# Tensor network formulation of massless lattice Schwinger model


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## Abstract


We construct a tensor network representation of the partition function for the massless Schwinger model on a two dimensional lattice using staggered fermions. The tensor network representation allows us to include a topological term. Using a particular implementation of the tensor renormalization group (HOTRG) we calculate the phase diagram of the theory. For a range of values of the coupling to the topological term $\theta$ and the gauge coupling $\beta$ we compare with results from hybrid Monte Carlo when possible and find good agreement.


## Motivation

In low dimensions tesnor network formulations can avoid the usual sign problems associated with negative or complex probability weights that plague Monte Carlo approaches, and can yield very efficient computational algorithms. For compact fields the general strategy has been to employ character expansions for all Boltzmann factors occurring in the partition function and subsequently to integrate out the original fields, yielding an equivalent formulation in terms of integer—or half-integer—valued fields. Typically local tensors can be built from these discrete variables and the partition function recast as the full contraction of all tensor indices. However, writing local tensors for models with relativistic lattice fermions is more complicated.One reason is tied to the Grassmann nature of the fermions which can induce additional, non-local sign problems which may be hard to generate from local tensor contractions. However, Gattringer et. al. have shown in Ref. [1] that a suitable dual formulation can be derived in the case of the massless Schwinger model which is free of these sign problems. Using this dual representation they have formulated a general Monte Carlo algorithm that can be used to simulate the model even in the presence of non-zero chemical potential and topological terms [2].

## Dual representation in terms of loops and dimers

We start with staggered action for fermions and standard Wilson gauge action. The first step is integration of Grassmann variables site by site.

$$Z_F = \int D[U]D[\bar\psi]D[\psi]\times \prod_x \sum_{k=0}^{1}\left(\frac{1}{2}\bar\psi(x)U_\mu(x)\psi(x+\mu)\right)^k \times \sum_{\bar k=0}^{1}\left(\frac{1}{2}\bar\psi(x+\mu)U_\mu^\dagger(x)\psi(x)\right)^{\bar k}. \tag{1}$$

This can be written as a sum over loops and dimers as follows

$$Z_F = \left(\frac{1}{2}\right)^V \sum_{l,d}(-1)^{N_L+\frac{1}{2}\sum_l L(l)+\sum_l W(l)}\times \prod_l\left[\prod_{x,\mu\in l} U_\mu^{k_\mu(x)}(x)\right]. \tag{2}$$

Now we expand the gauge action in terms of modified Bessels and dual characters. The link integration leads to the constraint

$$\int_{-\pi}^{\pi}\frac{dA_\ell}{2\pi}e^{i(m_p-m_{p'}+k_\ell-\bar k_\ell)A_\ell} = \delta_{m_p-m_{p'}+k_\ell-\bar k_\ell,0}. \tag{3}$$

This allows us to write the partition function as

$$Z = \sum_{\{m_p\}}\sum_{\{k_\ell,\bar k_\ell\}}\prod_\ell \delta_{m_p-m_{p'}+k_\ell-\bar k_\ell,0}\prod_p I_{m_p}(\beta)\times \prod_x T^x_{k_1\bar k_1 k_2\bar k_2 k_3\bar k_3 k_4\bar k_4}\times(-1)^{N_L+N_P+\frac{1}{2}\sum_l L(l)} \tag{4}$$

where tensor $T$ encodes all possibilities of Grassmann integrations at a given site.
We define

$$A_{m_i m_j k_a\bar k_a k_b\bar k_b} \equiv \delta_{m_i-m_j+k_a-\bar k_a,0}\delta_{k_a,k_b}\delta_{\bar k_a,\bar k_b}. \tag{5}$$

$$B_{m_1m_2m_3m_4} = I_m(\beta)\,\text{only if } m_1=m_2=m_3=m_4=m \tag{6}$$

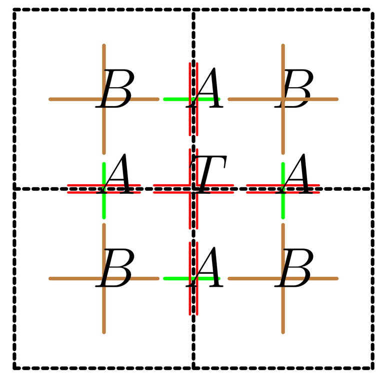
These definitions of the $A$ and $B$ tensors allow us to write the partition function as follows,

$$Z = \sum_{\{k,\bar k\}}\sum_{\{m_p\}}\left(\prod_p B_{m_i m_j m_k m_l}\right)\left(\prod_l A_{m_i m_j k_a\bar k_a k_b\bar k_b}\right)\times \left(\prod_x T_{k_a\bar k_a k_b\bar k_b k_c\bar k_c k_d\bar k_d}\right). \tag{7}$$

Starting with tensor $\mathcal{M}$ which is defined as

$$\mathcal{M}_{m_1m_2m_3m_4K_1K_2K_3K_4} = \sum_{m_1',m_2',\bar K_1,\bar K_2} B_{m_1m_1'm_2m_2'}\times A_{m_2'm_3K_1\bar K_1}T_{\bar K_1K_2\bar K_2K_3}A_{m_4m_1'\bar K_2K_4}. \tag{8}$$

we use HOTRG algorithm to extract different observables.



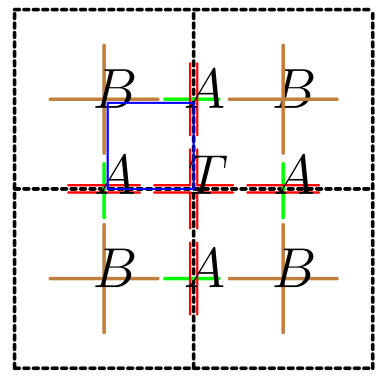


Figure 1: **Left**: Elementary tensors $T$, $A$, and $B$. When these tensors are contracted in the pattern shown here the world-line representation of the partition function is generated exactly. **Right**:Construction of tensor $\mathcal{M}$ shown as the four tensors sharing the blue loop. This is a possible single tensor which can be contracted with itself recursively to generate the partition function.

## Numerical Results

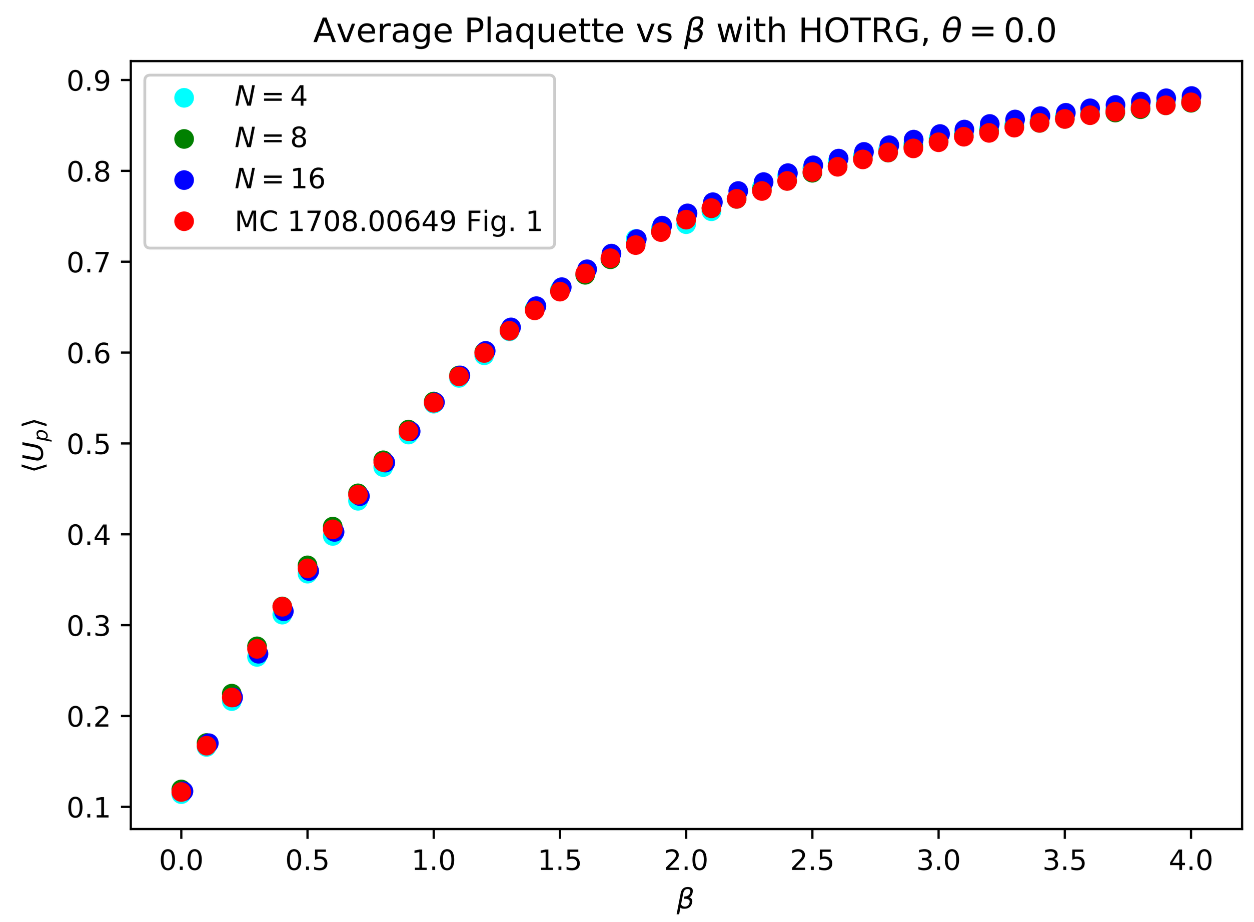



Figure 2: $\langle U_P(x)\rangle$

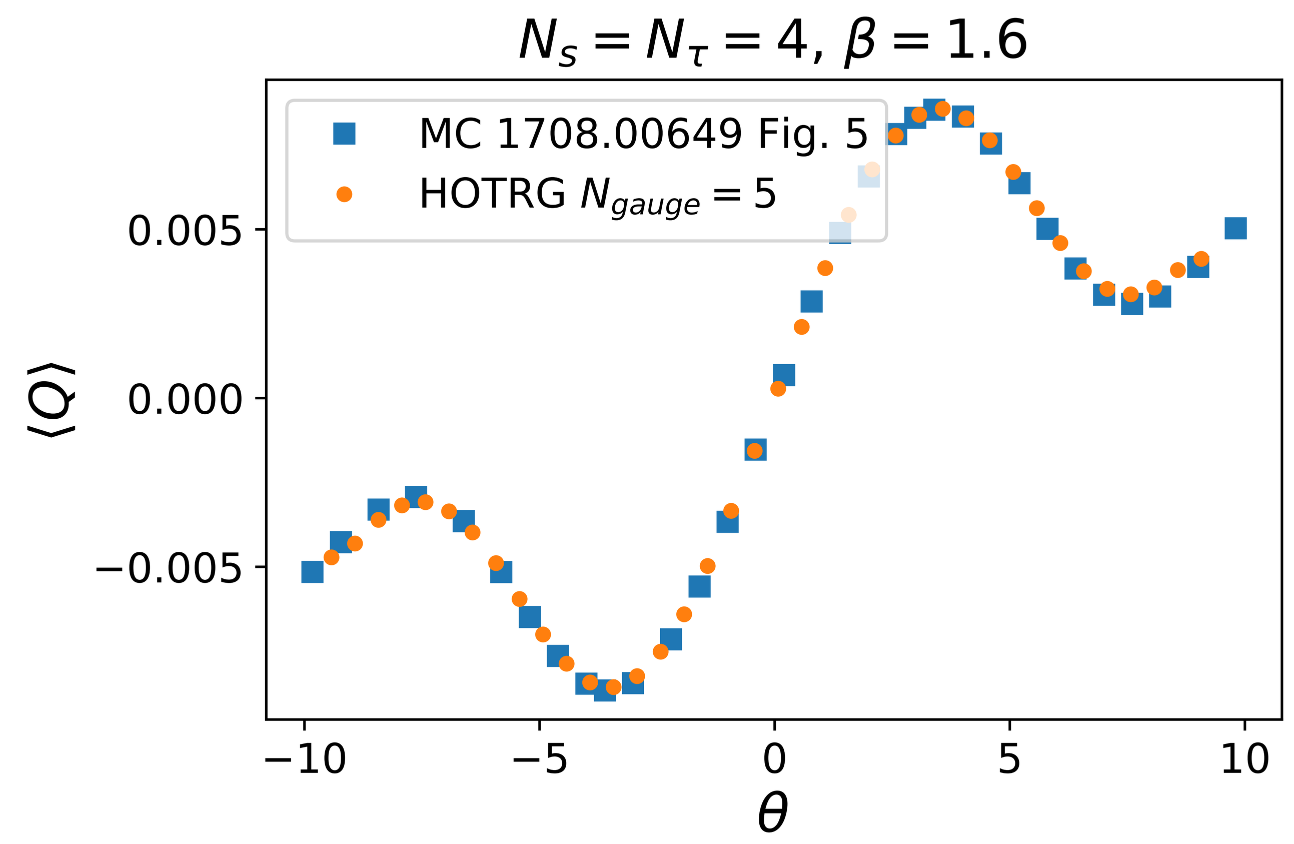



Figure 3: $\langle Q\rangle$

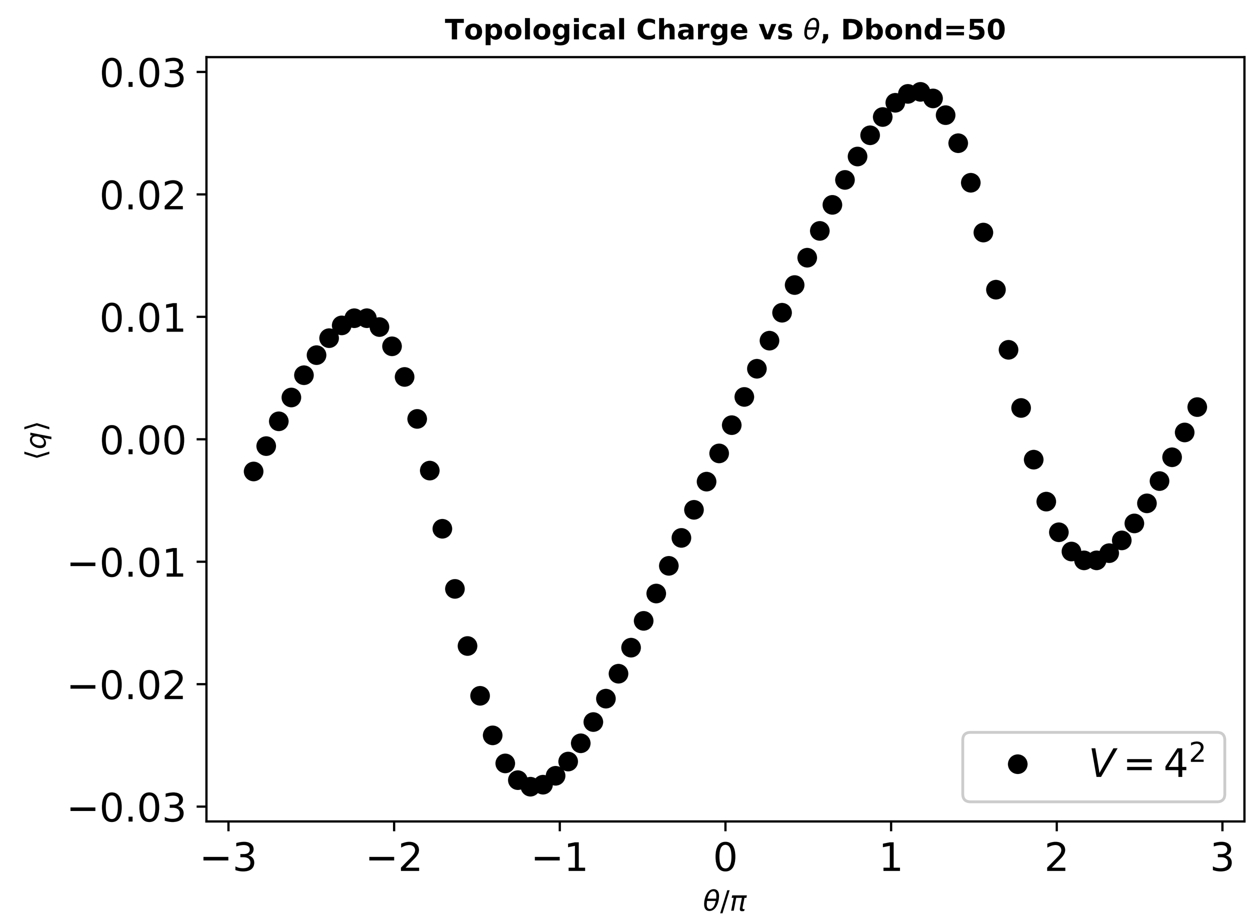



Figure 4: $\langle Q\rangle$ quenched

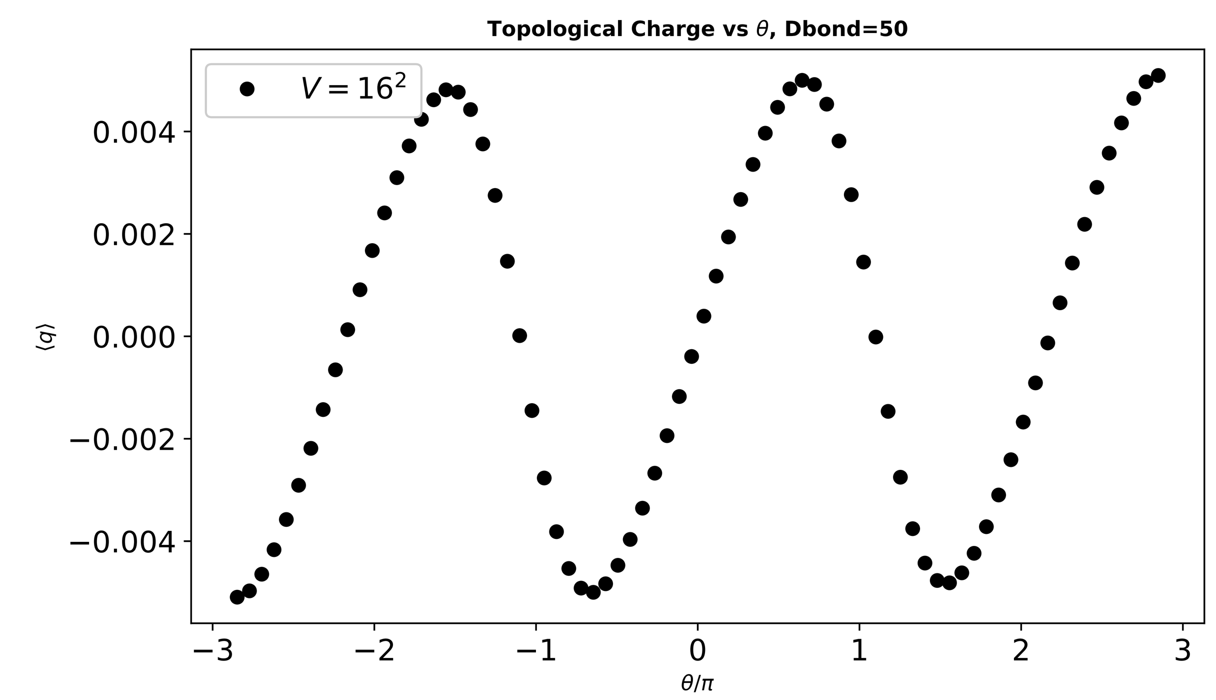



Figure 5: $\langle Q\rangle$ quenched

## References


[1] *Solving the sign problems of the massless lat-tice schwinger model with a dual formulation*, NuclearPhysics B, 897:732 748, 2015

[2] *Simulation strategies for the mass-less lattice schwinger model in the dual formulation*, Nu-clear Physics B, 924:63 85, 2017