The Torelon spectrum and the world-sheet Axion

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LATTICE 21

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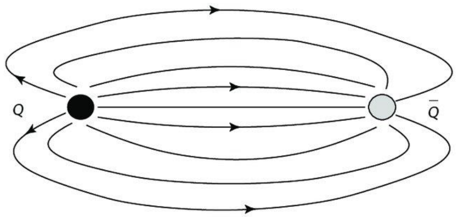
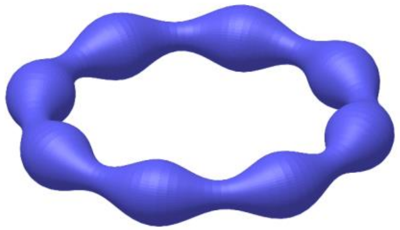
Generalities (1)



In QCD quarks are confined in bound states by forming flux-tubes of chromo-magnetic and chromo-electric flux.

Long flux-tubes behave pretty much like strings.
At some point they break (string breaking).
Of course we need fermions for string breaking.

There are $D-2$ obvious massless Goldstone modes arising from the broken translation invariance in the $D-2$ directions transverse to the flux tube.

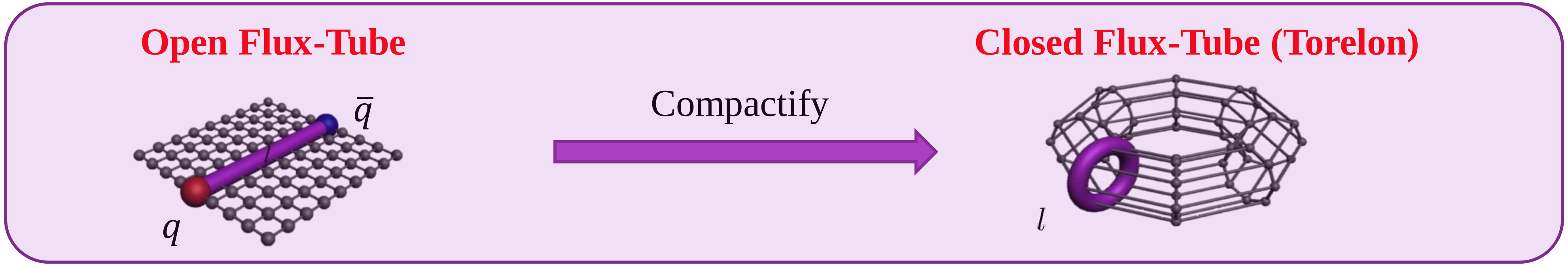


There should be a Low Energy Effective String Theory model describing the energy spectrum of the flux tube.

In addition to string modes are there other kind of (massive) excitations?

Generalities (2)

- Is there a theoretical description for the confining flux-tube in (dis)agreement with Lattice data?
- Confining flux-tube:



- Pure gauge phenomena are also present such as:
  - Glueball – Flux-Tube mixing
  - Flux-tube – anti-Flux-Tube mixing
- Possible low-energy effective string theoretical description?
  - Cannot capture pure gauge phenomena!
  - Might be possible in the Large-*N* limit!
- Investigate Closed Flux tubes in the Large-*N* limit
- So far A. A, B. Bringoltz and M. Teper, JHEP 1102:030,2011 [arXiv:1007.4720]

  A. A and M. Teper, PLB 771 (2017) 408 – 414 [arXiv:1702.03717]

Theoretical expectations: Nambu-Goto string

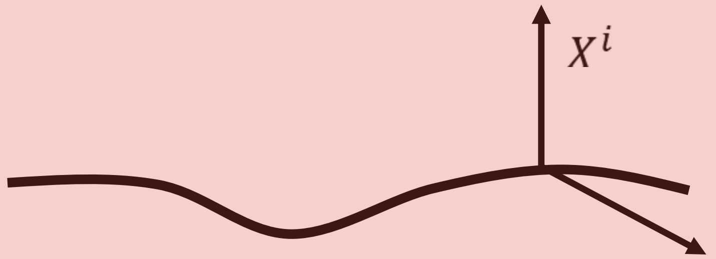
Contributions by

M. Lüscher ’81, J. Polchinski & A. Strominger ‘90, M. Lüscher & P. Weisz ’04, O. Aharony et al ’07 – 11, S. Dubovsky et al ’12 – 19

Quantize the Bosonic String (Nambu-Goto String):

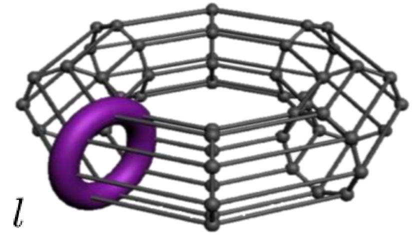
$$S_{\mathrm{NG}} = -\ell_s^{-2}\int d^2\sigma\,\sqrt{-\det\gamma} \qquad \gamma_{\alpha\beta} = \frac{\partial X^\mu}{\partial\sigma^\alpha}\frac{\partial X^\nu}{\partial\sigma^\beta}\eta_{\mu\nu}$$

$\ell_s^{-1} = \sqrt{\sigma}$ (string tension)



- The spectrum of a closed bosonic string compactified around a torus is:

$$E^2_{N_L,N_R,q,w} = (\sigma l w)^2 + 8\pi\sigma\left(\frac{N_L+N_R}{2} - \frac{D-2}{24}\right) + \left(\frac{2\pi q}{l}\right)^2 + p_\perp^2 .$$



- The spectrum is described by:
  1. The winding number w (w=1, 2, ...),
  2. The winding momentum $p_{||} = 2\pi q/l$ with $q = 0, \pm1, \pm2, \ldots$.
  3. The transverse momentum $p_\perp$ ($p_\perp = 0$),
  4. $N_L = \sum_{k>0} n_L(k)k$ and $N_R = \sum_{k'>0} n_R(k')k'$
  5. Level-matching constrain: $N_L - N_R = qw$.
- How do we construct the string states:

$$(\alpha^{i_1}_{-k_1})^{n_L(k_1)}\ldots(\alpha^{i_{m_L}}_{-k_{m_L}})^{n_R(k_{m_L})}(\bar{\alpha}^{i'_1}_{-k'_1})^{n_R(k'_1)}\ldots(\bar{\alpha}^{i'_{m_R}}_{-k'_{m_R}})^{n_R(k'_{m_R})}|0\rangle$$

For D =3

- Example: $\alpha_{-2}\alpha_{-1}\bar{\alpha}_{-1}|0\rangle$
  - ⋆ $N_L = 3$
  - ⋆ $N_R = 1$
  - ⋆ $q = 2$

Theoretical expectations: Effective String Theory

Contributions by

M. Lüscher ’81, J. Polchinski & A. Strominger ‘90, M. Lüscher & P. Weisz ’04, O. Aharony et al ’07 – 11, S. Dubovsky et al ’12 – 19

The energy (mass) of a closed flux-tube is expected to be described as:

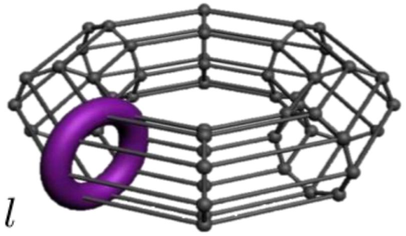
$$E_n \overset{l\to\infty}{=} \sigma l \qquad \text{linear confinement}$$

$$+ \frac{4\pi}{l}\left(n - \frac{D-2}{24}\right) \qquad \text{Lüscher 1980, Polchinski\&Strominger 1991}$$

$$- \frac{8\pi^2}{\sigma l^3}\left(n - \frac{D-2}{24}\right)^2 \qquad \text{Lüscher\&Weisz 2004, Drummond 2004}$$

$$+ \frac{32\pi^3}{\sigma^2 l^5}\left(n - \frac{D-2}{24}\right)^3 \qquad \text{Aharony\&Karzbrun 2009}$$

l



Relation to Nambu-Goto:

$$\frac{E_n(l)}{\sqrt{\sigma}} \overset{l\to\infty}{=} l\sqrt{\sigma} + \frac{c_1^{N.G}}{l\sqrt{\sigma}} + \frac{c_2^{N.G}}{(l\sqrt{\sigma})^3} + \frac{c_3^{N.G}}{(l\sqrt{\sigma})^5} + \mathcal{O}\left(\frac{1}{(l\sqrt{\sigma})^7}\right)$$

Sergei Dubovsky, Raphael Flauger, Victor Gorbenko’ 13 Phys. Rev. Lett. 111, 062006:

- By calculating the energy shifts due to phonon collisions, using approximate integrability and TBA, provided predictions for Short flux tubes
- Explain the deviations of 0^{--} ground state by introducing a worldsheet axion.

$$S_{int} = \frac{\alpha}{8\pi}\int d^2\sigma \phi K^i_{\alpha\gamma} K^{j\gamma}_{\beta} \epsilon^{\alpha\beta}\epsilon_{ij}$$

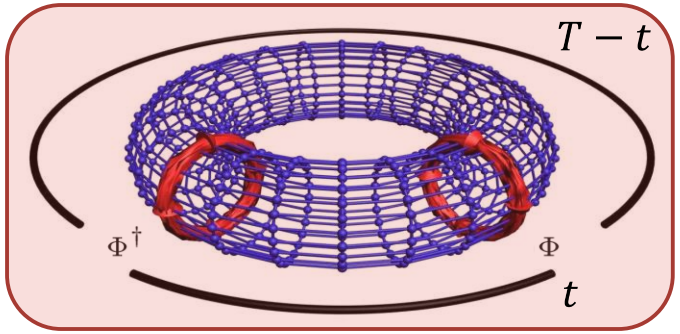
Lattice calculation (1)

Lattice Calculation

- We use the standard Wilson action:

$$S_L = \beta \sum_p \{1 - \frac{1}{N_c} \mathrm{ReTr} U_p\} \quad \text{with} \quad \beta = \frac{2N_c}{g^2}$$

- We calculate correlation functions of spatial Polyakov lines (operators)



$$\begin{aligned} C(t) &= \langle \Phi^\dagger(t)\Phi(0)\rangle = \langle \Phi^\dagger(0) e^{-Ht} \Phi(0)\rangle \\ &= |\langle 0|\Phi(0)|vac\rangle|^2 e^{-E_0 t} + \sum_{n=1} |\langle n|\Phi(0)|vac\rangle|^2 e^{-E_n t} \xrightarrow{t\to\infty} |\langle 0|\Phi(0)|vac\rangle|^2 e^{-E_0 t} \end{aligned}$$

- We calculate the effective energy: $\lim_{t\to\infty} \left[-\ln \left(\frac{C(t)}{C(t-a)} \right) \right] = aE_0$
- This provides only the ground state
- To extract the excitation spectrum we use GEVP

Lattice calculation (2)

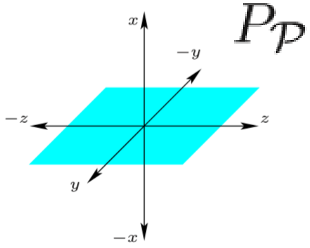
Generalized Eigenvalue Problem (GEVP):

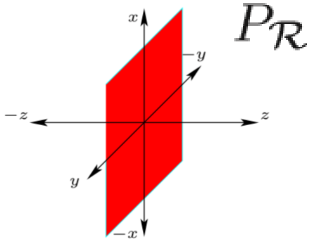
- Construct a large basis of Operators $\Phi_i : i = 1, 2, \ldots$ with **Right Quantum Numbers**
- Calculate the correlation function (Matrix) $C_{ij}(t) = \langle \Phi_i^\dagger(t) \Phi_j(0) \rangle$
- Diagonalize the matrix $C^{-1}(t=0)C(t=a)$
- Extract the eigenvectors
- Extract the correlator for each state $(\sim \mathrm{e}^{-E_n t})$
- By fitting the results, we extract the mass (energy) for each state

Quantum numbers

- Spin J: $\phi_L(J) = \sum_n e^{iJn\frac{\pi}{2}} \phi_{n\frac{\pi}{2}}$ Example $\phi_L(J=1) = \mathrm{Tr}$

- Parity:



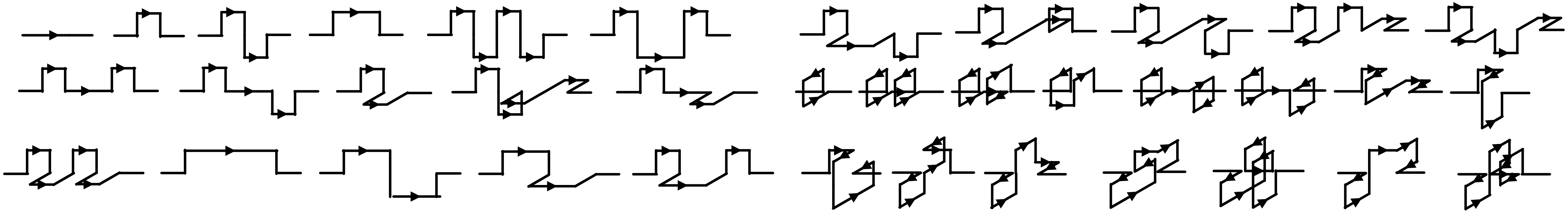


Example

Lattice calculation (3)

We build operators described by the quantum numbers of J, $P_{\mathcal{P}}$, $P_{\mathcal{R}}$

- We use a large basis of operators with transverse deformations:



- We project onto 5 irreducible representations × 2 ($P_{\mathcal{R}} = \pm$)

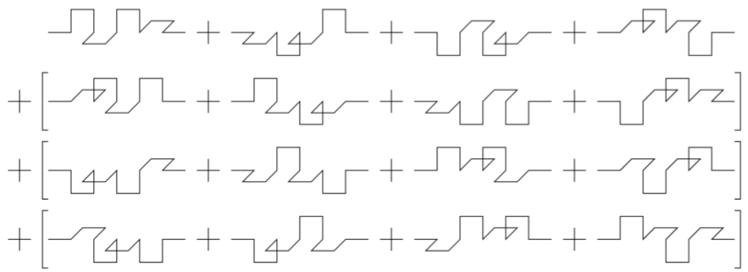
$\rightarrow A_1 \equiv (J = 0, 4, \ldots, 4N,\ \ P_{\mathcal{P}} = +),\ A_2 \equiv (J = 0, 4, \ldots, 4N,\ \ P_{\mathcal{P}} = -)$

$\rightarrow E \equiv (J = 1, 3, \ldots, 2N+1)$

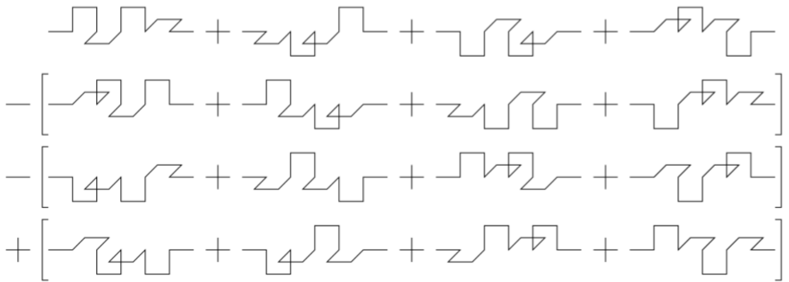
$\rightarrow B_1 \equiv (J = 2, 6, \ldots, 4N+2,\ \ P_{\mathcal{P}} = +),\ B_2 \equiv (J = 2, 6, \ldots, 4N+2,\ \ P_{\mathcal{P}} = -)$

- Example:

$J = 0,\ P_{\mathcal{P}} = +,\ P_{\mathcal{R}} = +$

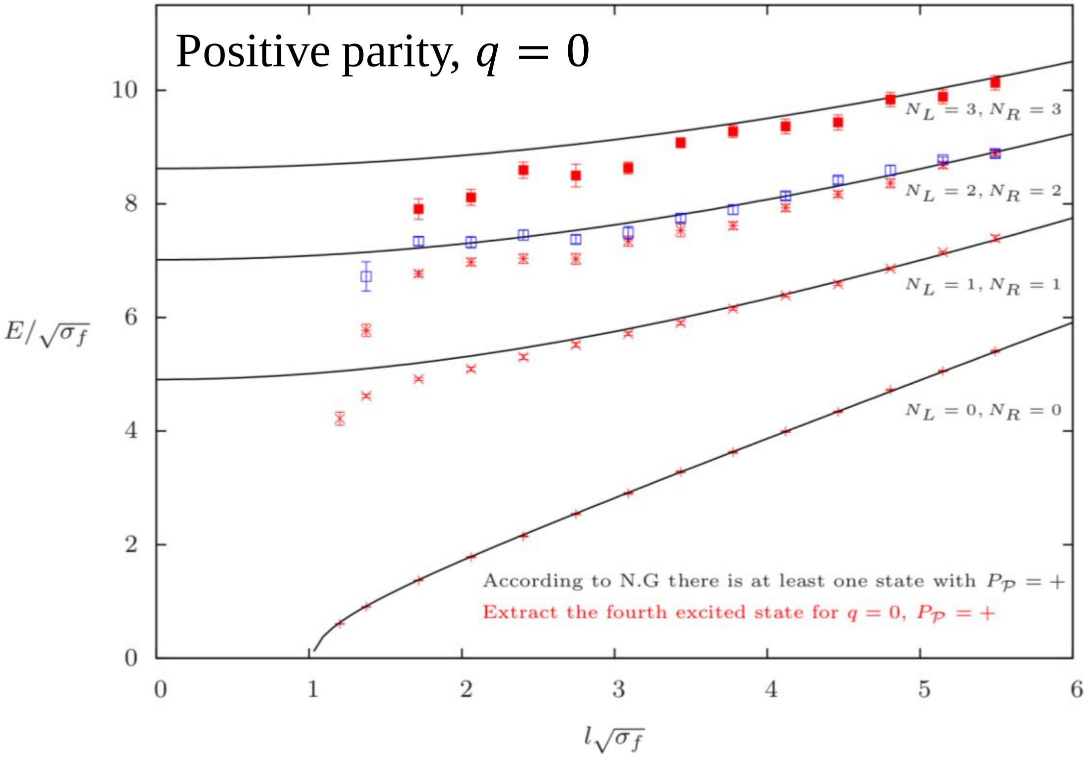


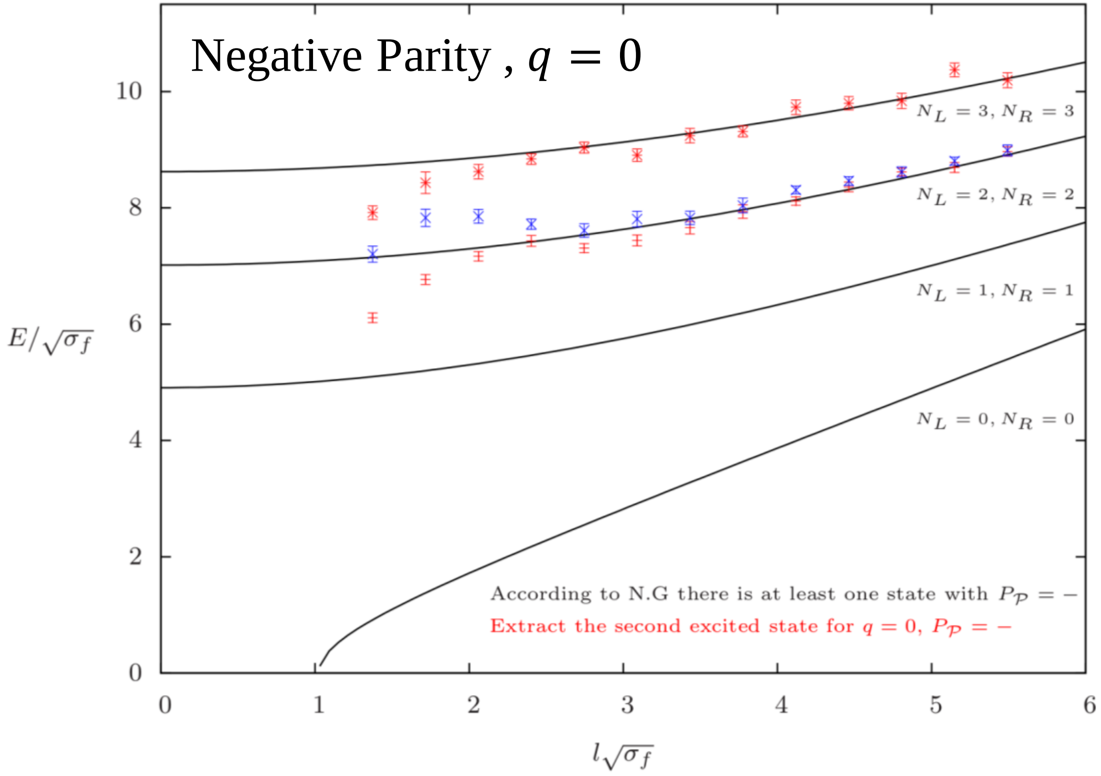
$J = 0,\ P_{\mathcal{P}} = -,\ P_{\mathcal{R}} = -$



Results, $D = 2 + 1$ (1)

Results for SU(6), $a\sqrt{\sigma} \sim 0.08582(5)$, AA & M. Teper, *JHEP* 05 (2011) 042



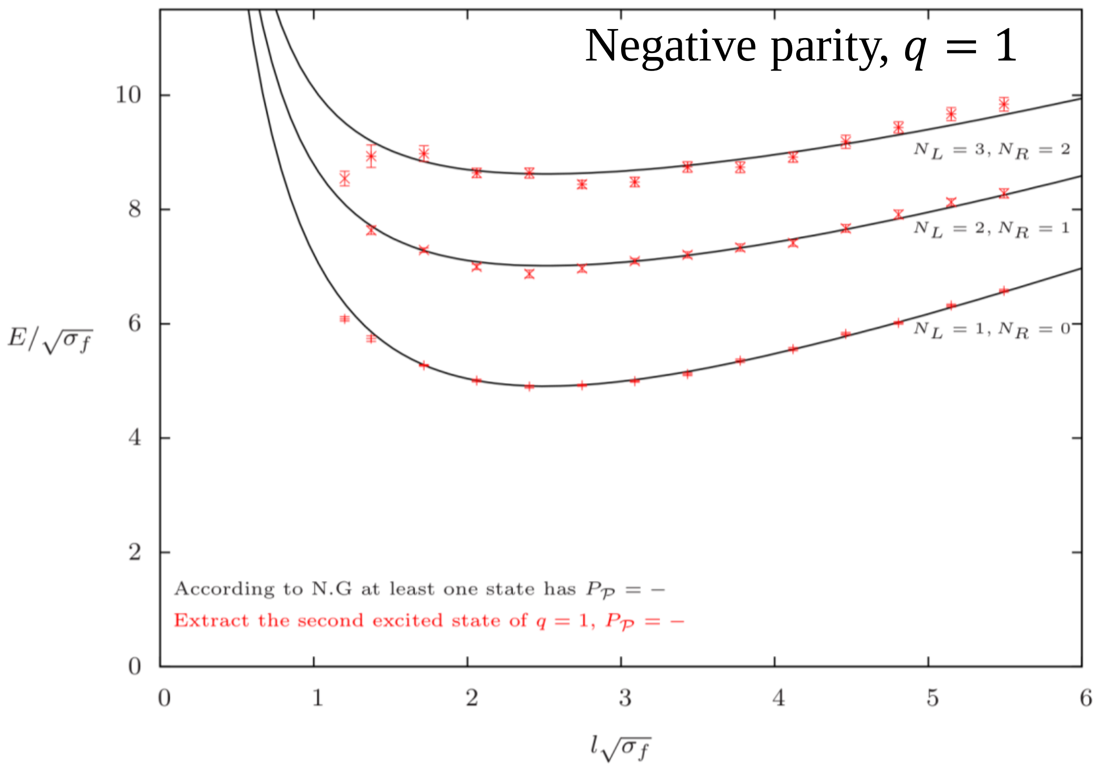


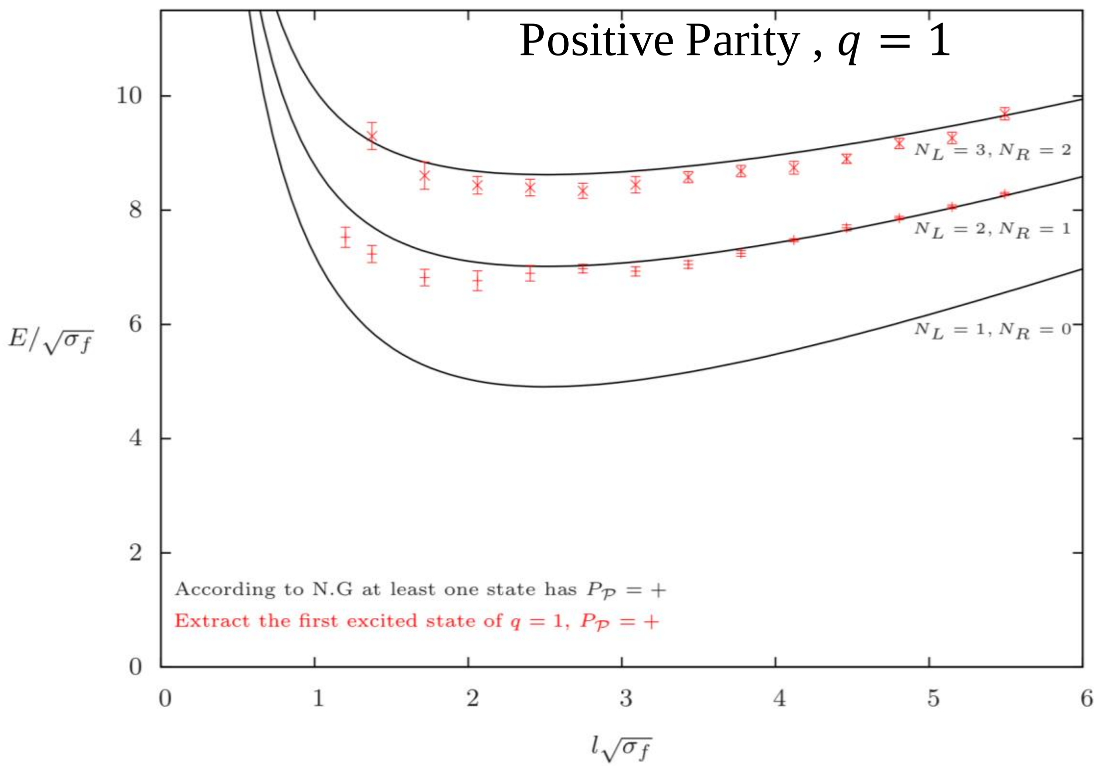
N_L, N_R	q	P	String State
$N_L = N_R = 0$	0	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	$a_1 a_{-1} \lvert 0\rangle$
$N_L = N_R = 2$	0	+	$a_2 a_{-2} \lvert 0\rangle$
		+	$a_1 a_1 a_{-1} a_{-1} \lvert 0\rangle$
		−	$a_2 a_{-1} a_{-1} \lvert 0\rangle$
		−	$a_1 a_1 a_{-2} \lvert 0\rangle$

N_L, N_R	q	P	String State
$N_L = N_R = 0$	0	+	$\lvert 0\rangle$
$N_L = N_R = 1$	0	+	$a_1 a_{-1} \lvert 0\rangle$
$N_L = N_R = 2$	0	+	$a_2 a_{-2} \lvert 0\rangle$
		+	$a_1 a_1 a_{-1} a_{-1} \lvert 0\rangle$
		−	$a_2 a_{-1} a_{-1} \lvert 0\rangle$
		−	$a_1 a_1 a_{-2} \lvert 0\rangle$

Results, $D = 2 + 1$ (2)

Results for SU(6), $a\sqrt{\sigma} \sim 0.08582(5)$, **AA & M. Teper,** *JHEP* 05 (2011) 042





- **Outcome: Spectrum in D=2+1 appears to be string like and has no signal of resonances.**
- **How about D=3+1 dimensions?**

Lattice parameters in $D = 3 + 1$

Simulations

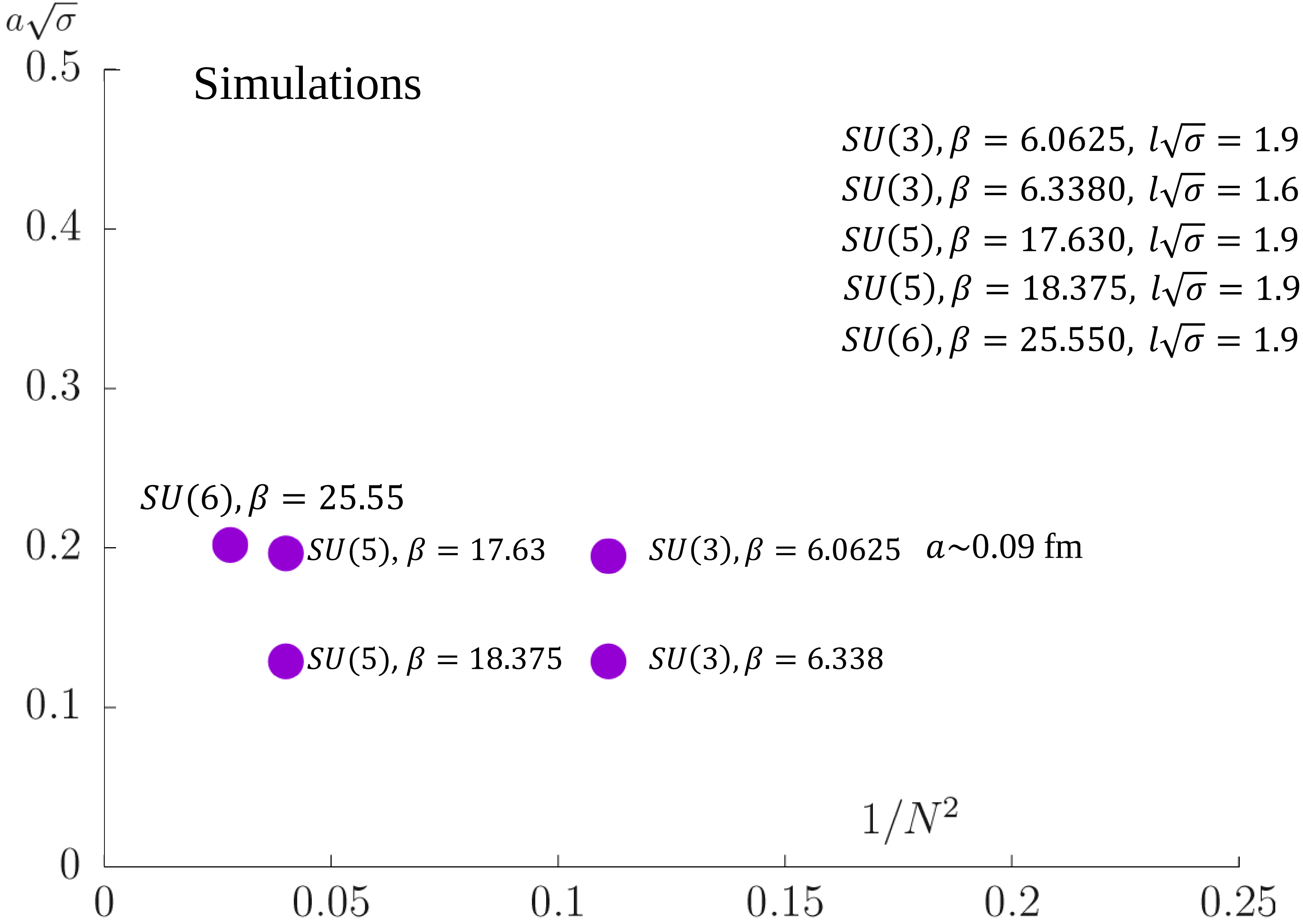
$SU(3), \beta = 6.0625,\ l\sqrt{\sigma} = 1.9 - 7$ 13 different flux tube lengths

$SU(3), \beta = 6.3380,\ l\sqrt{\sigma} = 1.6 - 5$ 14 different flux tube lengths

$SU(5), \beta = 17.630,\ l\sqrt{\sigma} = 1.9 - 4.8$ 8 different flux tube lengths

$SU(5), \beta = 18.375,\ l\sqrt{\sigma} = 1.9 - 4.8$ 5 different flux tube lengths

$SU(6), \beta = 25.550,\ l\sqrt{\sigma} = 1.9 - 4.8$ 8 different flux tube lengths

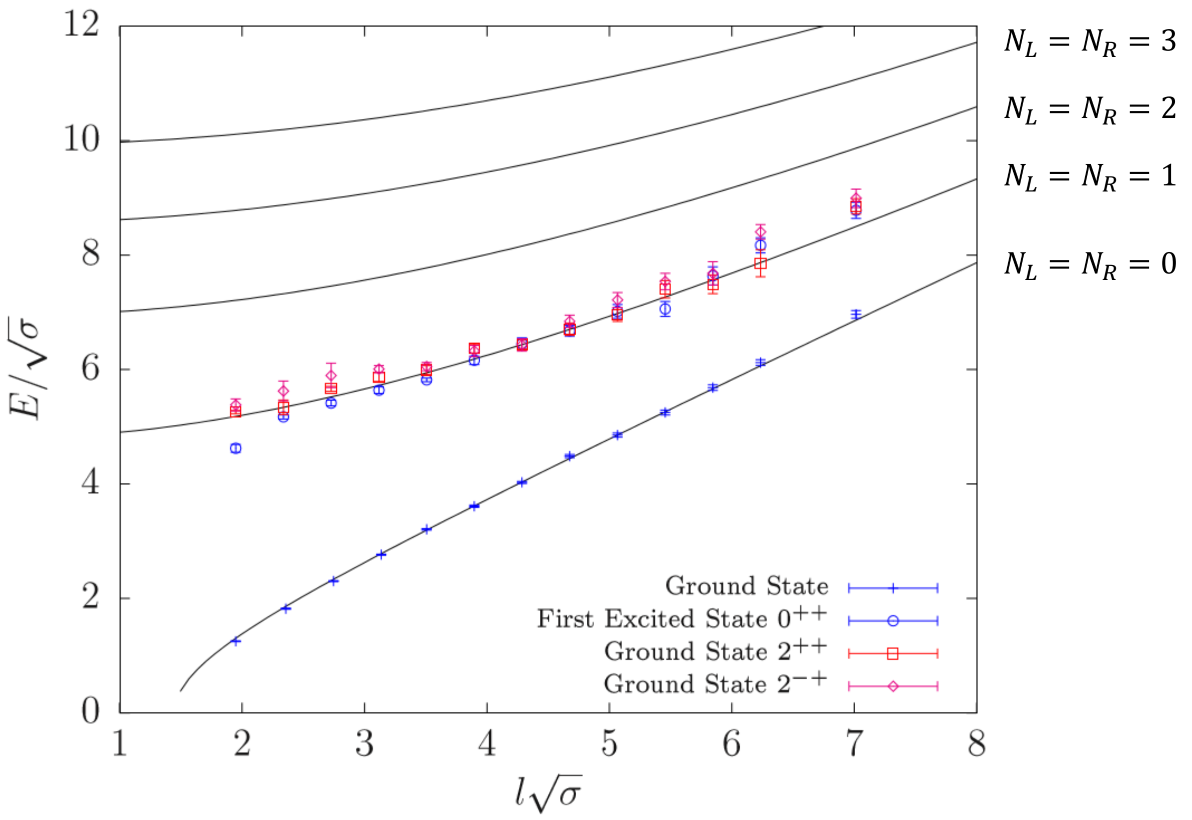


Results, $D = 3 + 1$ – first excited state $N_L = N_R = 1$ (1)

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	$P_\mathcal{P}$	$P_\mathcal{R}$	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	$+$		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	$\pm$		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$+$		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$-$		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$+$		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$+$		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

Results for the first excited state
Example for SU(3) and $\beta = 6.0625$



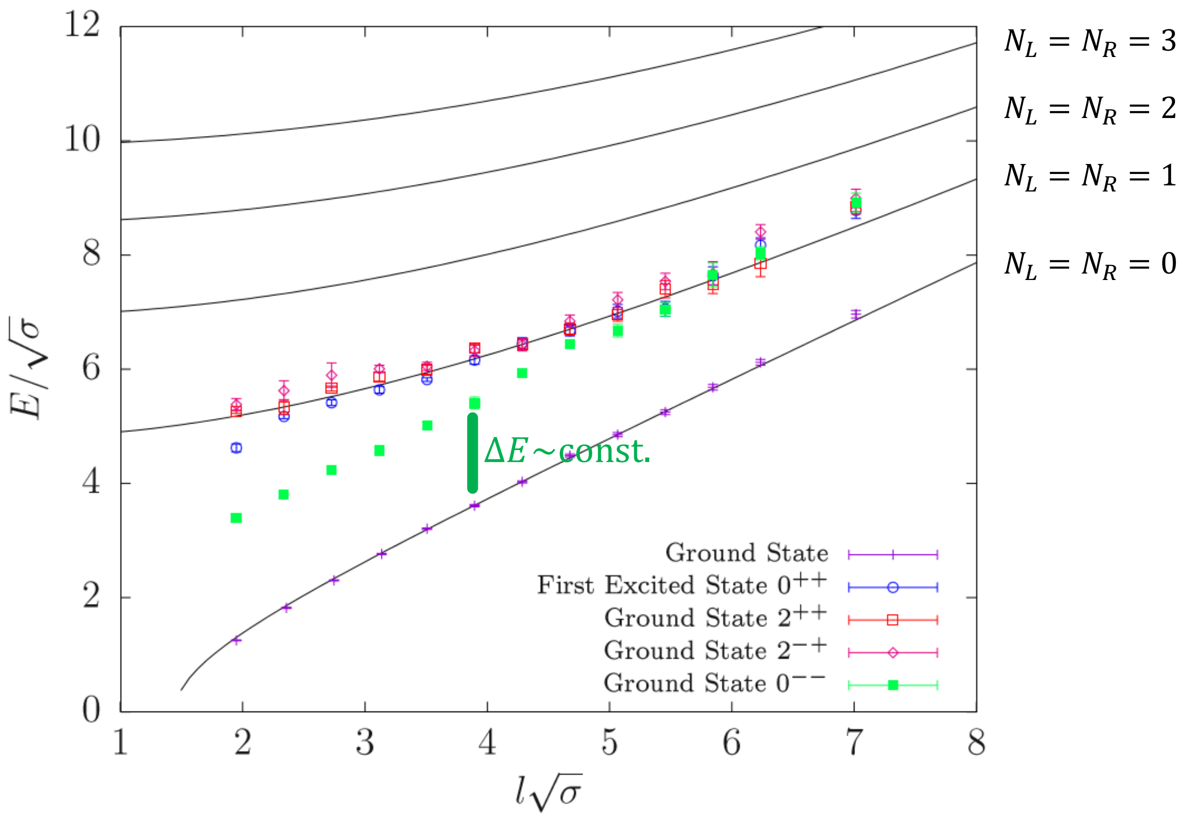
$$E^2 = (\sigma l)^2 + 8\pi\sigma\left(\frac{N_L + N_R}{2} - \frac{1}{12}\right) + \left(\frac{2\pi q}{l}\right)^2$$

Results, $D = 3+1$ – first excited state $N_L = N_R = 1$ (2)

String States & Quantum Numbers

N_L, N_R	$\lvert J\rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	$+$		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	$\pm$		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$+$		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$-$		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$+$		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$+$		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

Results for the ground state 0^{--}
Example for SU(3) and $\beta = 6.0625$



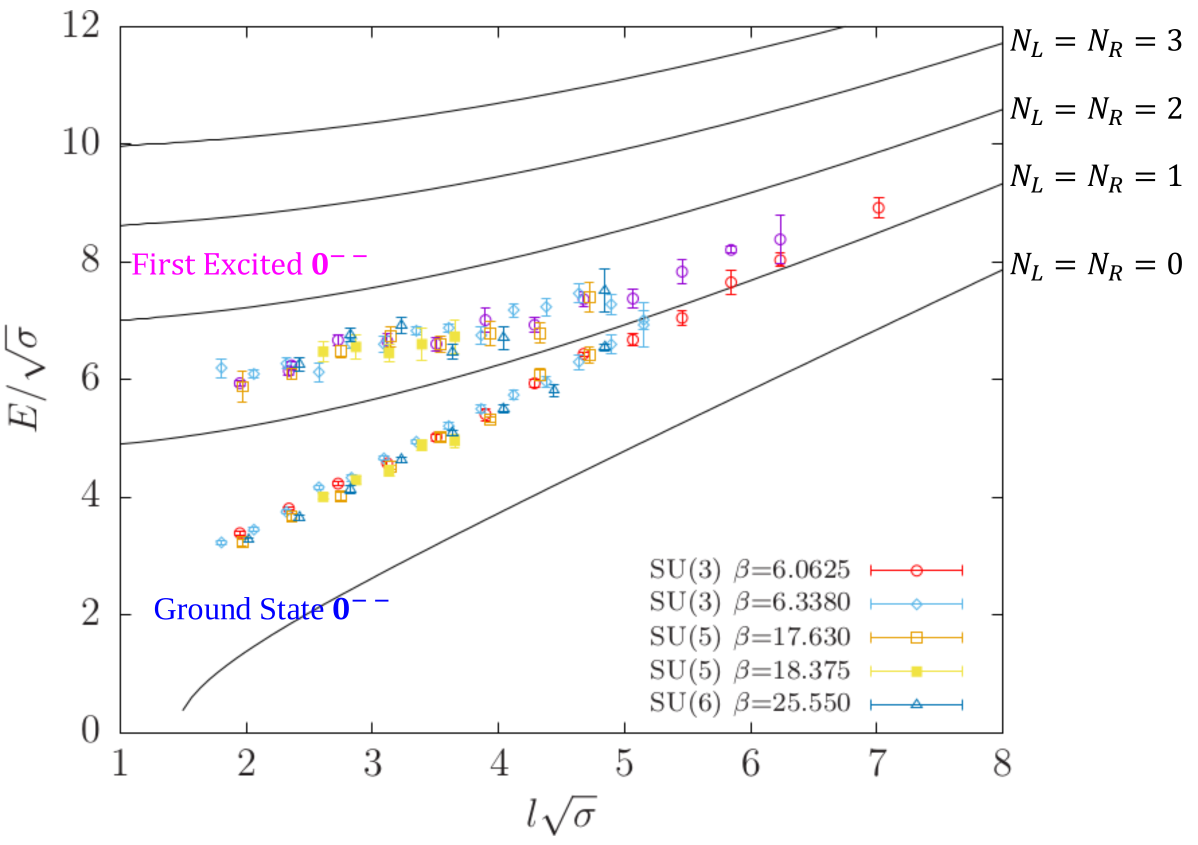
$$E^2 = (\sigma l)^2 + 8\pi\sigma\left(\frac{N_L+N_R}{2} - \frac{1}{12}\right) + \left(\frac{2\pi q}{l}\right)^2$$

Results, $D = 3+1$ – ground state $\mathbf{0}^{--}$ (1)

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	$+$	$+$	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	$-$	$-$	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	$+$	$+$	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	$-$	$-$	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	$+$	$+$	$a_1^+ a_1^- a_{-1}^+ a_{-1}^- \lvert 0\rangle$
	1	$\pm$	$+$	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	$\pm$	$-$	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	$\pm$	$+$	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	1	$\pm$	$-$	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	2	$+$	$+$	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	$-$	$+$	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	$+$	$+$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	$+$	$-$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	$-$	$+$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	$-$	$-$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	3	$\pm$	$+$	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	3	$\pm$	$-$	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	4	$+$	$+$	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$
	4	$-$	$-$	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$

Results for the ground and first excited state $\mathbf{0}^{--}$ For all ensembles



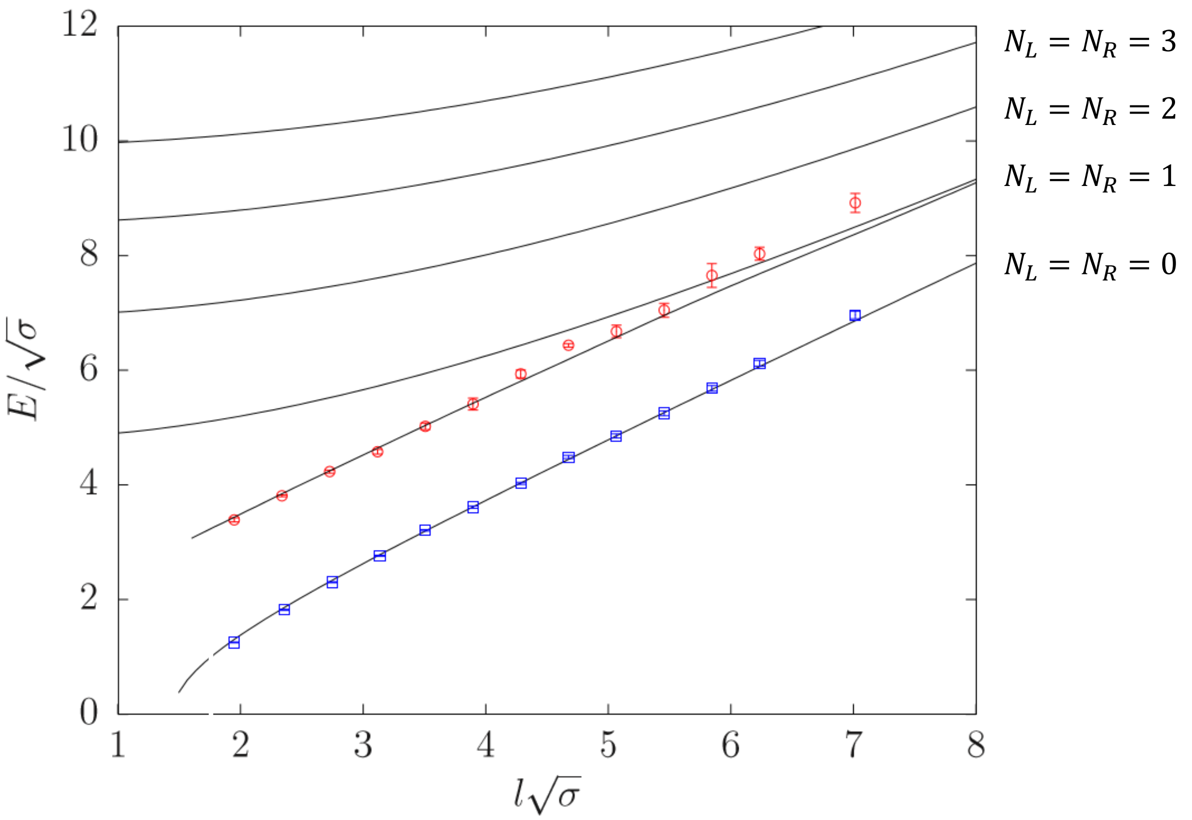
$$E^2 = (\sigma l)^2 + 8\pi\sigma\left(\frac{N_L+N_R}{2} - \frac{1}{12}\right) + \left(\frac{2\pi q}{l}\right)^2$$

Results, $D = 3 + 1$ – ground state $\mathbf{0}^{--}$ (2)

String States & Quantum Numbers

N_L, N_R	$\lvert J\rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	$+$		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	$\pm$		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$+$		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$-$		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$+$		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$+$		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

Results for the ground state 0^{--}
Example for SU(3) and $\boldsymbol{\beta} = 6.0625$



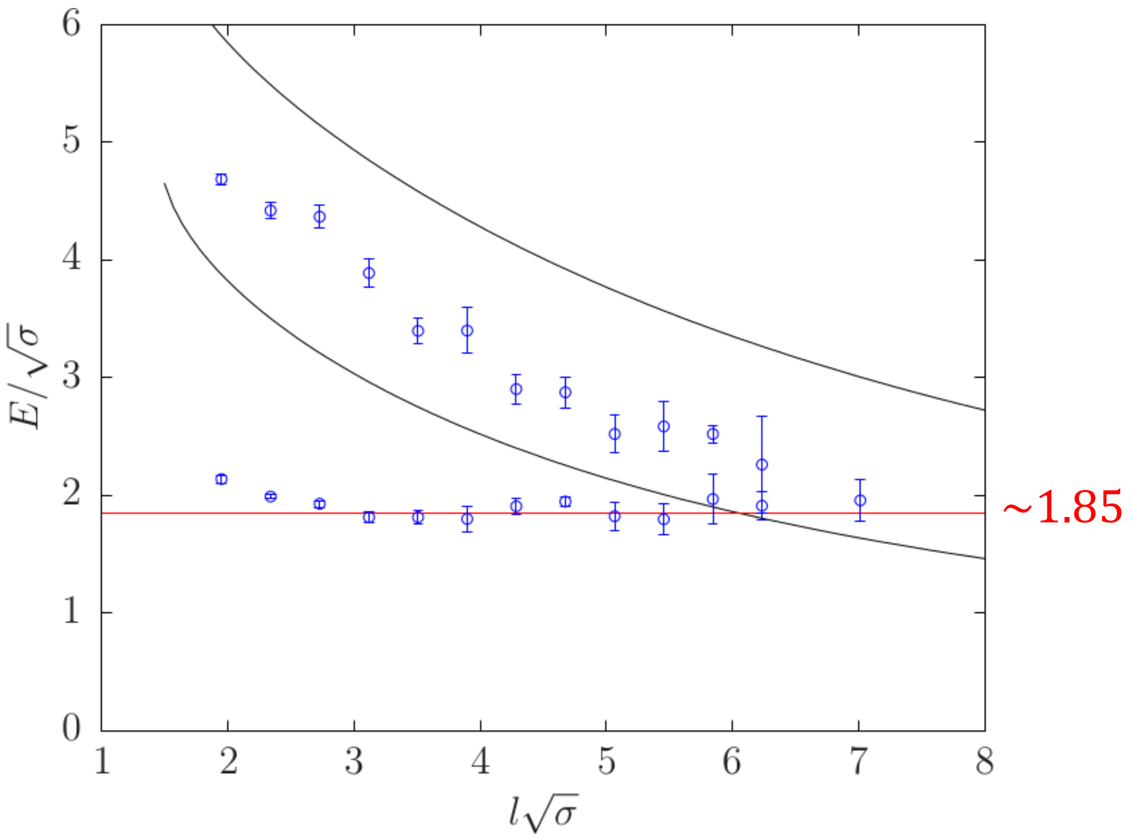
Including the pseudo-scalar field (axion)

Results, $D = 3 + 1$ – ground state $\mathbf{0}^{--}$ (3)

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	±		$\left(a_1^+ \pm a_1^-\right) \lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right) \lvert 0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right) \lvert 0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right) \lvert 0\rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right) \lvert 0\rangle$
	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right) \lvert 0\rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right) \lvert 0\rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right) \lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right) \lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^- \lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right] \lvert 0\rangle$
$N_L = 2, N_R = 2$	1	±	−	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right] \lvert 0\rangle$
	2	+	+	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right) \lvert 0\rangle$
	2	−	+	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right) \lvert 0\rangle$
	2	+	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	2	+	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	2	−	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	2	−	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0\rangle$
	3	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right] \lvert 0\rangle$
	3	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right] \lvert 0\rangle$
	4	+	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right) \lvert 0\rangle$
	4	−	−	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right) \lvert 0\rangle$

How about subtracting the absolute ground state for the ground and first excited state 0^{--}
Results presented for SU(3) and $\beta = 6.0625$

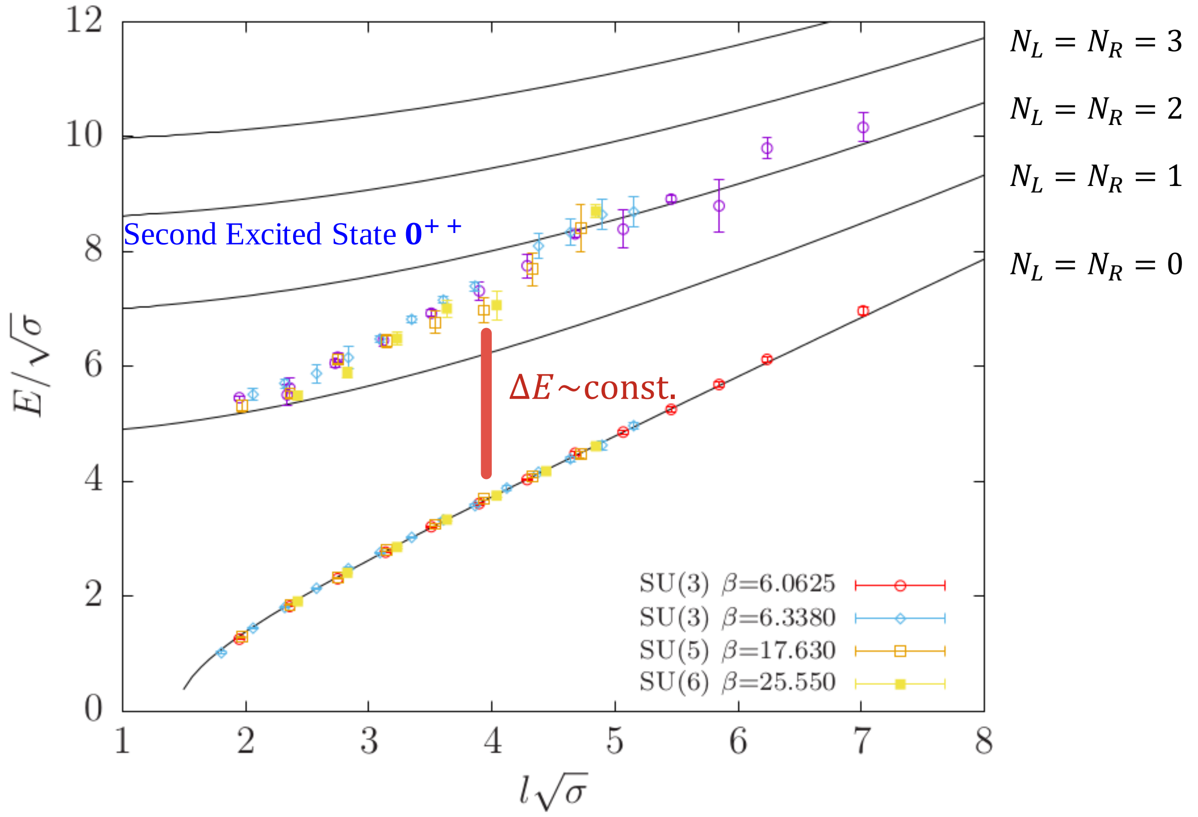


Results, $D = 3+1$ – second excited state 0^{++} (1)

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0 \rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right) \lvert 0 \rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = N_R = 1$	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = N_R = 1$	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 2$	0	$+$	$+$	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right) \lvert 0 \rangle$
	0	$-$	$-$	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right) \lvert 0 \rangle$
	0	$+$	$+$	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right) \lvert 0 \rangle$
	0	$-$	$-$	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right) \lvert 0 \rangle$
	0	$+$	$+$	$a_1^+ a_1^- a_{-1}^+ a_{-1}^- \lvert 0 \rangle$
	1	$\pm$	$+$	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right] \lvert 0 \rangle$
	1	$\pm$	$-$	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right] \lvert 0 \rangle$
	1	$\pm$	$+$	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right] \lvert 0 \rangle$
	1	$\pm$	$-$	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right] \lvert 0 \rangle$
	2	$+$	$+$	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right) \lvert 0 \rangle$
	2	$-$	$+$	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right) \lvert 0 \rangle$
	2	$+$	$+$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0 \rangle$
	2	$+$	$-$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0 \rangle$
	2	$-$	$+$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0 \rangle$
	2	$-$	$-$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0 \rangle$
	3	$\pm$	$+$	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right] \lvert 0 \rangle$
	3	$\pm$	$-$	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right] \lvert 0 \rangle$
	4	$+$	$+$	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right) \lvert 0 \rangle$
	4	$-$	$-$	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right) \lvert 0 \rangle$

Results for the ground and second excited state 0^{++}
Results presented for SU(3) and $\beta = 6.0625$



Absolute Ground State for $J = 0, P_{\mathcal{P}} = +, P_{\mathcal{R}} = +, q = 0.$

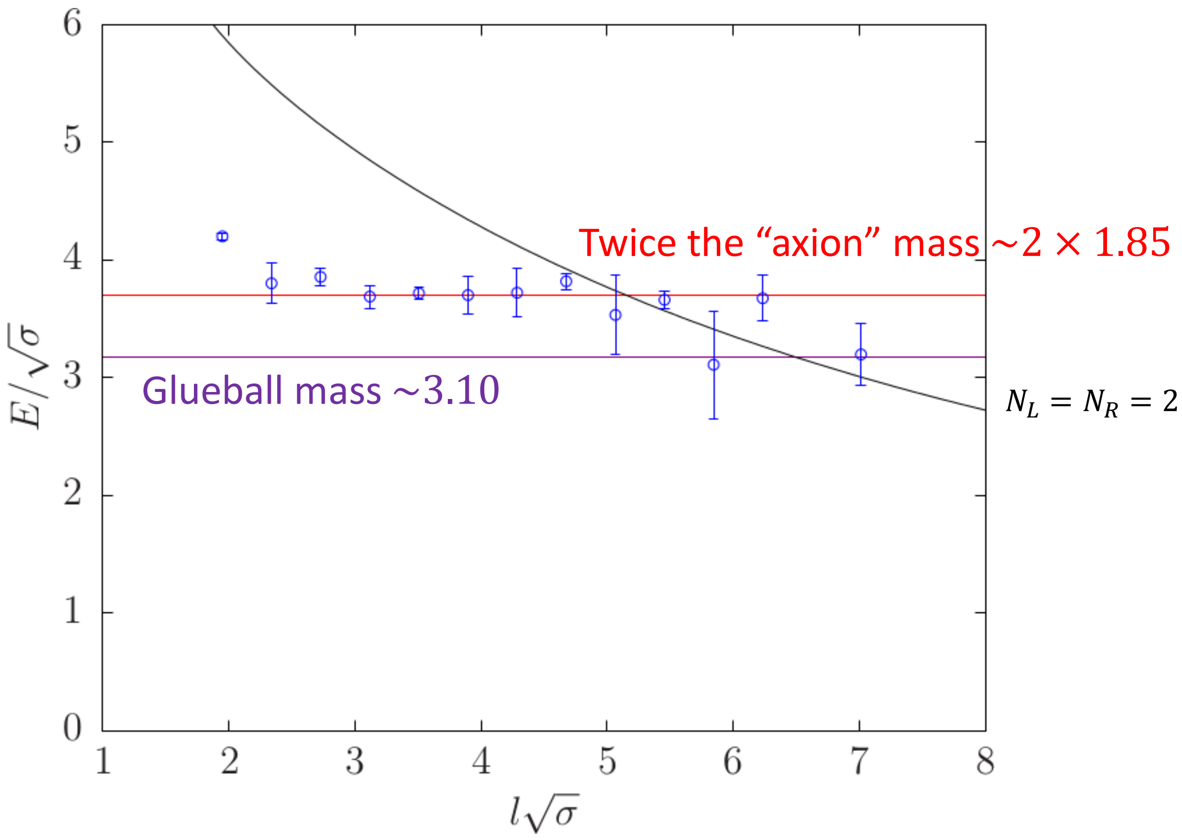
$$E^2 = (\sigma l)^2 + 8\pi\sigma\left(\frac{N_L + N_R}{2} - \frac{1}{12}\right) + \left(\frac{2\pi q}{l}\right)^2$$

Results, $D = 3 + 1$ – second excited state $\mathbf{0}^{++}$ (2)

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
	0	$+$	$+$	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	$-$	$-$	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	$+$	$+$	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	$-$	$-$	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	$+$	$+$	$a_1^+ a_1^- a_{-1}^+ a_{-1}^- \lvert 0\rangle$
	1	$\pm$	$+$	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	$\pm$	$-$	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	$\pm$	$+$	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
$N_L = 2, N_R = 2$	1	$\pm$	$-$	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	2	$+$	$+$	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	$-$	$+$	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	$+$	$+$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	$+$	$-$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	$-$	$+$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	$-$	$-$	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	3	$\pm$	$+$	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	3	$\pm$	$-$	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	4	$+$	$+$	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$
	4	$-$	$-$	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$

How about subtracting the absolute ground state? Results for the ground and second excited state $\mathbf{0}^{++}$. Results presented for SU(3) and $\boldsymbol{\beta} = \mathbf{6.0625}$



Absolute Ground State for $J = 0, P_{\mathcal{P}} = +, P_{\mathcal{R}} = +, q = 0.$

$$E^2 = (\sigma l)^2 + 8\pi\sigma\left(\frac{N_L + N_R}{2} - \frac{1}{12}\right) + \left(\frac{2\pi q}{l}\right)^2$$

Results, $D = 3+1$ – the $Q = 1$ sector (1)

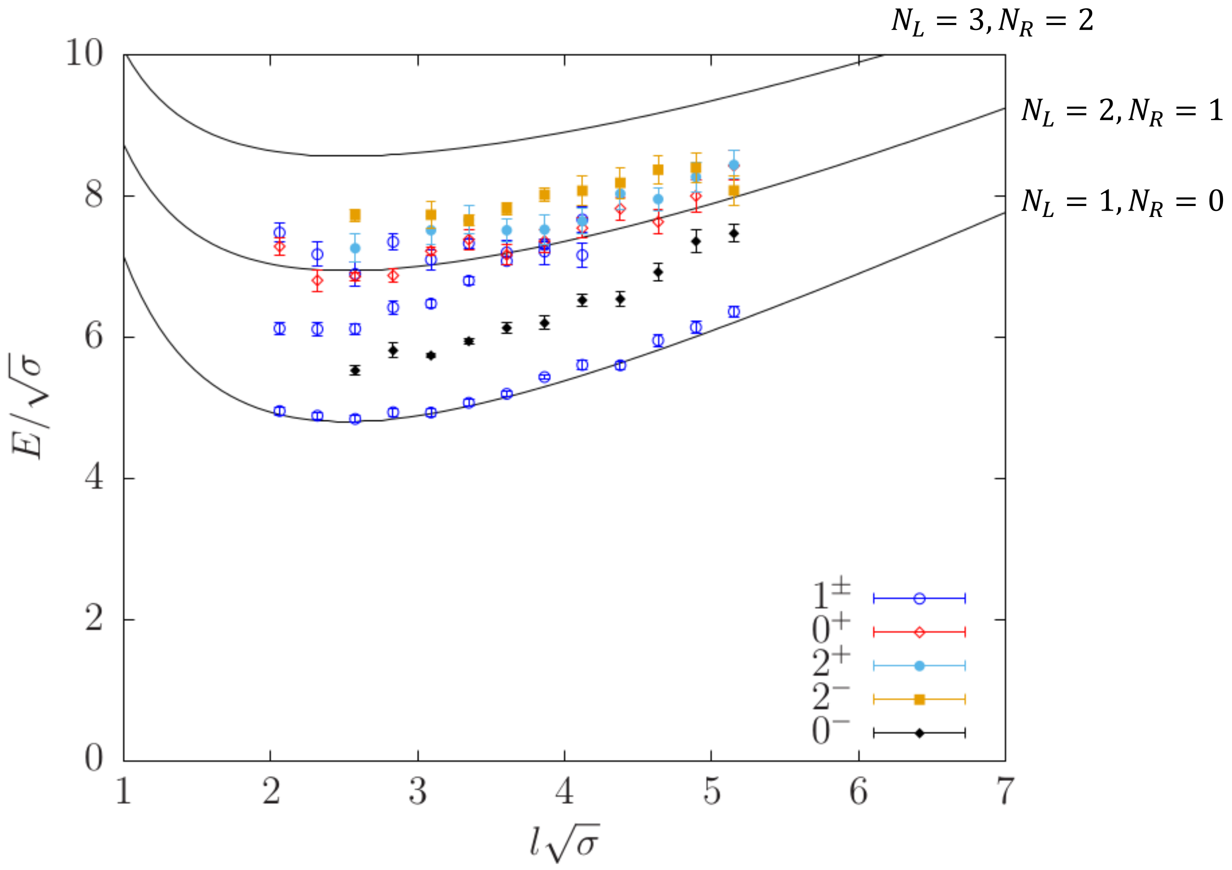
String States & Quantum Numbers

N_L, N_R	$\lvert J\rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	$+$		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	$\pm$		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$+$		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$-$		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$+$		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$+$		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

What are the states that correspond to $N_L = 2, N_R = 1$

State with 0^- behaves "anomalously"

Results presented for SU(3) and $\boldsymbol{\beta = 6.338}$?



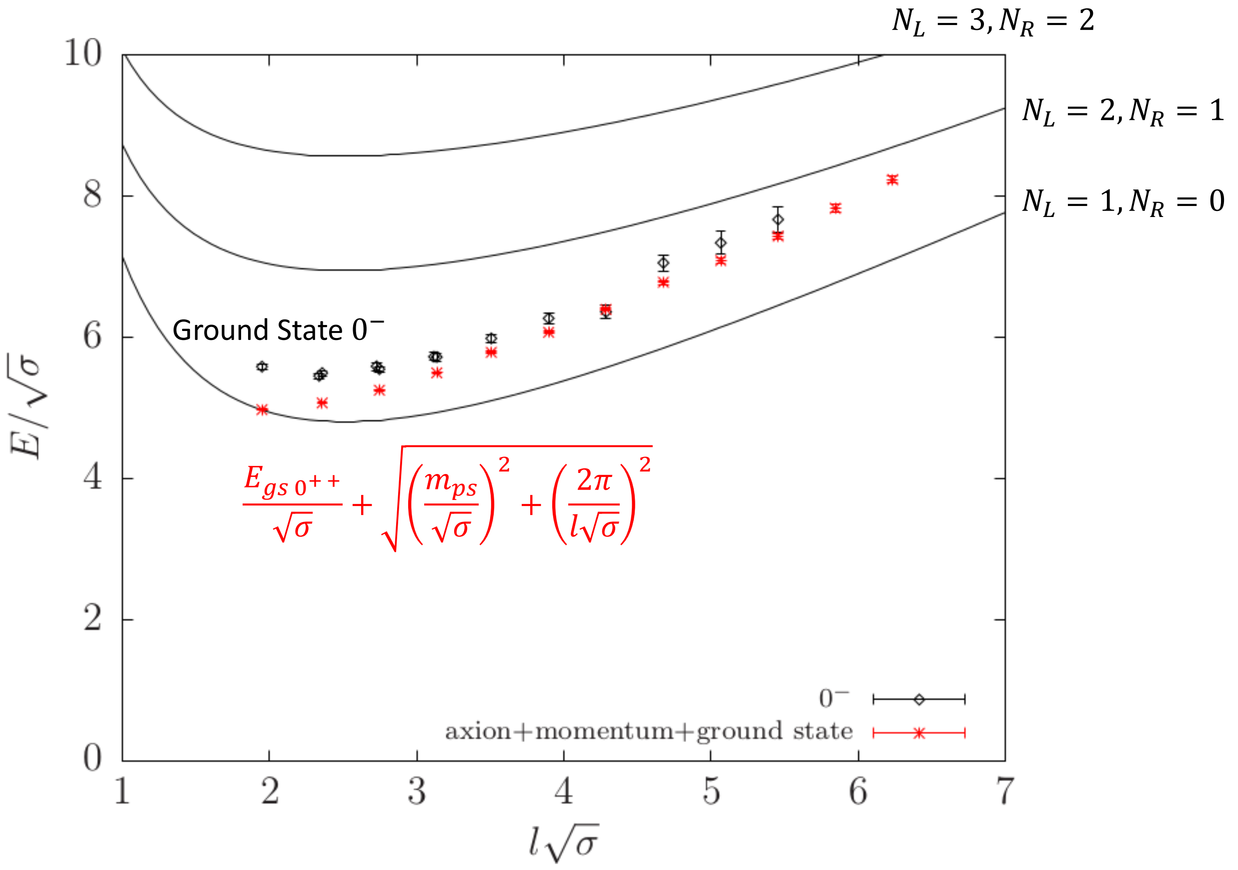
$$E^2 = (\sigma l)^2 + 8\pi\sigma\left(\frac{N_L+N_R}{2} - \frac{1}{12}\right) + \left(\frac{2\pi q}{l}\right)^2$$

Results, $D = 3 + 1$ – the $Q = 1$ sector (2)

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0 \rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right) \lvert 0 \rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = N_R = 1$	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = N_R = 1$	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 0$	0	$+$		$a_1^+ a_1^- \lvert 0 \rangle$
$N_L = 2, N_R = 0$	1	$\pm$		$\left(a_2^+ \pm a_2^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 0$	2	$+$		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 0$	2	$-$		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	0	$+$		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	2	$+$		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 1$	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right) \lvert 0 \rangle$

What if the state we observe is the "axion" with momentum on the ground state?
Results presented for SU(3) and $\beta = 6.0625$.



$$E^2 = (\sigma l)^2 + 8\pi\sigma\left(\frac{N_L + N_R}{2} - \frac{1}{12}\right) + \left(\frac{2\pi q}{l}\right)^2$$

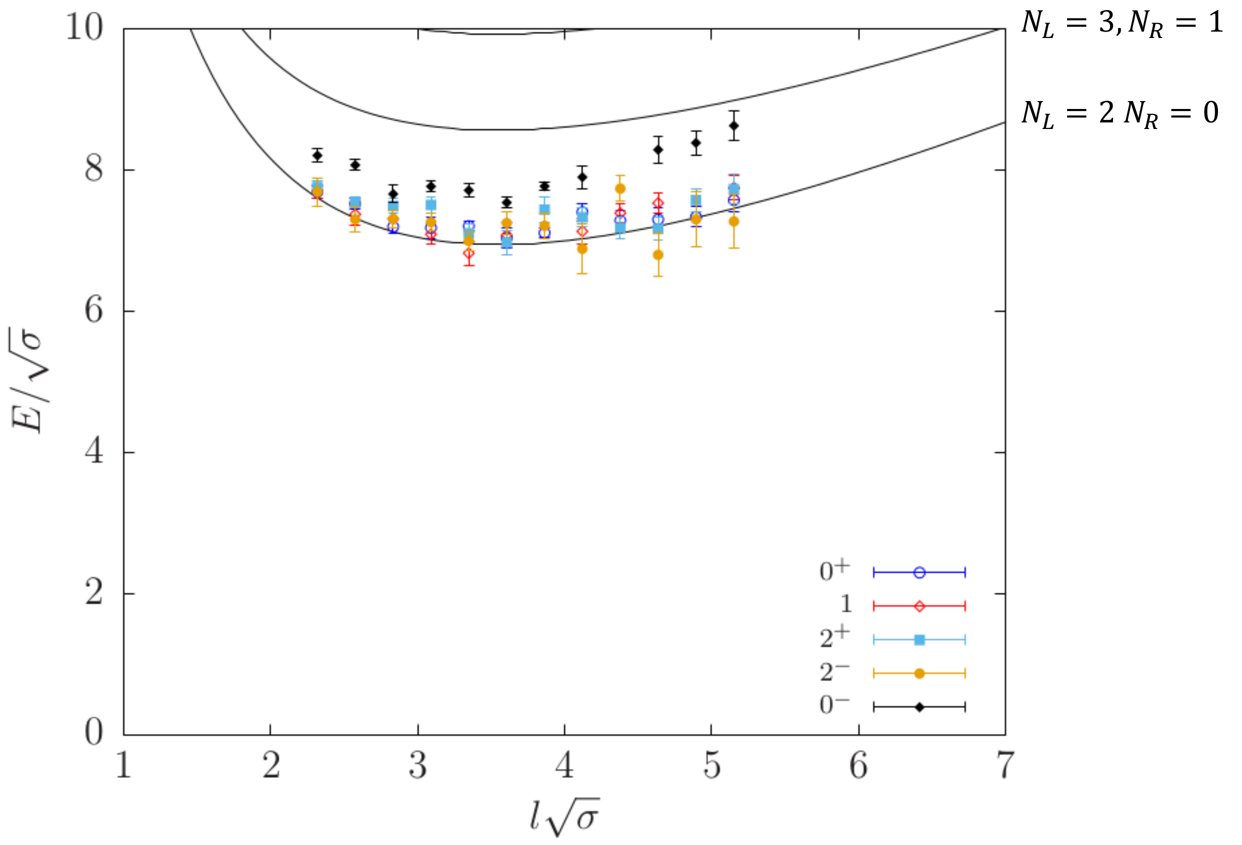
Results, $D = 3+1$ – the $Q = 2$ sector (1)

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	±		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	+		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	±		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	+		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	−		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	+		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	−		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	+		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	−		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	±		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

How does the ground state 0^- behave?

Results presented for SU(3) and $\boldsymbol{\beta = 6.338}$.



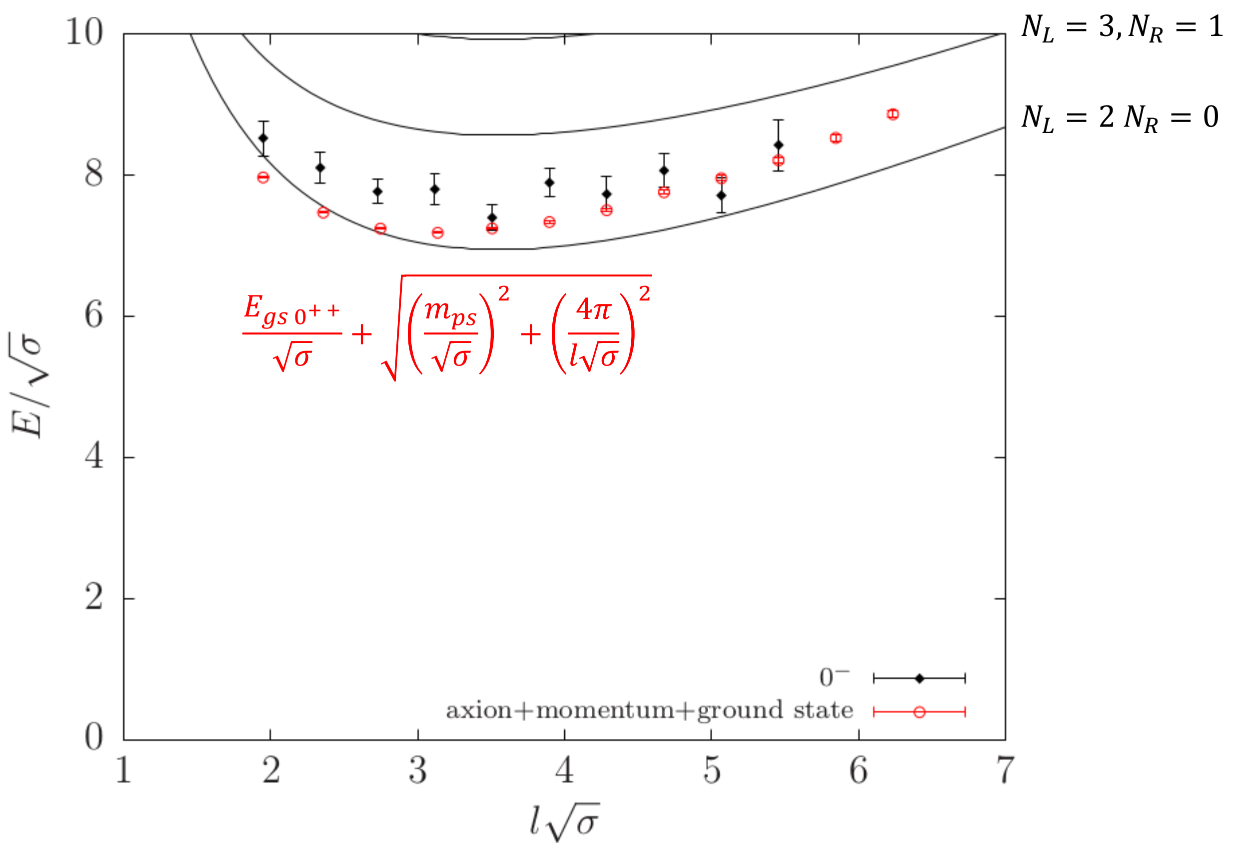
$$E^2 = (\sigma l)^2 + 8\pi\sigma\left(\frac{N_L + N_R}{2} - \frac{1}{12}\right) + \left(\frac{2\pi q}{l}\right)^2$$

Results, $D = 3 + 1$ – the $Q = 2$ sector (2)

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	$+$	$+$	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$+$	$+$	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	$-$	$-$	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$+$	$+$	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	$-$	$+$	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	0	$+$		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	$\pm$		$\left(a_2^+ \pm a_2^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$+$		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 0$	2	$-$		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$+$		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	0	$-$		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	1	$\pm$		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$+$		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	2	$-$		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 1$	3	$\pm$		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right)\lvert 0\rangle$

What if the state we observe is the "axion" with momentum on the ground state?
Results presented for SU(3) and $\beta = 6.0625$.

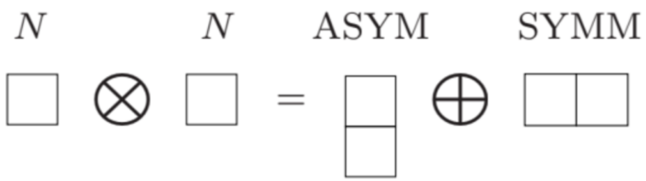


$$E^2 = (\sigma l)^2 + 8\pi\sigma\left(\frac{N_L+N_R}{2} - \frac{1}{12}\right) + \left(\frac{2\pi q}{l}\right)^2$$

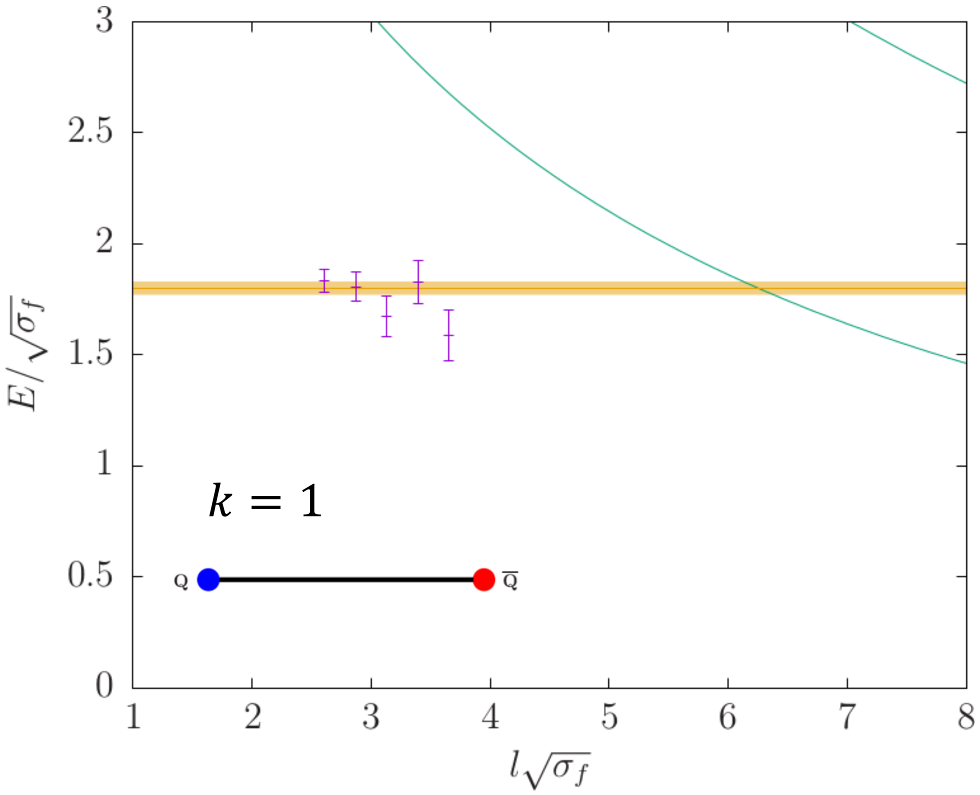
Results, $D = 3 + 1 - \boldsymbol{k} = \mathbf{2}$ string

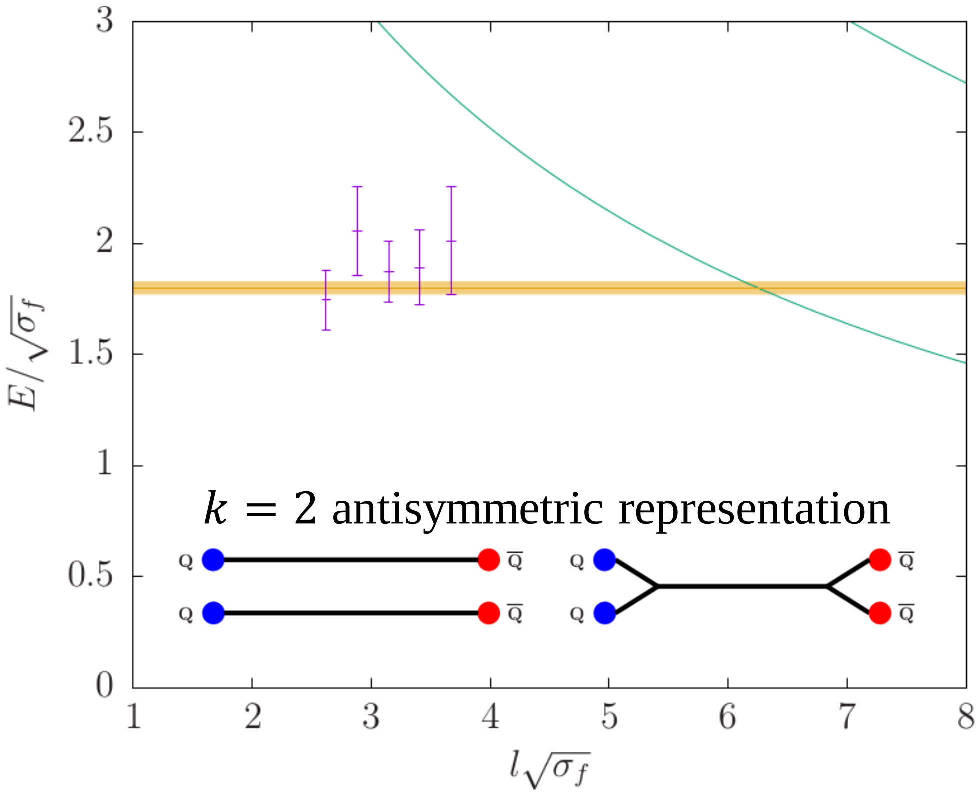
We can also extract the spectrum of flux-tubes in representations higher than the fundamental

- $k = 2$ antisymmetric representation for $SU(5)$



- We subtracted the absolute ground state from the ground state 0^{--}





Conclusions

- The flux tube looks pretty much like a bosonic NG string even for short flux tubes
  - This is striking for $D = 2 + 1$
  - Also holds for $D = 3 + 1$, albeit with some striking differences
- There is a massive "axion" particle with QM $\mathbf{0}^{--}$ on the worldsheet of the flux tube in $D = 3 + 1$ with mass approximately $m/\sqrt{\sigma} = 1.85$
- Looks like there is a massive state with mass of two axions with QM $\mathbf{0}^{++}$
- The "axion" appears also in the spectrum of the $k = 2$ string with the same mass
  - Universality?
- Signals for more massive modes

Thanks for your attention

Backup slide 1: Quantum Numbers – String States

→ Define α^{+}_{-k} and α^{-}_{-k} as $(x, y$ are the transverse directions):

- $\alpha^{+}_{-k} = \alpha^{x}_{-k} + i\alpha^{y}_{-k}$
- $\alpha^{-}_{-k} = \alpha^{x}_{-k} - i\alpha^{y}_{-k}$

→ Spin J.

- $J =\mid \#(+) - \#(-) \mid$

→ $\mathcal{P}$-Parity

- Under $\mathcal{P}$-Parity: $\alpha^{+}_{-k} \xleftrightarrow{P_{\mathcal{P}}} \alpha^{-}_{-k}$ & $\bar{\alpha}^{+}_{-k} \xleftrightarrow{P_{\mathcal{P}}} \bar{\alpha}^{-}_{-k}$

→ $\mathcal{R}$-Parity

- Under $\mathcal{R}$-Parity: $\alpha^{\pm}_{-k} \xleftrightarrow{P_{\mathcal{R}}} \bar{\alpha}^{\pm}_{-k}$

- Example: $(\alpha^{+}_{-1}\bar{\alpha}^{+}_{-1} \pm \alpha^{-}_{-1}\bar{\alpha}^{-}_{-1}) \mid 0\rangle$
  - $J = 2$
  - $P_{\mathcal{P}} = \pm$
  - $P_{\mathcal{R}} = +$

N_L, N_R	$\lvert J \rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	±		$\left(a_1^+ \pm a_1^-\right) \lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right) \lvert 0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right) \lvert 0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right) \lvert 0\rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right) \lvert 0\rangle$
$N_L = 2, N_R = 0$	0	+		$a_1^+ a_1^- \lvert 0\rangle$
$N_L = 2, N_R = 0$	1	±		$\left(a_2^+ \pm a_2^-\right) \lvert 0\rangle$
$N_L = 2, N_R = 0$	2	+		$\left(a_1^+ a_1^+ + a_1^- a_1^-\right) \lvert 0\rangle$
$N_L = 2, N_R = 0$	2	−		$\left(a_1^+ a_1^+ - a_1^- a_1^-\right) \lvert 0\rangle$
$N_L = 2, N_R = 1$	0	+		$\left(a_2^+ a_{-1}^- + a_2^- a_{-1}^+\right) \lvert 0\rangle$
$N_L = 2, N_R = 1$	0	−		$\left(a_2^+ a_{-1}^- - a_2^- a_{-1}^+\right) \lvert 0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^+ a_{-1}^- \pm a_1^- a_1^- a_{-1}^+\right) \lvert 0\rangle$
$N_L = 2, N_R = 1$	1	±		$\left(a_1^+ a_1^- a_{-1}^- \pm a_1^- a_1^+ a_{-1}^+\right) \lvert 0\rangle$
$N_L = 2, N_R = 1$	2	+		$\left(a_2^+ a_{-1}^+ + a_2^- a_{-1}^-\right) \lvert 0\rangle$
$N_L = 2, N_R = 1$	2	−		$\left(a_2^+ a_{-1}^+ - a_2^- a_{-1}^-\right) \lvert 0\rangle$
$N_L = 2, N_R = 1$	3	±		$\left(a_1^+ a_1^+ a_{-1}^+ \pm a_1^- a_1^- a_{-1}^-\right) \lvert 0\rangle$

Backup slide 2: Operator construction: irreducible representations

~ 1200 different operators with transverse deformations for different irreducible representations ($J^{P_{\mathcal{P}}\ P_{\mathcal{R}}}$)

The Correlation Matrix should Have a block diagonal form

$$C_{ij}(t) = \langle \Phi_i^\dagger(t)\Phi_j(0)\rangle = \begin{array}{cc} \left[\begin{array}{cccccccccc} [\boxed{0^{++}}] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & [\boxed{0^{+-}}] & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & [\boxed{0^{-+}}] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & [\boxed{0^{--}}] & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & [\boxed{1^{\pm+}}] & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & [\boxed{1^{\pm-}}] & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & [\boxed{2^{++}}] & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & [\boxed{2^{+-}}] & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & [\boxed{2^{-+}}] & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & [\boxed{2^{--}}] \end{array}\right] & \begin{array}{l} \sim 190 \\ \sim 50 \\ \sim 75 \\ \sim 110 \\ \sim 170 \\ \sim 160 \\ \sim 135 \\ \sim 100 \\ \sim 125 \\ \sim 60 \end{array} \end{array}$$

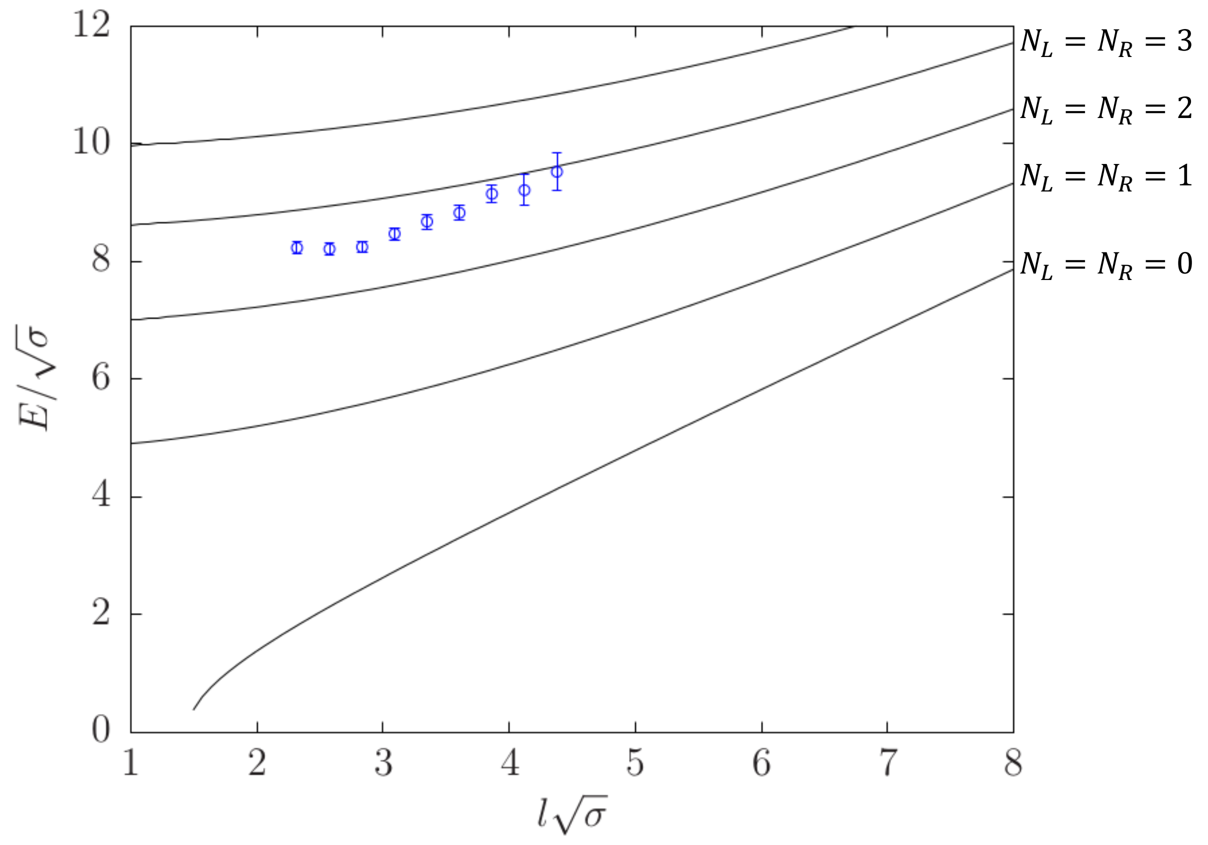
Backup slide 3: $D = 3 + 1$ – second excited state $\mathbf{0}^{-+}$

String States & Quantum Numbers

N_L, N_R	$\vert J\vert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	+	+	$\vert 0\rangle$
$N_L = 1, N_R = 0$	1	±		$\left(a_1^+ \pm a_1^-\right) \vert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right) \vert 0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right) \vert 0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right) \vert 0\rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right) \vert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right) \vert 0\rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right) \vert 0\rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right) \vert 0\rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right) \vert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^- \vert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right] \vert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right] \vert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right] \vert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right] \vert 0\rangle$
	2	+	+	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right) \vert 0\rangle$
	2	−	+	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right) \vert 0\rangle$
	2	+	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \vert 0\rangle$
	2	+	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \vert 0\rangle$
	2	−	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \vert 0\rangle$
	2	−	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \vert 0\rangle$
	3	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right] \vert 0\rangle$
	3	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right] \vert 0\rangle$
	4	+	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right) \vert 0\rangle$
	4	−	−	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right) \vert 0\rangle$

What is the mass of the ground state for $\mathbf{0}^{-+}$

Results presented for SU(3) and $\boldsymbol{\beta = 6.338}$



$$E^2 = (\sigma l)^2 + 8\pi\sigma\left(\frac{N_L+N_R}{2} - \frac{1}{12}\right) + \left(\frac{2\pi q}{l}\right)^2$$

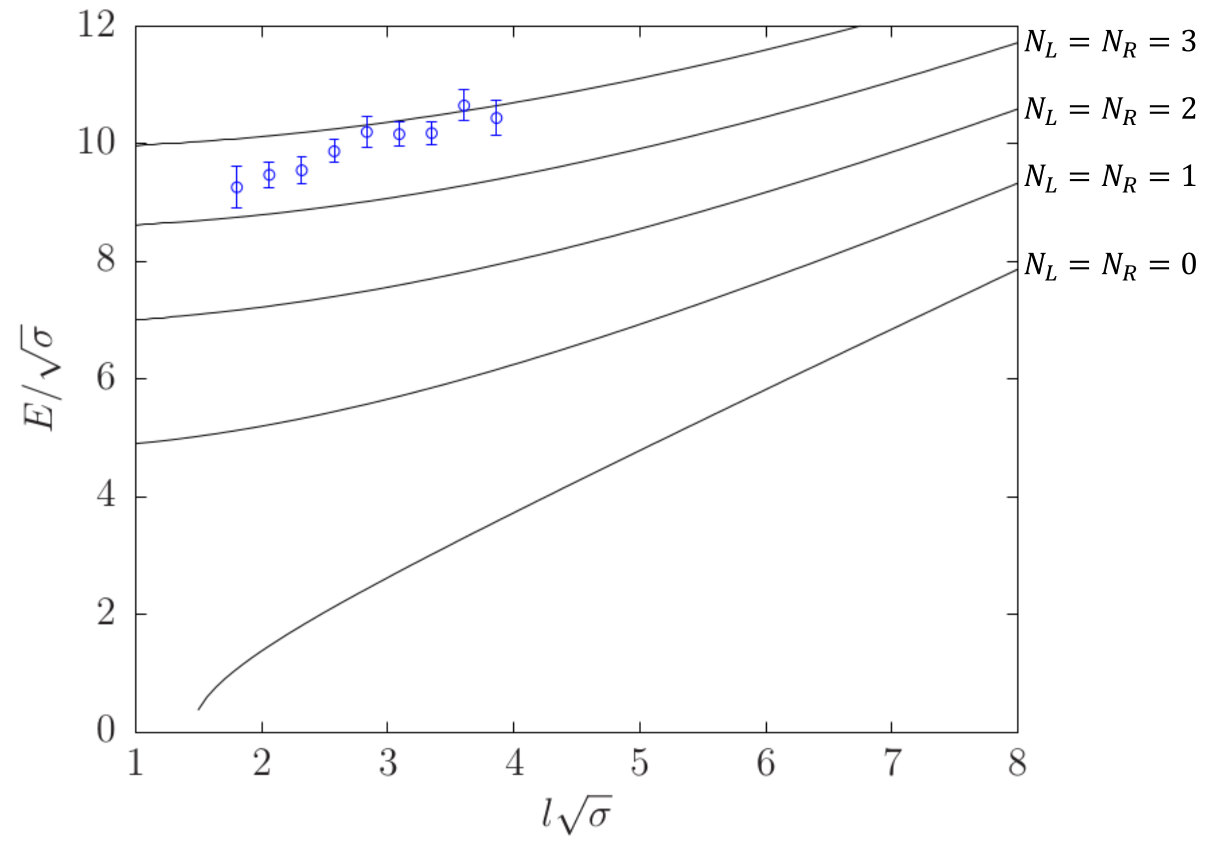
Backup slide 4: $D = 3 + 1$ – second excited state $\mathbf{0^{+\,-}}$

String States & Quantum Numbers

N_L, N_R	$\lvert J\rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	±		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^-\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
$N_L = 2, N_R = 2$	1	±	−	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	2	+	+	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	+	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	+	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	3	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	3	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	4	+	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$
	4	−	−	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$

What is the mass of the ground state for $\mathbf{0^{+\,-}}$

Results presented for SU(3) and $\boldsymbol{\beta = 6.338}$



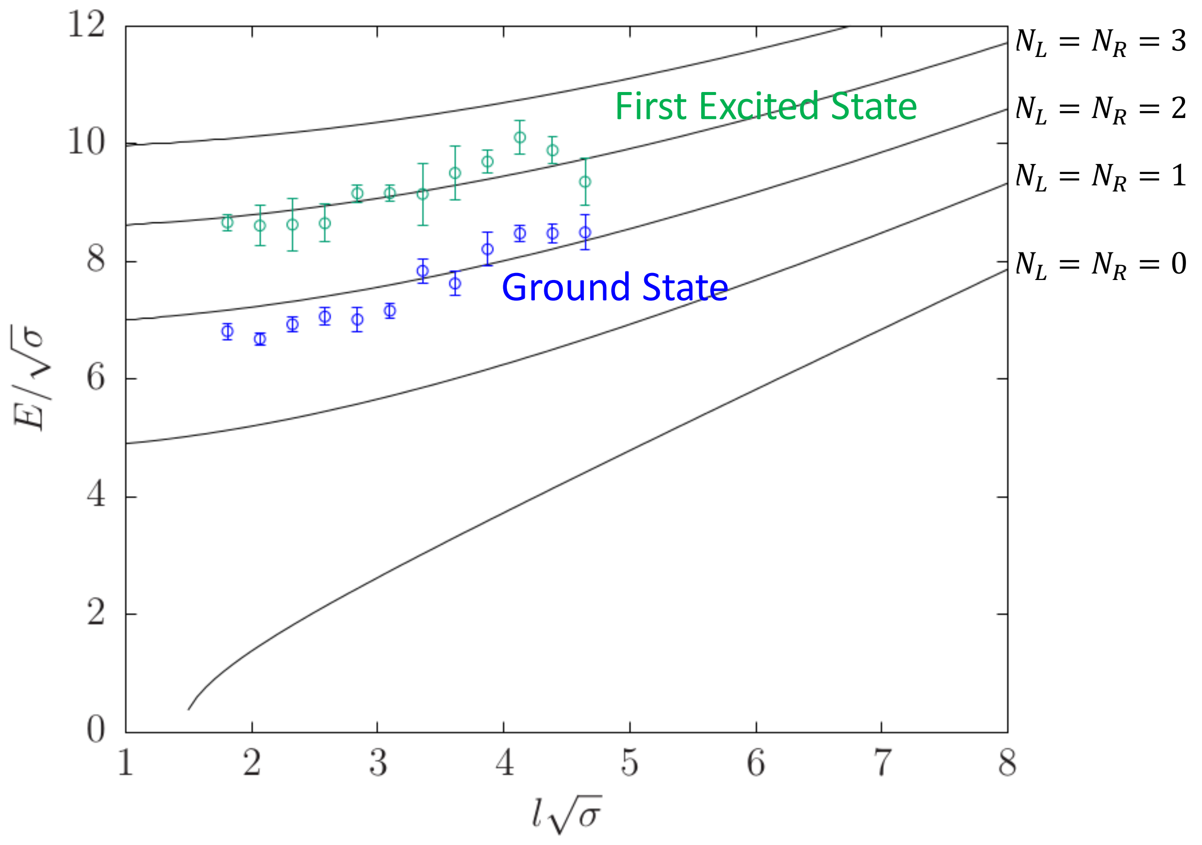
$$E^2 = (\sigma l)^2 + 8\pi\sigma\left(\frac{N_L + N_R}{2} - \frac{1}{12}\right) + \left(\frac{2\pi q}{l}\right)^2$$

Backup slide 5: $D = 3 + 1$ – states with $2^{+\,-}$

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	±		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = 2, N_R = 2$	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^- \lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	2	+	+	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	+	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	+	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	3	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	3	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	4	+	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$
	4	−	−	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$

What is the mass of the ground state for $2^{+\,-}$
Results presented for SU(3) and $\boldsymbol{\beta = 6.338}$



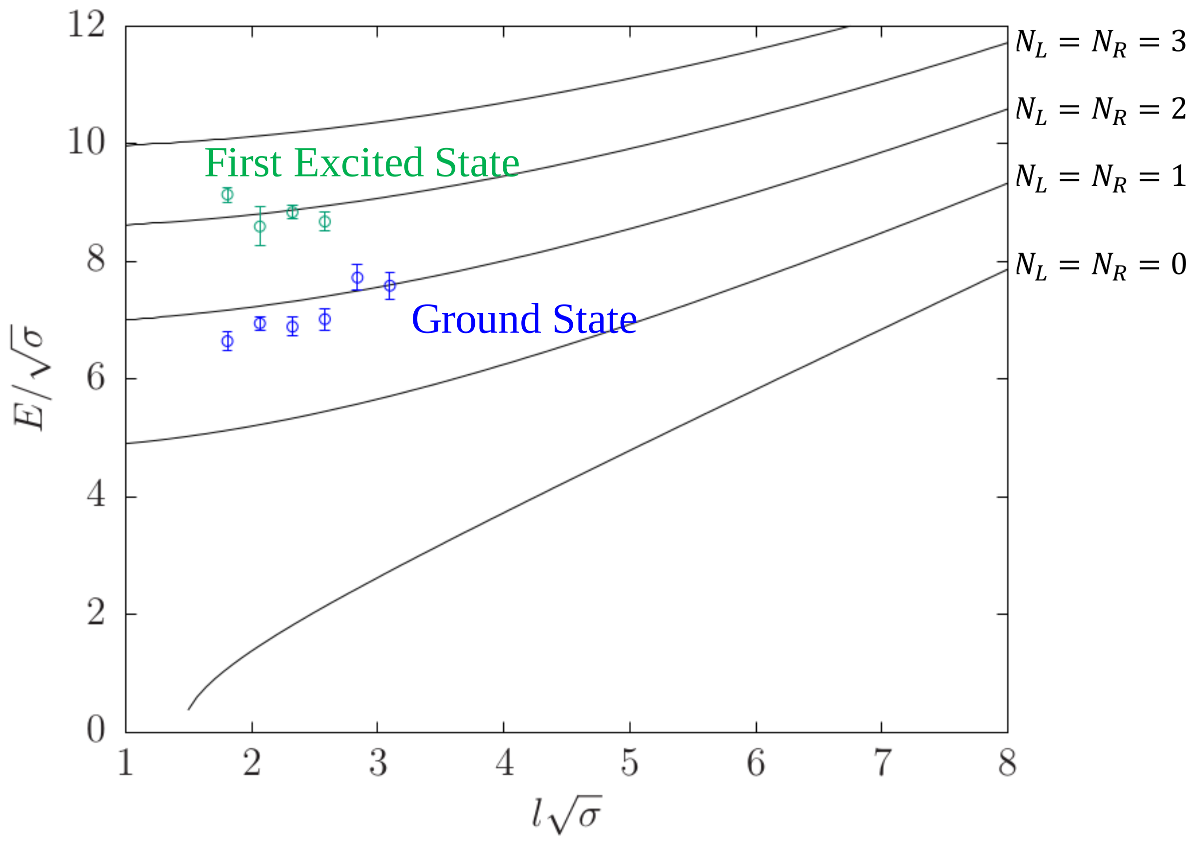
$$E^2 = (\sigma l)^2 + 8\pi\sigma\left(\frac{N_L+N_R}{2} - \frac{1}{12}\right) + \left(\frac{2\pi q}{l}\right)^2$$

Backup slide 6: $D = 3 + 1$ – states with 2^{--}

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0 \rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right) \lvert 0 \rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right) \lvert 0 \rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right) \lvert 0 \rangle$
$N_L = 2, N_R = 2$	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right) \lvert 0 \rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right) \lvert 0 \rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right) \lvert 0 \rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right) \lvert 0 \rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^- \lvert 0 \rangle$
	1	$\pm$	+	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right] \lvert 0 \rangle$
	1	$\pm$	−	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right] \lvert 0 \rangle$
	1	$\pm$	+	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right] \lvert 0 \rangle$
	1	$\pm$	−	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right] \lvert 0 \rangle$
	2	+	+	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right) \lvert 0 \rangle$
	2	−	+	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right) \lvert 0 \rangle$
	2	+	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0 \rangle$
	2	+	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0 \rangle$
	2	−	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0 \rangle$
	2	−	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right] \lvert 0 \rangle$
	3	$\pm$	+	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right] \lvert 0 \rangle$
	3	$\pm$	−	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right] \lvert 0 \rangle$
	4	+	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right) \lvert 0 \rangle$
	4	−	−	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right) \lvert 0 \rangle$

What is the mass of the ground state for 2^{--}
Results presented for SU(3) and $\beta = 6.338$



$$E^2 = (\sigma l)^2 + 8\pi\sigma\left(\frac{N_L + N_R}{2} - \frac{1}{12}\right) + \left(\frac{2\pi q}{l}\right)^2$$

Backup slide 7: $D = 3 + 1$ – ground and first excited states 1^+

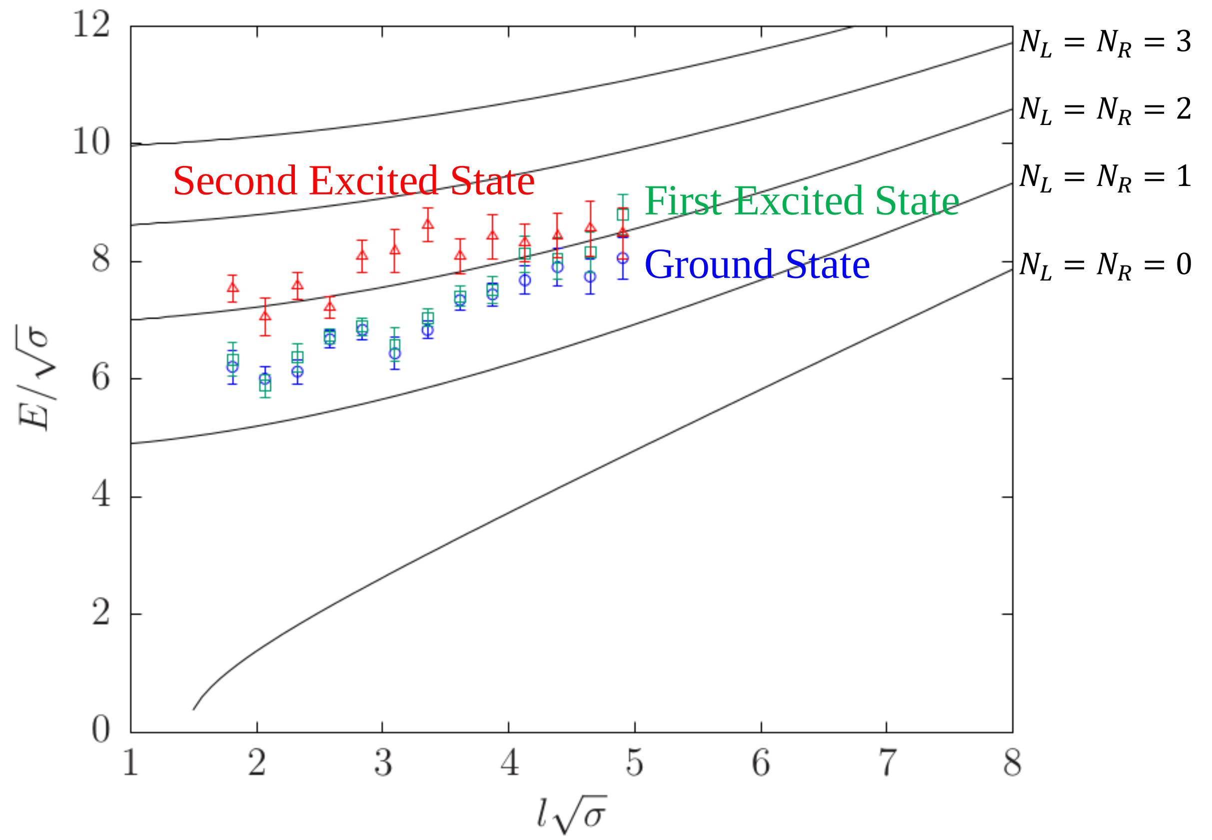
String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	$\pm$		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$

$N_L = 2, N_R = 2$	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^- \lvert 0\rangle$
	1	$\pm$	+	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	$\pm$	−	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	$\pm$	+	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	1	$\pm$	−	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	2	+	+	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	+	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	+	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
???	3	$\pm$	+	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	3	$\pm$	−	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	4	+	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$
	4	−	−	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$

What is the mass of the ground state and first excited state for 1^+

Results presented for SU(3) and $\beta = 6.338$



$$E^2 = (\sigma l)^2 + 8\pi\sigma\left(\frac{N_L + N_R}{2} - \frac{1}{12}\right) + \left(\frac{2\pi q}{l}\right)^2$$

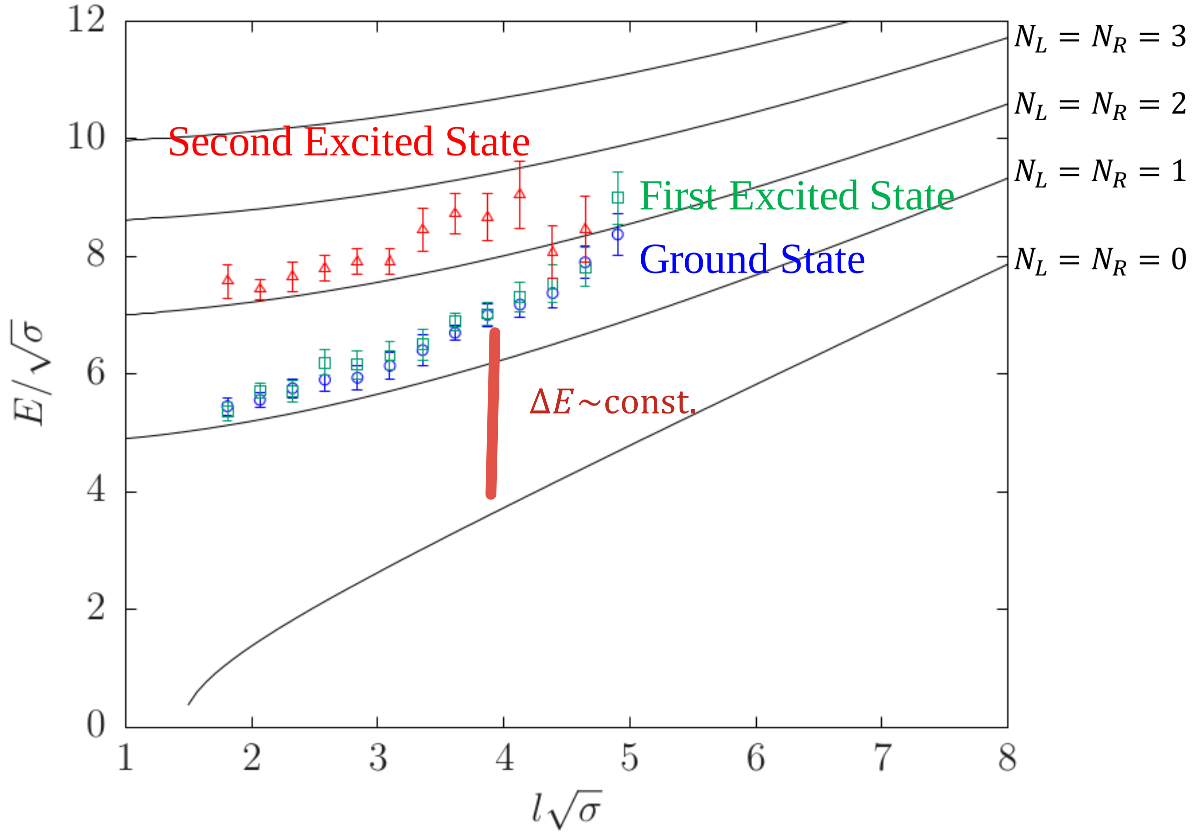
Backup slide 8: $D = 3 + 1$ – ground and first excited states 1^-

String States & Quantum Numbers

N_L, N_R	$\lvert J \rvert$	$P_{\mathcal{P}}$	$P_{\mathcal{R}}$	String State
$N_L = N_R = 0$	0	+	+	$\lvert 0\rangle$
$N_L = 1, N_R = 0$	1	±		$\left(a_1^+ \pm a_1^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	+	+	$\left(a_1^+ a_{-1}^- + a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	0	−	−	$\left(a_1^+ a_{-1}^- - a_1^- a_{-1}^+\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	+	+	$\left(a_1^+ a_{-1}^+ + a_1^- a_{-1}^-\right)\lvert 0\rangle$
$N_L = N_R = 1$	2	−	+	$\left(a_1^+ a_{-1}^+ - a_1^- a_{-1}^-\right)\lvert 0\rangle$
	0	+	+	$\left(a_2^+ a_{-2}^- + a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_2^+ a_{-2}^- - a_2^- a_{-2}^+\right)\lvert 0\rangle$
	0	+	+	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- + a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	−	−	$\left(a_1^+ a_1^+ a_{-1}^- a_{-1}^- - a_1^- a_1^- a_{-1}^+ a_{-1}^+\right)\lvert 0\rangle$
	0	+	+	$a_1^+ a_1^- a_{-1}^+ a_{-1}^- \lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ + a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^+ - a_2^- a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	1	±	+	$\left[\left(a_1^+ a_1^- a_{-2}^+ + a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- + a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
$N_L = 2, N_R = 2$	1	±	−	$\left[\left(a_1^+ a_1^- a_{-2}^+ - a_2^+ a_{-1}^- a_{-1}^+\right) \pm \left(a_1^- a_1^+ a_{-2}^- - a_2^- a_{-1}^+ a_{-1}^-\right)\right]\lvert 0\rangle$
	2	+	+	$\left(a_2^+ a_{-2}^+ + a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	−	+	$\left(a_2^+ a_{-2}^+ - a_2^- a_{-2}^-\right)\lvert 0\rangle$
	2	+	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	+	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- + a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- + a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	+	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) + \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	2	−	−	$\left[\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^- - a_1^- a_1^- a_{-1}^- a_{-1}^+\right) - \left(a_1^+ a_1^- a_{-1}^- a_{-1}^- - a_1^- a_1^+ a_{-1}^+ a_{-1}^+\right)\right]\lvert 0\rangle$
	3	±	+	$\left[\left(a_1^+ a_1^+ a_{-2}^+ + a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- + a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
???	3	±	−	$\left[\left(a_1^+ a_1^+ a_{-2}^+ - a_2^+ a_{-1}^+ a_{-1}^+\right) \pm \left(a_1^- a_1^- a_{-2}^- - a_2^- a_{-1}^- a_{-1}^-\right)\right]\lvert 0\rangle$
	4	+	+	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ + a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$
	4	−	−	$\left(a_1^+ a_1^+ a_{-1}^+ a_{-1}^+ - a_1^- a_1^- a_{-1}^- a_{-1}^-\right)\lvert 0\rangle$

What is the mass of the ground state and first excited state for 1^-

Results presented for SU(3) and $\beta = 6.338$



$$E^2 = (\sigma l)^2 + 8\pi\sigma\left(\frac{N_L+N_R}{2} - \frac{1}{12}\right) + \left(\frac{2\pi q}{l}\right)^2$$