Properties of the η and η' mesons
Part II: gluonic matrix elements

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Definitions: fermionic bilinears and their renormalization

We define ($\psi = (u, d, s)^\intercal$, $N_f = 3$)

$$P^a = \bar{\psi} t^a \gamma_5 \psi, \quad A^a_\mu = \bar{\psi} t^a \gamma_\mu \gamma_5 \psi, \quad t^0 = \frac{1}{\sqrt{2N_f}} \mathbb{1}, \quad t^8 = \frac{\lambda^8}{2}.$$

Then ($A^q_\mu = \bar{q}\gamma_\mu\gamma_5 q$, $m_\ell = m_u = m_d$, $A^\ell_\mu = (A^u_\mu + A^d_\mu)/\sqrt{2}$)

$$A^8_\mu = \frac{1}{12}\left(A^u_\mu + A^d_\mu - 2A^s_\mu\right) = \frac{1}{\sqrt{6}} A^\ell_\mu - \frac{1}{\sqrt{3}} A^d_\mu,$$

$$A^0_\mu = \frac{1}{6}\left(A^u_\mu + A^d_\mu + A^s_\mu\right) = \frac{1}{\sqrt{3}} A^\ell_\mu + \frac{1}{\sqrt{6}} A^d_\mu.$$

Renormalization (ignoring $O(a)$ improvement terms):

$$\widehat{A}^8_\mu = Z_A(g^2) A^8_\mu, \qquad \widehat{P}^8(\mu) = Z_P(g^2, \mu a) P^8,$$

$$\widehat{A}^0_\mu(\mu) = Z^s_A(g^2, \mu a) A^0_\mu, \qquad \widehat{P}^0(\mu) = Z^s_P(g^2, \mu a) P^0.$$

$\widehat{A}^0_\mu$: 1-loop anomalous dimension vanishes, i.e $\widehat{A}^0_\mu(\infty)/\widehat{A}^0_\mu(\mu) =$ finite
$\Rightarrow$ natural to set $\mu = \infty$ in this case.

$r_P = Z^s_P / Z_P = 1 + O(g^6)$ only depends on g^2, not on $\ln(\mu a)$.

Axial Ward Identities

$$\partial_\mu \widehat{A}^a_\mu = \widehat{\left(\overline{\psi}\gamma_5\{M, t^a\}\psi\right)} + \sqrt{2N_f}\,\delta^{a0}\widehat{\omega}.$$

Mass matrix: $M = \mathrm{diag}(m_\ell, m_\ell, m_s)$. Topological charge density:

$$\omega(x) = -\frac{1}{32\pi^2} F^a_{\mu\nu}(x)\widetilde{F}^a_{\mu\nu}(x) = -\frac{1}{64\pi^2}\epsilon_{\mu\nu\rho\sigma} F^a_{\mu\nu}(x) F^a_{\rho\sigma}(x).$$

Flavour-basis AWIs are popular (FKS model) because flavour-diagonal:

$$\partial_\mu \widehat{A}^s_\mu = 2\widehat{m}_s \widehat{P}^s + 2\,\widehat{\omega}, \quad \partial_\mu \widehat{A}^\ell_\mu = 2\widehat{m}_\ell \widehat{P}^\ell + 2\sqrt{2}\,\widehat{\omega}.$$

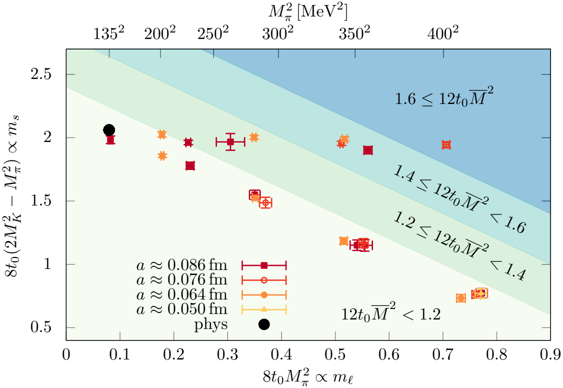
Both AWIs contain $\widehat{\omega}$ and have (in the standard $\overline{\mathrm{MS}}$ scheme) a non-trivial renormalization scale-dependence.

Here: AWIs in the octet/singlet basis ($\delta\widehat{m} = \widehat{m}_s - \widehat{m}_\ell$):

$$\partial_\mu \widehat{A}^8_\mu = \frac{2}{3}\left(\widehat{m}_\ell + 2\widehat{m}_s\right)\widehat{P}^8 - \frac{2\sqrt{2}}{3}\delta\widehat{m}\,\widehat{P}^0,$$

$$\partial_\mu \widehat{A}^0_\mu = \frac{2}{3}\left(2\widehat{m}_\ell + \widehat{m}_s\right)\widehat{P}^0 - \frac{2\sqrt{2}}{3}\delta\widehat{m}\,\widehat{P}^8 + \sqrt{6}\,\widehat{\omega}.$$

21 gauge ensembles analysed (37 for topol. suscept.)

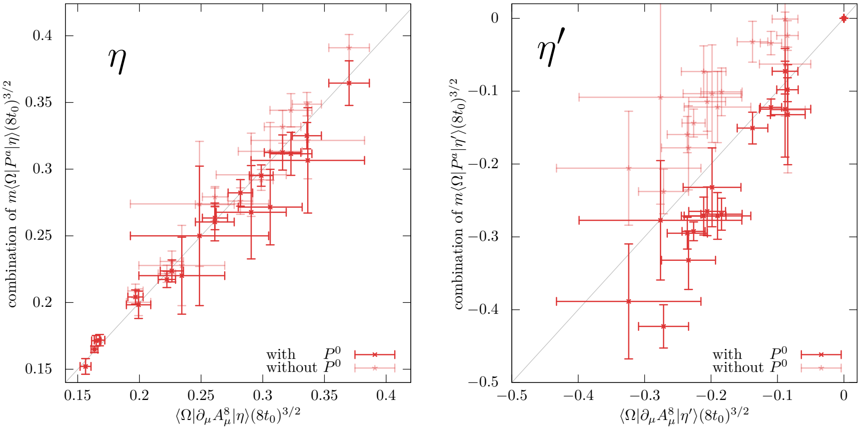


Coordinated Lattice Simulations (CLS): Berlin, CERN, TC Dublin, Kraków, Mainz, UA Madrid, Milano Bicocca, Münster, Odense, Regensburg, Roma I and II, Wuppertal, DESY-Zeuthen.

⋆ Non-perturbatively improved Sheikholeslami-Wohlert-Wilson fermion action and tree-level Lüscher-Weisz gauge action.

⋆ $N_f = 2 + 1$, $LM_\pi \gtrsim 4$.

Test of the octet AWI for $\eta^{(\prime)}$ mesons

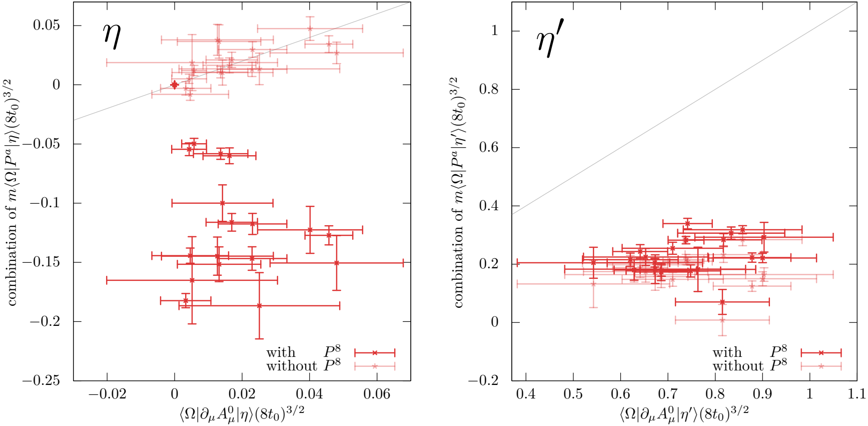


$$\partial_\mu\langle\Omega|A^8_\mu|\mathcal{M}\rangle = \frac{2}{3}\left(\widetilde{m}_\ell + 2\widetilde{m}_s\right)\langle\Omega|P^8|\mathcal{M}\rangle - \frac{2\sqrt{2}}{3}\delta\widetilde{m}\, r_P\langle\Omega|P^0|\mathcal{M}\rangle + \mathcal{O}(a).$$

We assume $r_P = 1$. $\widetilde{m}$: bare AWI mass (renormalizes with Z_A/Z_P).

Deviations due to incomplete $\mathcal{O}(a)$ improvement and $\mathcal{O}(a^2)$ terms are expected.

Confirmation that $\exists$ gluonic contribution to singlet AWI



$$\partial_\mu \langle\Omega|A^0_\mu|\mathcal{M}\rangle = \frac{2}{3}\left(2\widetilde{m}_\ell + \widetilde{m}_s\right)\langle\Omega|P^0|\mathcal{M}\rangle - \frac{2\sqrt{2}}{3}\delta\widetilde{m}\langle\Omega|P^8|\mathcal{M}\rangle$$

(missing:) $\quad + \sqrt{\frac{3}{2}}\langle\Omega|2\widehat{\omega}|\mathcal{M}\rangle + \mathcal{O}(a)$

Gluonic matrix elements from fermions

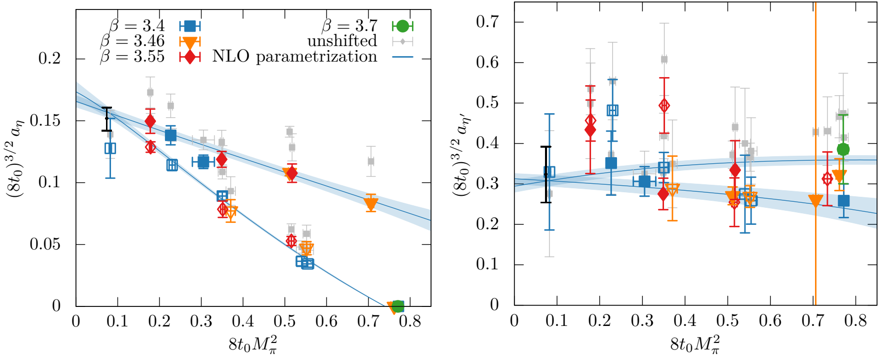
We can obtain renormalized gluonic matrix elements through the singlet AWI from the singlet decay constants $F^0_{\mathcal{M}}$ ($\mathcal{M} \in \{\eta, \eta'\}$) and pseudoscalar matrix elements $H^0_{\mathcal{M}}$ and $H^8_{\mathcal{M}}$

$$Z^s_A(\mu)\partial_\mu \langle \Omega | A^0_\mu | \mathcal{M} \rangle = M^2_{\mathcal{M}} F^0_{\mathcal{M}}(\mu), \quad H^a_{\mathcal{M}}(\mu) = Z^{(s)}_P(\mu) \langle \Omega | P^a | \mathcal{M} \rangle,$$

$$\begin{aligned} a_{\mathcal{M}}(\mu) &:= \langle \Omega | 2\widehat{\omega} | \mathcal{M} \rangle \\ &= \sqrt{\frac{2}{3}} M^2_{\mathcal{M}} F^0_{\mathcal{M}}(\mu) + \frac{4}{3\sqrt{3}} \delta \widehat{m}\, H^8_{\mathcal{M}} - \frac{2}{3}\sqrt{\frac{2}{3}} (2\widehat{m}_\ell + \widehat{m}_s) H^0_{\mathcal{M}}. \end{aligned}$$

Note that $\widehat{m} H^8_{\mathcal{M}} = Z_A \widetilde{m} \langle \Omega | P^8 | \mathcal{M} \rangle$, $\widehat{m} H^0_{\mathcal{M}} = Z_A r_P \widetilde{m} \langle \Omega | P^0 | \mathcal{M} \rangle$, $r_P = 1 + \mathcal{O}(g^6)$.
We will later check if the above results from fermionic matrix elements are consistent with the gluonic definition.

Gluonic matrix elements from the singlet AWI



Parametrization is NLO U(3) Large-N_c ChPT. 6 LECs (with priors from analysis of decay constants → Part I) plus 3 parameters to account for $\mathcal{O}(a)$ effects. $\chi^2/N_{\mathrm{df}} \approx 34/31$. At $\mu = \infty$:

$$(8t_0^{\mathrm{ph}})^{3/2} a_\eta = 0.1564\,(^{37}_{63}) \quad \text{and} \quad (8t_0^{\mathrm{ph}})^{3/2} a_{\eta'} = 0.308\,(^{16}_{17})\,.$$

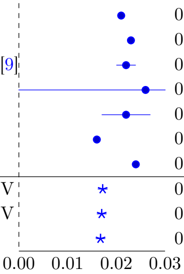
The NLO Large-N_c ChPT prediction from the decay constants alone reads:

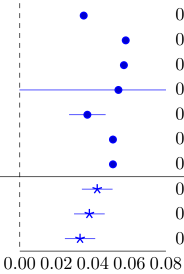
$$(8t_0^{\mathrm{ph}})^{3/2} a_\eta = 0.1609\,(^{17}_{27}) \quad \text{and} \quad (8t_0^{\mathrm{ph}})^{3/2} a_{\eta'} = 0.383\,(^{11}_{17})\,.$$

We take the difference as our systematic error (black error bar).

Comparison with the literature

ref	a_η/GeV^3	$a_{\eta'}/\mathrm{GeV}^3$
Novikov et al. [116]	0.021	0.035
Feldmann [97]	0.023	0.058
Beneke and Neubert [9]	0.022(2)	0.057(2)
Cheng et al. [118]	0.026(28)	0.054(57)
Singh [117]	0.0220(50)	0.037(10)
Qin et al. [119]	0.016	0.051
Ding et al. [106]	0.024	0.051
this work at $\mu = 1\mathrm{GeV}$	0.0172(10)	0.0424(84)
this work at $\mu = 2\mathrm{GeV}$	0.0170(10)	0.0381(84)
this work at $\mu = \infty$	0.0168(10)	0.0330(83)





Systematics from parametrization, renormalization and scale setting included.
If anomaly dominates [Novikov et al., NPB165(80)55]:

$$R(J/\psi) = \frac{\Gamma[J/\psi \to \eta'\gamma]}{\Gamma[J/\psi \to \eta\gamma]} \approx \frac{a_{\eta'}^2}{a_\eta^2} \left(\frac{k_{\eta'}}{k_\eta} \right)^3$$

with $k_{\mathcal{M}}$: momentum of the meson in the rest frame of J/ψ. From this:

$$R(J/\psi, \mu = 2\,\mathrm{GeV}) = 5.03\,\binom{19}{45}_{\mathrm{stat}}\,(1.94)_{\mathrm{syst}}\,, \quad \text{PDG:} \quad R(J/\psi) = 4.74(13).$$

Renormalization of the topological charge density

In the massless limit:

$$\partial_\mu \widehat{A}^0_\mu = \sqrt{2N_f}\,\widehat{\omega}.$$

Remark: we use $\mathscr{L} = \frac{1}{4g^2} F^a_{\mu\nu} F^a_{\mu\nu} + \cdots$. In pQCD $\mathscr{L} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \cdots$. Then $\omega \sim g^2 F\widetilde{F}$ (instead of $\omega \sim F\widetilde{F}$) and $F\widetilde{F}$ runs with the inverse β-function. It is then explicit that the anomaly vanishes as $g^2 \to 0$.

ω on the lattice is protected by topology and cannot acquire an anomalous dimension:

$$Q = \int \mathrm{d}^4 x\, \omega(x) \in \mathbb{Z}.$$

But $\partial_\mu A^0_\mu$ has the same dimension and symmetries as ω. Hence:

$$\widehat{\omega}(\mu) = Z_\omega \omega + Z_{\omega A}(\mu) \partial_\mu A^0_\mu.$$

Mixing with $a^{-1} P^0$ is removed if ω is defined via the gradient flow (or from the overlap operator). It is likely that $Z_\omega = 1$ when ω is defined through the gradient flow/cooling, e.g., [Del Debbio & Pica, hep-lat/0309145].

The topological susceptibility

$$\hat{\tau} = \sum_x \langle \widehat{\omega}(0)\widehat{\omega}(x)\rangle = \frac{1}{V}\sum_{x,y}\langle \widehat{\omega}(x)\widehat{\omega}(y)\rangle = \frac{\langle \widehat{Q}^2\rangle}{V} = Z_\omega^2 \frac{\langle Q^2\rangle}{V}$$

We determine this for the gradient flow time $t_0 = t_0^*$, $\sqrt{8t_0^*} \approx 0.413\,\text{fm}$.

For ensembles with open boundary conditions in time, a distance $\sim 1.9\,\text{fm}$ is kept from the boundaries.

We see no evidence of mass-dependent lattice spacing effects but cut-off effects are substantial.

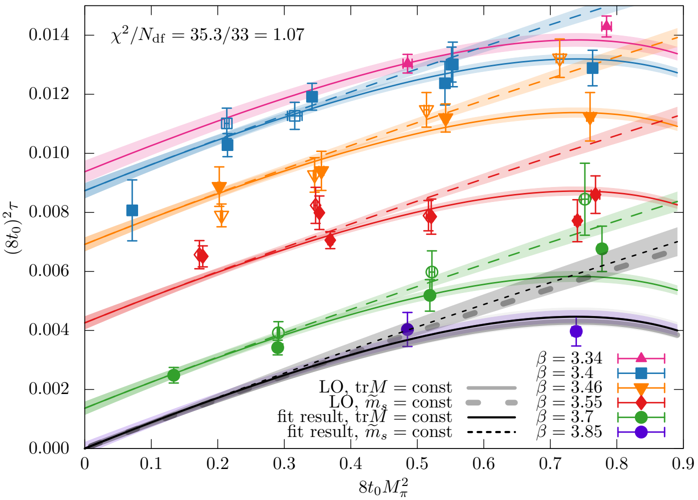
Leading order ChPT expectation plus lattice effects:

$$(8t_0)^2\tau = \frac{(8t_0)^2F^2}{2Z_\omega^2}\left(\frac{1}{2M_K^2 - M_\pi^2} + \frac{2}{M_\pi^2}\right)^{-1} + l_\tau^{(2)}\frac{a^2}{t_0^*} + l_\tau^{(3)}\frac{a^3}{(t_0^*)^{3/2}} + l_\tau^{(4)}\frac{a^4}{(t_0^*)^2}.$$

Fit to 37 CLS ensembles with free F/Z_ω:

$$l_\tau^{(2)} = -0.072(10), \quad l_\tau^{(3)} = 0.355(34), \quad l_\tau^{(4)} = -0.324(30).$$

The topological susceptibility 2



We conclude that $Z_\omega = 1$.

Problem: $Z_{\omega A}$ is unknown (except for its scale dependence)!

Renormalizing the gluonic definition

From the eigenvectors of the GEVP of Part I, we can compute $\langle\Omega|2\omega(0)|\mathcal{M}\rangle$, $\mathcal{M}\in\{\eta,\eta'\}$. Then the renormalized matrix element is given as

$$a_{\mathcal{M}}(\mu) = Z_\omega\langle\Omega|2\omega|\mathcal{M}\rangle + 2\frac{Z_{\omega A}}{Z_A^s}M_{\mathcal{M}}^2 F_{\mathcal{M}}^0(\mu).$$

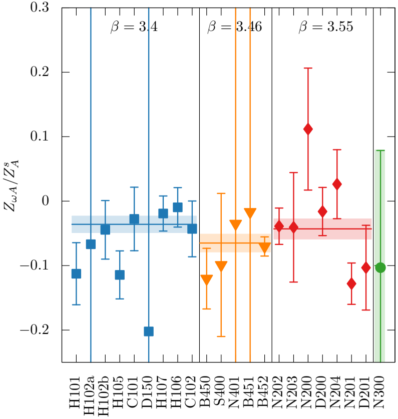
Problem: $Z_{\omega A}$ is unknown. However, only one $Z_{\omega A}$ per lattice spacing.
$\rightarrow$ Isolate the scale independent combination $Z_{\omega A}/Z_A^s$ ($Z_\omega = 1$):

$$\frac{Z_{\omega A}}{Z_A^s} = \frac{a_{\mathcal{M}} - \langle\Omega|2\omega|\mathcal{M}\rangle}{2M_{\mathcal{M}}^2 F_{\mathcal{M}}^0}.$$

Shown for $\mathcal{M} = \eta'$.

Denominator near zero for the η.

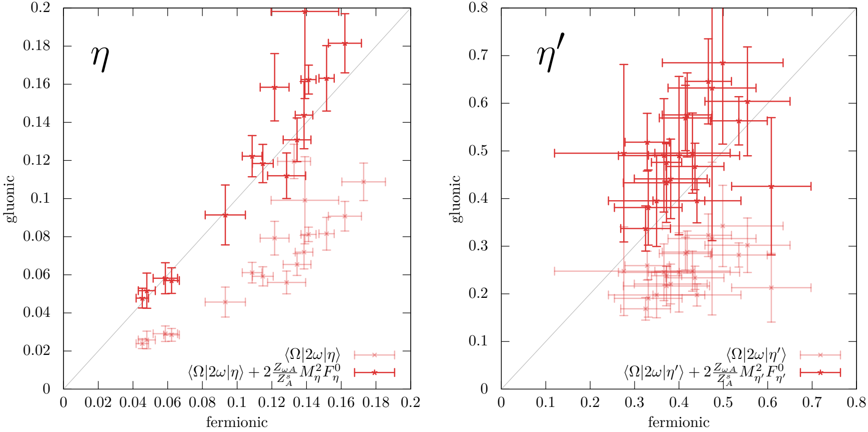
Consistent with constant values, slowly varying with g^2.



Comparison between fermionic and gluonic determinations

$Z_{\omega A} \neq 0$ is necessary!

Light symbols: without $Z_{\omega A}$, dark symbols: with $Z_{\omega A}$.



Summary

- First verification of the singlet AWI with Wilson fermions.
- First lattice determination of the gluonic matrix elements.
- Consistent results found, using fermionic and gluonic definitions, when appropriately renormalized (not specific to Wilson fermions).
- Dependence on the meson masses derived to NLO in U(3) Large-N_c ChPT and very reasonable agreement found.
- Scale dependence of $a_\eta(\mu)$, $a_{\eta'}(\mu)$ included for the first time.
- The topological susceptibility seems well described by Large-N_c ChPT, albeit with large cut-off effects (also observed for other fermion actions: [MILC, arXiv:1003.5695], [Chowdury et al., arXiv:1110.6013], [ETMC, arXiv:1312.5161], [ALPHA, arXiv:1406.5363], [Bonati et al., arXiv:1512.06746], [BMWc, arXiv:1606.07494].)
- Also $\exists$ results on PS fermionic η and η' matrix elements $\rightarrow$ arXiv:2106.05398.