Fuzzy Sphere Regularization of the 1+1d Sigma Model

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Quantum Simulation and Universality

- To perform simulations of QFTs on quantum computers, a regularization of the QFT is needed that can be mapped onto qubits. In general, two regularizations are needed:
  - Spacetime must be discretized into a lattice, and
  - The Hilbert space at each site must be finite dimensional.
- Fermionic theories have finite dimensional one-site Hilbert spaces, but bosonic theories typically have infinite dimensional one-site Hilbert spaces and therefore require some sort of truncation / regularization.
- *Example*: the 1+1d lattice sigma model has Hamiltonian operator

$$\hat{H} = \frac{g^2}{2} \sum_x \hat{\boldsymbol{L}}^2(x) - \frac{1}{g^2} \sum_x \hat{\boldsymbol{n}}(x) \cdot \hat{\boldsymbol{n}}(x+1) \quad (1)$$

  where $\hat{\boldsymbol{L}}(x)$ is the orbital angular momentum operator at site x and $\hat{\boldsymbol{n}}(x)$ is the "position" operator at site x. The spectrum of $\hat{\boldsymbol{L}}^2$ is $l(l+1)$ for $l \in \mathbb{Z}_+$, so the $|l, m\rangle$ basis is infinite dimensional. But the obvious truncation (choosing some l_{max}) breaks $O(3)$ symmetry.
- For any proposed regularization, one wants the same physics as the full model. There is usually a parameter that recovers the true lattice theory in some limit, e.g. $l_{\text{max}} \to \infty$ above. But one can also ask whether, for a *fixed* regularization, the same physics can be obtained by tuning to *quantum* criticality.

The Fuzzy Sphere

- In the $\boldsymbol{n}$-space representation of the sigma model, $\hat{\boldsymbol{n}} \mapsto \boldsymbol{n}$ a unit vector, and $\hat{\boldsymbol{L}}^2 \mapsto -\Delta$ is the Laplacian on the 2-sphere. Wave functions for a 1-site system are functions $\psi(\boldsymbol{n})$, which may be expanded:

$$\psi(\boldsymbol{n}) = \psi_0 + \psi_i n_i + \frac{1}{2}\psi_{ij} n_i n_j + \cdots \tag{2}$$

- The *fuzzy sphere* is a mapping of the function space on the sphere to a vector space of matrices defined by

$$n_i \mapsto \mathbb{J}_i, \quad \frac{\partial}{\partial n^i} \mapsto \mathrm{ad}_{\mathbb{J}_i} = [\mathbb{J}_i, \cdot], \tag{3}$$

$$\psi(\boldsymbol{n}) \mapsto \boldsymbol{\Psi} = \psi_0 \mathbb{I} + \psi_i \mathbb{J}_i + \frac{1}{2}\psi_{ij}\mathbb{J}_i\mathbb{J}_j + \cdots \tag{4}$$

where the $\mathbb{J}_i$ are any spin-j representation of $\mathfrak{su}(2)$. They are normalized such that $\sum_k \mathbb{J}_k^2 = \mathbb{I}$. For any particular j, the sum above eventually truncates.

- For $j = 1/2$, one finds $\mathbb{J}_k = \sigma_k/\sqrt{3}$, with σ_k the usual Pauli matrices. Wave functions are of the form

$$\psi = \psi_0 \mathbb{I} + \psi_k \mathbb{J}_k, \tag{5}$$

so the Hilbert space of a single fuzzy spin is 4-dimensional.

$j = 1/2$ Fuzzy Representation

- The fuzzy Laplacian operator $\sum_k \mathrm{ad}^2_{\mathbb{J}_k}$ has $\{\mathbb{I}, \mathbb{J}_k\}$ as its eigenvectors, and reproduces the $l = 0, 1$ subspace of the true $-\Delta$ eigenspace.
- The Hamiltonian operator for a system of N $j = 1/2$ fuzzy spins is obtained by mapping the sigma model Hamiltonian according to the fuzzy map:

$$\hat{H} = \frac{\kappa g^2}{2} \sum_x \sum_k \mathrm{ad}^2_{\mathbb{J}_k(x)} - \frac{\kappa}{g^2} \sum_x \mathbb{J}_k(x)\mathbb{J}_k(x+1) \tag{6}$$

- Rotations: any $R \in O(3)$ acts on states by the adjoint action

$$\Pi_R \Psi = U_R \Psi U_R^\dagger, \quad U_R \in SU(2). \tag{7}$$

  The Hamiltonian is covariant under rotations in the sense that

$$\hat{H}(\Pi_R \Psi) = \Pi_R(\hat{H}\Psi) \tag{8}$$

- Because the fuzzy model preserves $O(3)$ symmetry, we expect (hope) that it will be in the same universality class as the sigma model [1]. (Note: a truncation of the ordinary sigma model Hilbert space by some $l_{\max}$ does *not* preserve $O(3)$ symmetry.)

How to Simulate the Fuzzy Model

- A standard basis $|a\rangle$, $a = 1, 2, 3, 4$ can be defined with inner product

$$\langle a|b\rangle := \mathrm{tr}\ \mathbb{T}_a^\dagger \mathbb{T}_b = \delta_{ab}, \quad \text{where} \quad \{\mathbb{T}_a\} = \{\tfrac{i}{\sqrt{2}}\mathbb{I}, \sqrt{\tfrac{3}{2}}\mathbb{J}_k\}, \tag{9}$$

  leading to a 4×4 matrix representation of all 1-site operators. The many-body state on a timeslice t is then

$$|\boldsymbol{a}_t\rangle = |a_{1,t}\rangle \otimes \cdots \otimes |a_{N,t}\rangle \tag{10}$$

  for an N-site chain.

- The partition function can be written as

$$Z = \sum_{\{a_{x,t}\}} \langle \boldsymbol{a}_1|\mathrm{e}^{-\epsilon H}|\boldsymbol{a}_{N_t}\rangle \cdots \langle \boldsymbol{a}_2|\mathrm{e}^{-\epsilon H}|\boldsymbol{a}_1\rangle, \tag{11}$$

  where $\epsilon = \beta/N_t$ is the timestep.

- One can attempt to compute expectation values by MCMC sampling from the above distribution. The system turns out to have a *sign problem*, however, which makes a direct MC simulation difficult. (Perhaps a worm algorithm would work.)
- Instead of a MC simulation, we have instead used the machinery of **matrix product states** ...

Matrix Product States (MPS)

- The state vectors of an N-site quantum chain with 1-site Hilbert space $\mathscr{H}_1$ of dimension d are of the form

$$|\psi\rangle = \sum_{i_1,\ldots,i_N=1}^{d} c_{i_1\ldots i_N}|i_1,\ldots,i_N\rangle, \tag{12}$$

  and they span a 2^N-dimensional vector space $\mathscr{H}_N$.
- A *matrix product state* for such a chain is a state of the form [3]

$$|\psi_A\rangle = \sum_{i_1,\ldots,i_N=1}^{d} \mathrm{tr}(A_1^{i_1}\cdots A_N^{i_N})|i_1,\ldots,i_N\rangle, \tag{13}$$

  where the $A_x^{i_x}$ are $D \times D$ complex matrices. Any state in $\mathscr{H}_N$ can be written as an MPS, for D large enough (rigorously proven).
- There is reason to believe, however, that ground states of *gapped, local* systems are well-approximated by an MPS, for D quite small.
- For example, the ground state of the spin-1 AKLT model with Hamiltonian

$$H = \sum_x \left[\boldsymbol{S}(x)\cdot\boldsymbol{S}(x+1) + \frac{1}{3}\big(\boldsymbol{S}(x)\cdot\boldsymbol{S}(x+1)\big)^2\right] \tag{14}$$

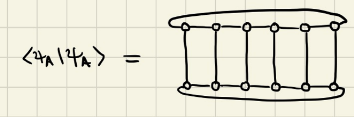
  is known to be an MPS with $D = 2$.

Variational MPS Algorithm (DMRG)

- The idea is to minimize the expectation value of H,

$$\langle \psi_A | H | \psi_A \rangle \tag{15}$$

with respect to the coefficients $(A_x^{i_x})_{ab}$ of the MPS matrices, subject to the constraint that $\langle \psi_A | \psi_A \rangle = 1$. It is assumed that the minimization can be done for one matrix at a time.



- The minimization with respect to the matrices A_x^i at site x is equivalent to solving a generalized eigenvalue problem,

$$(H_{\text{eff}})_{i,a,b;i',a',b'} (A_x^{i'})_{a'b'} = \lambda (N_{\text{eff}})_{i,a,b;i',a',b'} (A_x^{i'})_{a'b'}, \tag{16}$$

where the effective environment matrices are defined by

$$(A_x^{*i})_{ab} (H_{\text{eff}})_{i,a,b;i',a',b'} (A_x^{i'})_{a'b'} := \langle \psi_A | H | \psi_A \rangle, \tag{17}$$

$$(A_x^{*i})_{ab} (N_{\text{eff}})_{i,a,b;i',a',b'} (A_x^{i'})_{a'b'} := \langle \psi_A | \psi_A \rangle, \tag{18}$$

i.e. A_x^i has been "extracted" out of the tensor network.

- Excited states $|\psi_k\rangle$ and their energies E_k can also be obtained by adding constraints $\langle \psi_k | \psi_0 \rangle = 0$ to the minimization.

Testing Equivalence to the Sigma Model

- We need some way to compare the physics of the two models. One way is to compute the finite volume mass gap at various mL values and compare with MC or TBA predictions.
- Another check: The sigma model has a triplet of massive particles in its spectrum. The $2 \to 2$ S-matrix for these particles is known exactly, and the phase shift turns out to be exactly that of 2 particles interacting by a delta function potential,

$$\delta(k) = \frac{i}{2} \log \frac{ik + \alpha}{ik - \alpha} \tag{19}$$

  where α is the interaction strength. From the 1d Lüscher formula,

$$\frac{2\pi n}{L} = k_n + 2\delta(k_n), \tag{20}$$

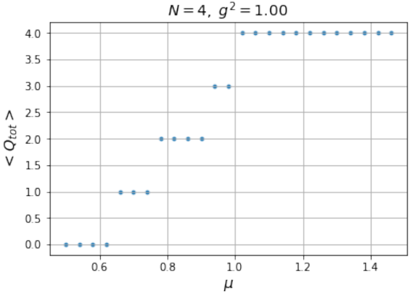
  one may compute the allowed relative momenta k_n of the two particles.
- So we see if the fuzzy model reproduces the correct 2-particle energies. How can we do that? One way:
  - Finite chemical potential (see Bruckmann, et al., [2]).

Finite Chemical Potential

- There is no sign problem in the MPS framework, so we can compute the ground states of the grand canonical potential

$$\Omega = H - \mu Q, \quad Q = \sum_x \mathrm{ad}_{J_3(x)} \tag{21}$$

- We look at the expectation of the charge operator as chemical potential μ varies:



- The sum of the critical values $\mu_2 + \mu_1$ is the minimal 2-particle energy W. For a short-ranged potential, we assume free particles

$$W = 2\sqrt{k^2 + m^2} \tag{22}$$

So we measure W and compare with that implied by the Lüscher formula above.

Renormalizing the speed of light

- This assumes that the gap $\Delta_1 = E_1 - E_0$ of the Hamiltonian is equal to the mass, i.e. that the single particle dispersion is relativistic.
- In general, however, hamiltonian systems may be anisotropic, and a rescaling of the time or space axes is required to ensure that the energies coincide $E_1 - E_0 = m$. This is called *renormalization of the speed of light*.
- If we can measure both the gap Δ_1 and the mass m (in infinite volume), then we can define their ratio

$$\eta := m/\Delta_1 \tag{23}$$

  and rescale the other energies E_k by η.
- Δ_1 is easy to measure by performing the variational MPS minimization.
- m can be measured in several ways:
  1. Obtain the ground state ψ_0, then measure the spatial correlator

$$C(x) = \langle \psi_0 | \mathbb{J}_3(x) \mathbb{J}_3(x+z) | \psi_0 \rangle \tag{24}$$

     and extract the mass as usual (difficult for the MPS).
  2. Extract the mass from finite volume effects:

$$M_L - m = 4\pi m \int_{-\infty}^{\infty} \mathrm{d}\theta \ \cosh(\theta) \ \frac{\mathrm{e}^{-mL\cosh(\theta)}}{\theta^2 + 9\pi^2/4} + O(\mathrm{e}^{-\sqrt{3}mL}) \tag{25}$$

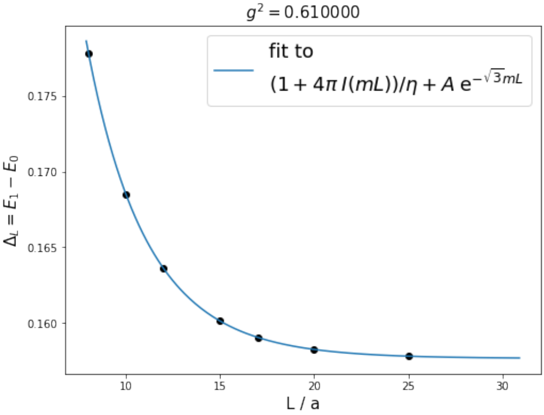
     and $M_L = \eta \Delta_L$, where $\Delta_L = E_1 - E_0$ we can measure: this also gives us an η estimate.

Mass extraction

- Extract the mass from finite volume effects:

$$M_L - m = 4\pi m \int_{-\infty}^{\infty} \mathrm{d}\theta \, \cosh(\theta) \, \frac{\mathrm{e}^{-mL\cosh(\theta)}}{\theta^2 + 9\pi^2/4} + O(\mathrm{e}^{-\sqrt{3}mL}) \tag{26}$$

and $M_L = \eta\Delta_L$, where $\Delta_L = E_1 - E_0$.

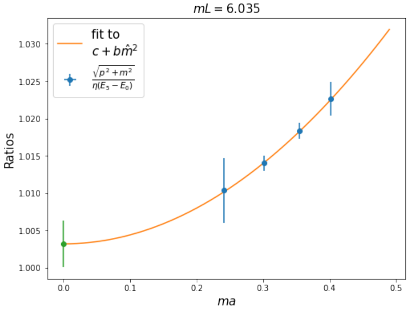


Testing relativity

- Once the speed of light is renormalized, we check for relativistic invariance by checking the disperion relation

$$\omega(p) = \sqrt{p^2 + m^2} \tag{27}$$

at various fixed $p/m \propto 1/mL$. We expect that the fifth state in the spectrum is the lowest nonzero momentum 1-particle state, so we compare $\omega(p)$ to $\eta(E_5 - E_0)$.



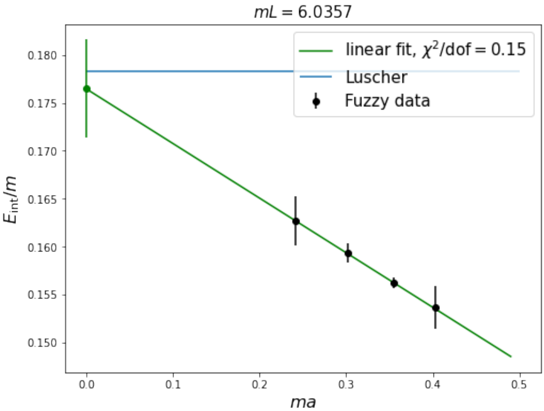
Minimal 2-particle interaction energy

- We define the interaction energy by

$$E_{\rm int} := W - 2m = \mu_2 - \mu_1 \qquad (28)$$

and look at its scaling toward the the continuum, comparing with the prediction of the Lüscher formula, at fixed mL.

- For $mL = 6.037$:



Final thoughts / future work

- While we have indication that the low energy ($p \lesssim m$) regime of the fuzzy model in the continuum is consistent with the low-energy sigma model, but there seems to be deviation at higher energy ($p \gg m$ or $mL \lesssim 1$). (One really wants consistency even in the "UV" ($m \ll p \ll a^{-1}$)).
- Perhaps this is not surprising, since the fuzzy model with $g^2 > 0$ is ferromagnetic:
  - The *Haldane conjecture* asserts that spin-1 *antiferromagnets* are equivalent to the $O(3)$ sigma model.
- Thus, we will repeat the analysis above for $g^2 < 0$.
- We are also trying the Heisenberg comb of Singh et al. [4], and will likely try the spin-1 Heisenberg antiferromagnet directly.

References:

1. Alexandru, et al., *Sigma models of quantum computers*. Phys. Rev. Lett. 123, 090501 (2019).
2. Bruckmann, et al., *Grand canonical ensemble, multi-particle wave functions, scattering data, and lattice field theories*. Phys. Rev. Lett. 115, 231601 (2015).
3. Orús, *A practical introduction to tensor networks: Matrix product states and projected entangled pair states*. arxiv:1306.2164.
4. Bhattacharya et al., *Qubit regularization of asymptotic freedom*. Phys. Rev. Lett. 126, 172001 (2021).