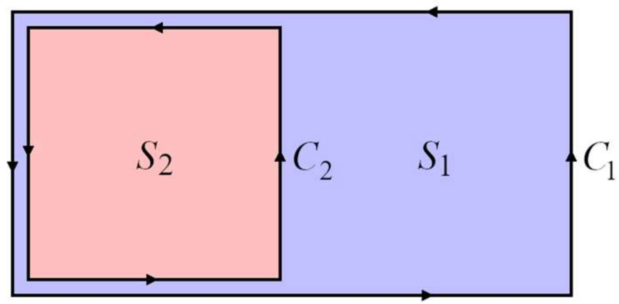
Greensite and Höllwieser introduced a double-winding Wilson loop operator $W(C_1 \cup C_2)$ in the lattice gauge theory (LGT) to examine the quark confinement mechanism:
([1]Phys. Rev. D91, 054509 (2015))

$$W(C_1 \cup C_2) \equiv \mathrm{tr}\left[\prod_{\ell \in C_1 \cup C_2} U_\ell\right]$$

$W(C_1 \cup C_2)$ is a trace of the path-ordered product of gauge link variables $U_\ell \in Z_N$ along a closed loop C composed of two loops C_1, C_2 .



"*shifted*" , $W(C_1 \cup C_2)_{R \neq 0}$



"*coplanar*" , $W(C_1 \cup C_2)_{R = 0}$

■ The area dependence of the expectation value $\langle W(C_1 \cup C_2) \rangle$

- In the lattice SU(2) Yang-Mills model, "difference-of-areas law" for $\langle W(C_1 \cup C_2) \rangle_{R=0}$ has been first showed in [1]: $\langle W(C_1 \cup C_2) \rangle_{R=0} \simeq \exp[-\sigma ||S_1| - |S_2||]$

- In the continuum SU(N) Yang-Mills model, Matsudo and Kondo showed that there is a novel "max-of-areas law" which is neither difference-of-areas law nor sum-of-areas law for multiple-winding Wilson loop average. ([2]Phys. Rev. D96, 105011 (2017))

- In the lattice SU(N) Yang-Mills model, Kato et al. showed that there is an N-dependent area law falloff for $\langle W(C_1 \cup C_2) \rangle_{R=0}$: ([3]Phys. Rev. D102, 094521 (2020))

  for N=2 , "difference-of-areas law" : $\langle W(C_1 \cup C_2) \rangle_{R=0} \simeq \exp[-\sigma ||S_1| - |S_2||]$

  for N=3 , "max-of-areas law" : $\langle W(C_1 \cup C_2) \rangle_{R=0} \simeq \exp[-\sigma \max(|S_1|, |S_2|)]$

  for N≥4 , "sum-of-areas law" : $\langle W(C_1 \cup C_2) \rangle_{R=0} \simeq \exp[-\sigma (|S_1| + |S_2|)]$

■ Center group dominance and quark confinement

- Fröhlich showed that the Wilson loop average in the non-Abelian LGT with the gauge group G is bounded from above by the same Wilson loop average in the Abelian LGT with the center gauge group Z(G): ([4]Phys. Lett. 83B, 195 (1979))

$$|\langle W_{R(G)}(C)\rangle_G(\beta)| \leq 2\mathrm{tr}(\mathbf{1})\langle W_{R(Z(G))}(C)\rangle_{Z(G)}(2\dim(G)\beta)$$

- From the result of [4], confinement in the Z_N LGT implies confinement in the SU(N) LGT. Moreover, confinement in the U(1) LGT implies confinement in the U(N) LGT. (due to the property $Z[SU(N)] = Z_N$ and $Z[U(N)] = U(1)$)

- The above statement can be extended to the double-winding Wilson loop average, beyond the case of the ordinary single-winding Wilson loop average:

$$|\langle W_{R(G)}(C_1 \cup C_2)\rangle_G(\beta)| \leq 2\mathrm{tr}(\mathbf{1})\langle W_{R(Z(G))}(C_1 \cup C_2)\rangle_{Z(G)}(2\dim(G)\beta)$$

■ Motivation

Examination of the center group dominance for a double-winding Wilson loop average

- We introduce the *character expansion* to the weight $e^{S_G[U]}$ from the action and perform the group integration, in order to estimate the expectation value in the Z_N LGT.

- We evaluate the double-winding Wilson loop average up to the leading contribution to show that the N-dependent area law falloff in the SU(N) LGT can be reproduced by using the (Abelian) Z_N LGT.

- By taking the limit N→∞, We also investigate the center group dominance for a double-winding Wilson loop average in the U(N) LGT through the U(1) LGT.

- We extend the above arguments for the Z_N LGST. On the "analytic region", we estimate the area law falloff, the string tension, the mass gap by using the *cluster expansion*.

2. Lattice Z_N gauge theory

■ Character expansion

The action of the lattice Z_N gauge theory:

$$S_G[U] = \beta \sum_{p\in\Lambda} \mathrm{Re}\, U_p \ , \quad U_p := \prod_{\ell\in\partial p} U_\ell$$

We apply the *character expansion* :

$$\langle \mathscr{F} \rangle_\Lambda = Z_\Lambda^{-1} \int \prod_{\ell\in\Lambda} dU_\ell \ e^{S_G[U]} \mathscr{F} = Z_\Lambda^{-1} \int \prod_{\ell\in\Lambda} dU_\ell \ \prod_{p\in\Lambda} \sum_{n=0}^{N-1} b_n(\beta) U_p^n \mathscr{F}$$

where $b_n(\beta) := \frac{1}{N} \sum_{\zeta\in Z_N} \zeta^{-n} e^{\beta \mathrm{Re}\, \zeta}$, we define $c_n(\beta) = b_n(\beta)/b_0(\beta)$.

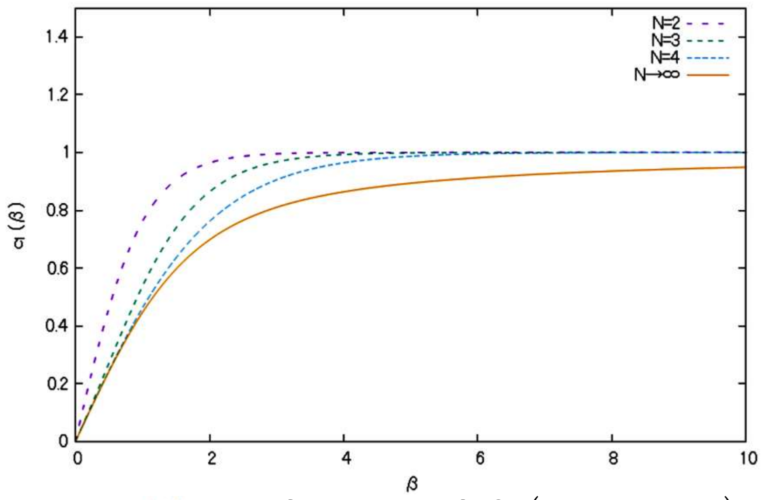
$$c_1(\beta) = \frac{e^\beta - e^{-\beta}}{e^\beta + e^{-\beta}} \quad \text{(N=2)} \qquad c_1(\beta) = \frac{e^\beta - e^{-\beta/2}}{e^\beta + 2e^{-\beta/2}} = c_2(\beta) \quad \text{(N=3)}$$

$$c_1(\beta) = \frac{e^\beta - e^{-\beta}}{e^\beta + 2 + e^{-\beta}} \ , \quad c_2(\beta) = \frac{e^\beta - 2 + e^{-\beta}}{e^\beta + 2 + e^{-\beta}} \quad \text{(N=4)}$$

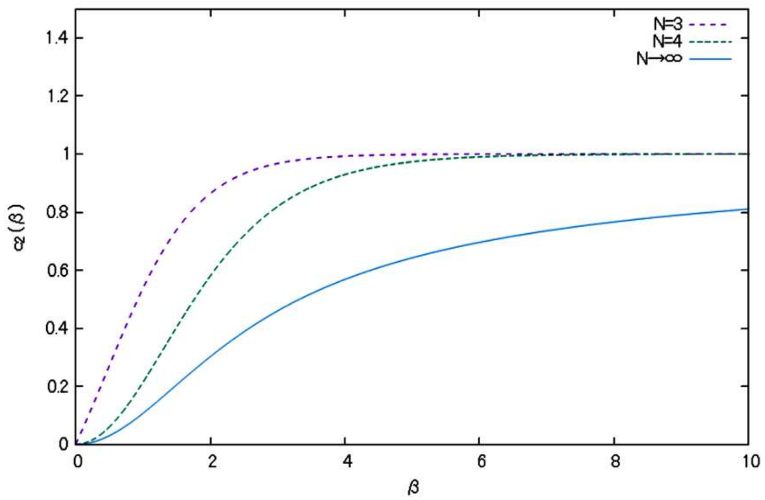
$$c_1(\beta) = \frac{I_1(\beta)}{I_0(\beta)} \ , \quad c_2(\beta) = \frac{I_2(\beta)}{I_0(\beta)} \quad (\mathrm{N}=\infty)$$

Note that in the region $\beta \ll 1$,

$c_1(\beta) \sim \mathcal{O}(\beta)$ for N≥2 , $c_2(\beta) \sim \mathcal{O}(\beta^2)$ for N≥4



$c_1(\beta)$ as a function of β (N=2,3,4,∞)

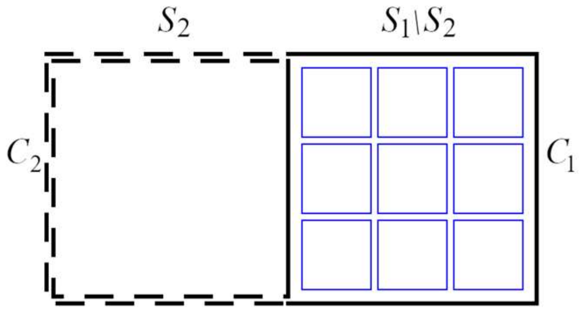


$c_2(\beta)$ as a function of β (N=3,4,∞)

■ A *coplanar* double-winding Wilson loop (Z_N LGT)

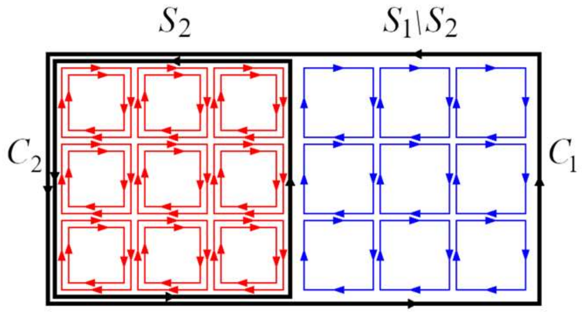
The leading contribution to a *coplanar* double-winding Wilson loop average is given by the tiling by a planar set of plaquettes, as shown in the following diagrams.
(These result are exact for all β when D=2, while valid for $\beta \ll 1$ when D>2):

□ N=2



$\simeq c_1(\beta)^{|S_1|-|S_2|}$

□ N≥3



$\simeq c_2(\beta)^{|S_2|} c_1(\beta)^{|S_1|-|S_2|}$

The result for general N:

$$\langle W(C_1 \cup C_2) \rangle_{R=0} \simeq \begin{cases} c_1(\beta)^{|S_1|-|S_2|} & (N=2) \\ c_1(\beta)^{|S_1|} & (N=3) \\ c_2(\beta)^{|S_2|} c_1(\beta)^{|S_1|-|S_2|} & (N \geqslant 4) \end{cases}$$

The string tension:

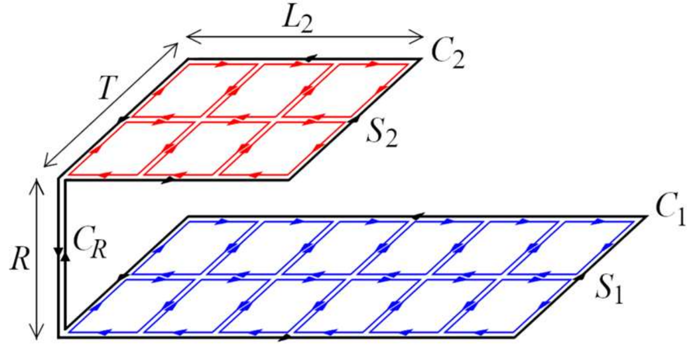
$$\sigma(\beta) \simeq \ln \frac{1}{c_1(\beta)} > 0$$

- In the strong coupling region, this result reproduces the area law falloff in the SU(N) LGT obtained in [3].
- In the continuous group limit N→∞, the area law for N≥4 persists in the U(1) LGT.

■ A *shifted* double-winding Wilson loop (Z_N LGT)

The leading contribution to a *shifted* double-winding Wilson loop average can be given by the 2 types of tiling by a set of plaquettes, as shown in following diagrams:

□ type I : R-independent

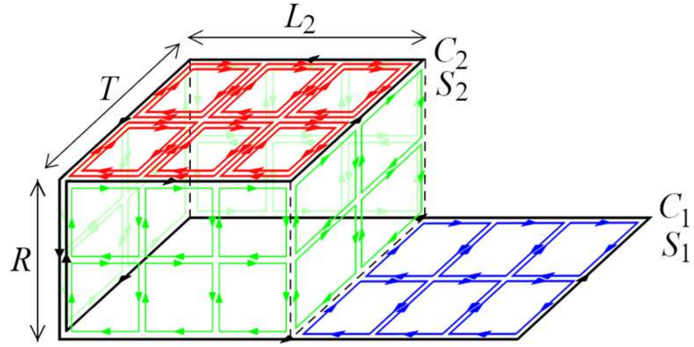


The result for general N :

$$\langle W(C_1 \cup C_2)\rangle_{R\neq 0} \simeq \begin{cases} c_1(\beta)^{|S_1|+|S_2|} + c_1(\beta)^{2R(L_2+T)} \cdot c_1(\beta)^{|S_1|-|S_2|} & (N=2) \\ c_1(\beta)^{|S_1|+|S_2|} + c_1(\beta)^{2R(L_2+T)} \cdot c_1(\beta)^{|S_1|} & (N=3) \\ c_1(\beta)^{|S_1|+|S_2|} + c_1(\beta)^{2R(L_2+T)} \cdot c_2(\beta)^{|S_2|} c_1(\beta)^{|S_1|-|S_2|} & (N \geqslant 4) \end{cases}$$

type I type II

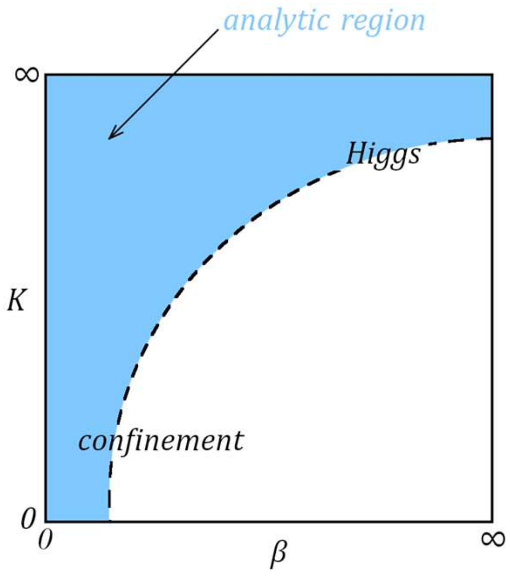
□ type II : R-dependent



- This result reproduces the R-dependence of the shifted double-winding Wilson loop average in [3].
- The non-zero mass gap $\Delta(\beta)$ is obtained from the case of $S_1 = S_2 = 1$ and $R \gg 1$:

$$\Delta(\beta) = 4 \ln \frac{1}{c_1(\beta)} > 0$$

3. Lattice Z_N gauge scalar theory



■ Cluster expansion

The Z_N LGT with the "fundamental" scalar field $\varphi_x \in Z_N$:

$$S[U,\varphi] = \beta \sum_{p\in\Lambda} \mathrm{Re}\ U_p + K \sum_{\ell\in\Lambda} \mathrm{Re}\left(\varphi_x U_\ell \varphi^*_{x+\ell}\right)$$

$$\langle \mathscr{F} \rangle_\Lambda = Z_\Lambda^{-1} \int \prod_{\ell\in\Lambda} dU_\ell\ h[U]\ e^{\beta \sum_{p\in\Lambda} \mathrm{Re}\ U_p} \mathscr{F}\ ,\quad h[U] := \int \prod_{x\in\Lambda} d\varphi_x\ e^{K \sum_{\ell\in\Lambda} \mathrm{Re}(\varphi_x U_\ell \varphi^*_{x+\ell})}$$

We introduce $d\mu_\Lambda := \dfrac{\prod_{\ell\in\Lambda} dU_\ell\ h[U]}{\int \prod_{\ell\in\Lambda} dU_\ell\ h[U]}$ and $\rho_p := e^{\beta \mathrm{Re}\ U_p} - 1$. Then,

$\langle \mathscr{F} \rangle_\Lambda$ has the *cluster expansion* : ([5]K. Osterwalder and E. Seiler, Annals Phys. 110, 440 (1978))

$$\langle \mathscr{F} \rangle_\Lambda = \frac{\int d\mu_\Lambda \prod_{p\in\Lambda} (1+\rho_p)\, \mathscr{F}}{\int d\mu_\Lambda \prod_{p\in\Lambda} (1+\rho_p)} = \sum_{Q(Q_0)\subset\Lambda} \int d\mu_\Lambda\ \mathscr{F} \prod_{p\in Q(Q_0)} \rho_p \cdot \frac{Z_{[Q(Q_0)\cup Q_0]^c}}{Z_\Lambda}$$

where Q_0 is the set of plaquettes which is the support of $\mathscr{F}$, $Q(Q_0)$ is the set of plaquettes which is connected to Q_0. Note that $\rho_p \sim \mathcal{O}(\beta)$ for $\beta \ll 1$.

Fradkin and Shenker showed that the confinement region $(0 \leqslant \beta \ll 1, K \ll 1)$ and the Higgs region $(\beta \gg 1, K_c \leqslant K < \infty)$ are connected in the analytic region, where the cluster expansion converges uniformly. ([6]Phys. Rev. D19, 3682-3697 (1979))

■ The evaluation of h[U]

We apply the *character expansion* and perform the group integration. Ignoring the contributions from multiple plaquettes, we obtain the expression which is valid up to the lowest plaquettes order:

$$h[U] = \int \prod_{x\in\Lambda} d\varphi_x \prod_{\ell\in\Lambda} \left[b_0(K) + b_1(K)\varphi_x U_\ell \varphi^*_{x+\ell} + \cdots + b_{N-1}(K)\left(\varphi_x U_\ell \varphi^*_{x+\ell}\right)^{N-1} \right] = N^{|\Lambda|} \prod_{p\in\Lambda} \sum_{n=0}^{N-1} b_n(K)^4 U_p^n + \cdots$$

■ The estimation of the upper bound of $\langle W(C_1 \cup C_2)\rangle$ (Z_N LGST)

To estimate the leading contribution to $\langle W(C_1 \cup C_2)\rangle$ with the above h[U], we also apply the *character expansion* for ρ_p and evaluate the upper bound of the *cluster expansion* by using the binominal expansion.

There is an analogy between the results for the Z_N LGT and for the Z_N LGST:

$$c_n(\beta) \mapsto a_n(\beta, K) := \frac{\{b_0(\beta) - e^\beta\} c_n(K)^4 + b_1(\beta) c_{n+1}(K)^4 + \cdots + b_{N-1}(\beta) c_{N+n-1}(K)^4}{b_0(\beta) + b_1(\beta) c_1(K)^4 + \cdots + b_{N-1}(\beta) c_{N-1}(K)^4} + c_n(K)^4 \qquad \begin{array}{c} \bmod N \\ n = 1, \cdots, N-1 \end{array}$$

Note that $a_n(\beta, 0) = c_n(\beta)$ and $a_n(\beta, \infty) = 1$.

This evaluation is valid only for the range R where the string breaking does not occur and for the values of parameter β and K on the analytic region.

■ A *coplanar* double-winding Wilson loop (Z_N LGST)

The *coplanar* double-winding Wilson loop average and the string tension:

$$\langle W(C_1 \cup C_2)\rangle_{R=0} \lesssim \begin{cases} a_1(\beta,K)^{|S_1|-|S_2|} & (N=2) \\ a_1(\beta,K)^{|S_1|} & (N=3) \\ a_2(\beta,K)^{|S_2|} a_1(\beta,K)^{|S_1|-|S_2|} & (N \geqslant 4) \end{cases}, \quad \sigma(\beta,K) \gtrsim \ln\frac{1}{a_1(\beta,K)} > 0$$

- The area law falloff in the Z_N LGT persists in the Z_N LGST. (K→0 limit is consistent)
- For σ(β,K), K→0 limit agree with σ(β) in the Z_N LGT.
  In K→∞ limit, σ(β,K)→0. (The string tension is non-zero on the analytic region)

■A *shifted* double-winding Wilson loop (Z_N LGST)

The *shifted* double-winding Wilson loop average and the mass gap:

$$\langle W(C_1 \cup C_2)\rangle_{R\neq 0} \lesssim \begin{cases} a_1(\beta,K)^{|S_1|-|S_2|} + a_1(\beta,K)^{2R(L_2+T)} \cdot a_1(\beta,K)^{|S_1|-|S_2|} & (N=2) \\ a_1(\beta,K)^{|S_1|-|S_2|} + a_1(\beta,K)^{2R(L_2+T)} \cdot a_1(\beta,K)^{|S_1|} & (N=3) \\ a_1(\beta,K)^{|S_1|-|S_2|} + a_1(\beta,K)^{2R(L_2+T)} \cdot a_2(\beta,K)^{|S_2|} a_1(\beta,K)^{|S_1|-|S_2|} & (N \geqslant 4) \end{cases}$$

$$\Delta(\beta,K) \gtrsim 4\ln\frac{1}{a_1(\beta,K)} > 0$$

- For Δ(β,K), K→0 limit agree with Δ(β) in the Z_N LGT.
  In K→∞ limit, Δ(β,K)→0. (The mass gap is non-zero on the analytic region)

4. Conclusion

We studied the area law falloff of the double-winding Wilson loops in the Z_N LGT and Z_N LGST, where the gauge group is the center group of the original SU(N).

- We evaluated the N-dependent area law falloff for $\langle W(C_1 \cup C_2)\rangle_{R=0}$ up to the leading contribution. We found the area law falloff in the Z_N LGT, which reproduces the area law falloff in the SU(N) LGT theory obtained in [3].

- We also checked the limit N→∞, the area law falloff for N≥4 persists in the U(1) LGT. This result implies that $\langle W(C_1 \cup C_2)\rangle_{R=0}$ in the U(N) LGT and the SU(N) (N≥4) LGT obeys the same area law up to the leading contribution.

- We also considered $\langle W(C_1 \cup C_2)\rangle_{R\neq 0}$ up to the leading contributions. This result reproduces the R-dependent behavior in the SU(N) LGT obtained in [3]. We obtained the (non-zero) mass gap $\Delta(\beta)$ from this result.

- We extended the above study for the Z_N LGST on the analytic region. We found that the area law falloff in the Z_N LGT persists in the Z_N LGST. We discovered that the string tension and the mass gap are non-zero on the analytic region.