Hybrid stochastic method for the tensor renormalization group

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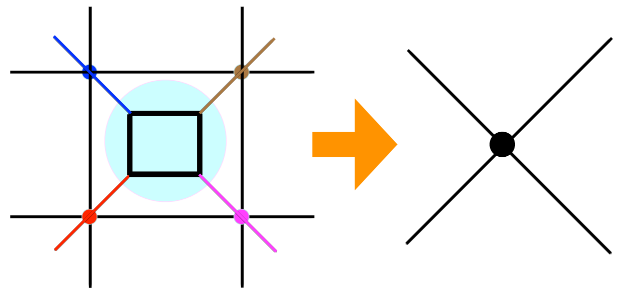
Abstract
We study a hybrid stochastic method for the tensor renormalization group (TRG) approach. We use a noise vector for the low-rank approximation with combining the truncated singular value decomposition (SVD), by which the truncation error is replaced with a statistical error due to noise, and an improvement of the error estimation could be expected. We test this method in the classical Ising model in comparison with the original TRG method, and also discuss other applications of the method for further error reductions.

1. Introduction

The computational cost for TRG significantly increases as the bond dimension increases, and there is a systematic error due to truncation.



CPU Cost : $\mathcal{O}(D^6)$
D: bond dimension

Can we reduce D and improve the error estimation?

2. Our strategy: noise vectors

SVD of the matrix M (rank R)

$$M = \sum_{i=1}^{D_{svd}} \sqrt{s_i}u_i\sqrt{s_i}v_i + \sum_{i=D_{svd}+1}^{R} \sqrt{s_i}u_i\sqrt{s_i}v_i$$

(neglected in the original TRG)

$$\simeq \sum_{i=1}^{D_{svd}} \sqrt{s_i}u_i\sqrt{s_i}v_i + \sum_{i,j=D_{svd}+1}^{R} \sqrt{s_i}u_i \left[\sum_{r=1}^{N_r}\left(\frac{\eta_{ir}}{\sqrt{N_r}}\right)\left(\frac{\eta^*_{rj}}{\sqrt{N_r}}\right)\right]\sqrt{s_j}v_j$$

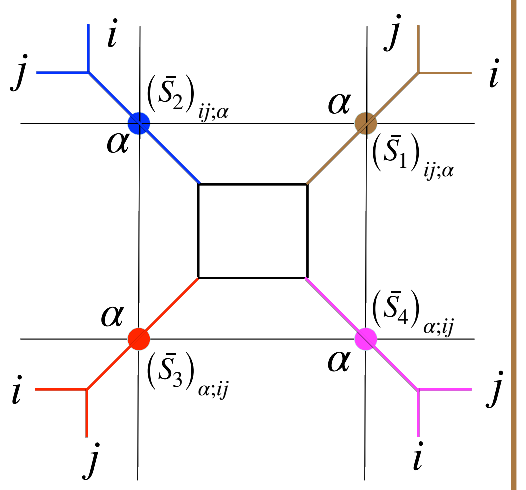
(inserting 1 approximated by noise vector)

$$\sum_{r=1}^{N_r}\frac{1}{N_r}\eta_r\cdot\eta_r^\dagger = \mathbf{1}_{R\times R} + \mathcal{O}(1/\sqrt{N_r})$$ (random noise vector)

Modified three-order tensors

$$(\bar{S}_1)_\alpha = \begin{cases} \sqrt{s_\alpha}u_\alpha & (1\le\alpha\le D_{svd}) \\ \sum_{i=D_{svd}+1}^{R}\sqrt{s_i}u_i\frac{\eta_{i\alpha}}{\sqrt{N_r}} & (D_{svd}+1\le\alpha\le D_{svd}+N_r)\end{cases}$$

$$(\bar{S}_3)_\alpha = \begin{cases} \sqrt{s_\alpha}v_\alpha & (1\le\alpha\le D_{svd}) \\ \sum_{i=D_{svd}+1}^{R}\frac{\eta^*_{\alpha i}}{\sqrt{N_r}}\sqrt{s_i}v_i & (D_{svd}+1\le\alpha\le D_{svd}+N_r)\end{cases}$$

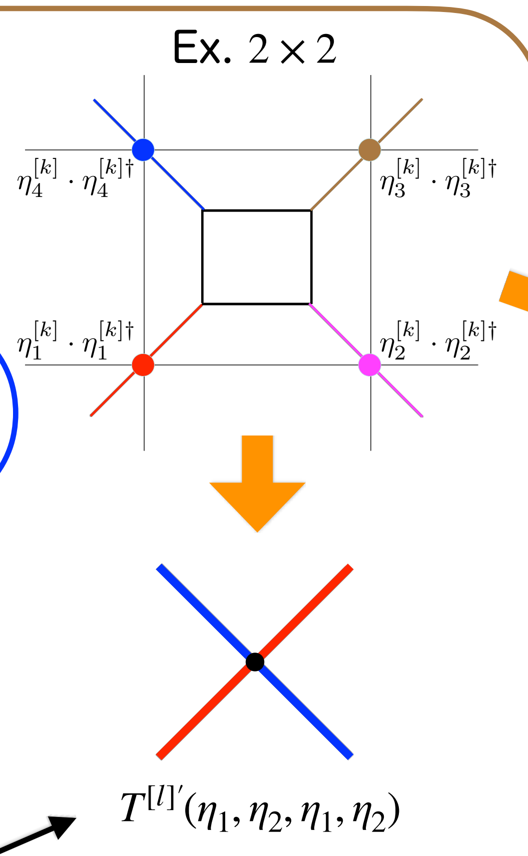


$(\bar{S}_{2,4})$ are also defined similarly.

3. Hybrid stochastic method

Generation of tensor configurations with a multiple set of noise vectors for site i,

$\{\eta_i^{[1]}, \eta_i^{[2]}, \cdots, \eta_i^{[N]}\}$, where N is the number of statistics.



Renormalized tensor

$$T^{[l]'} = \mathrm{tr}\left[\bar{S}_1(\eta_1^{[l]})\bar{S}_2(\eta_2^{[l]})\bar{S}_3(\eta_3^{[l]})\bar{S}_4(\eta_4^{[l]})\right]$$

$D_{cut} = D_{svd} + N_r$: Total bond dimensions
D_{svd} : Number of SVDs ($0 \ll D_{svd} \ll R$)
N_r : Dimensions for noise vectors ($0 \ll N_r$)
N : Number of statistics

To preserve the isotropy in RG steps, we constrain the noise vectors to $\eta_1 = \eta_3,\ \ \eta_2 = \eta_4$ (common noise)

Truncation error → Statistical error up to noise contamination

Exact in $N, N_r \to \infty$.

4. Numerical test (common noise)

We test this method for 2-D Ising model.

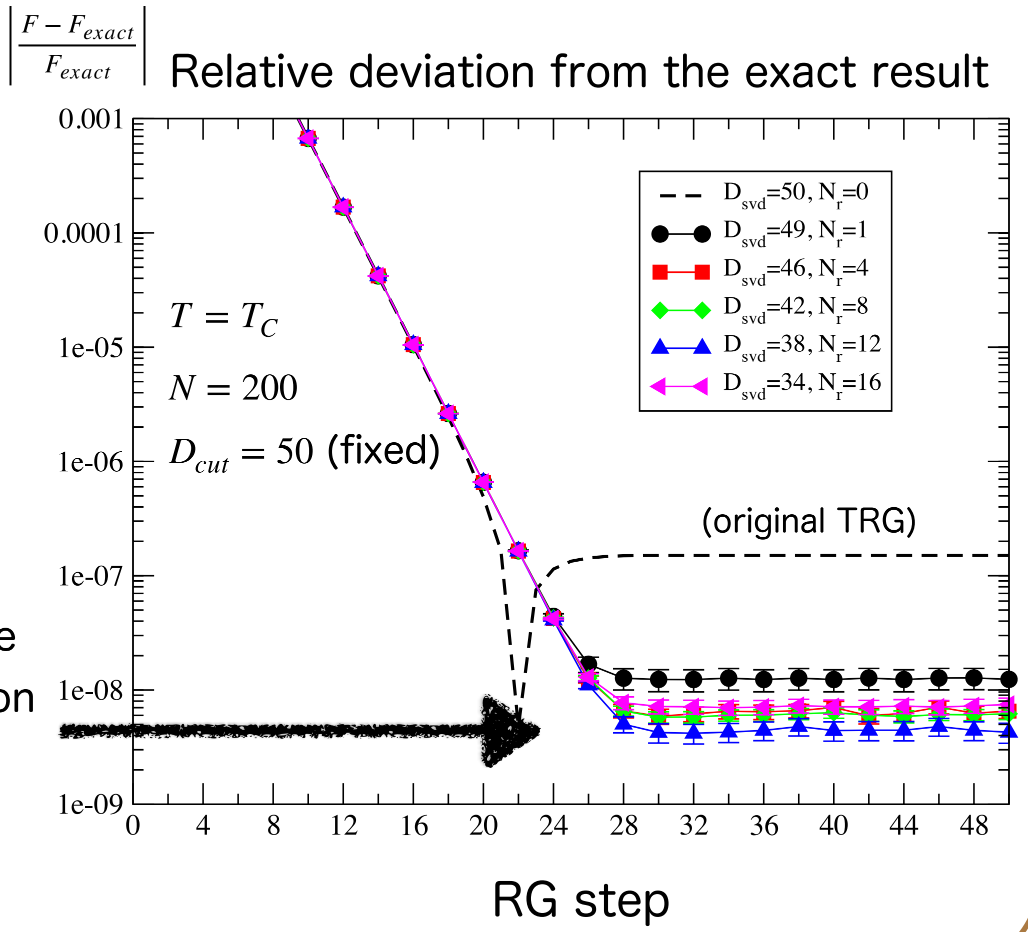
Partition function

$$Z(T^{[l]}) = \mathrm{Tr}[T^{[l]}]$$

Free energy

$$F = \frac{1}{N}\sum_{l=1}^{N}\left(-T\log Z(T^{[l]})\right)$$

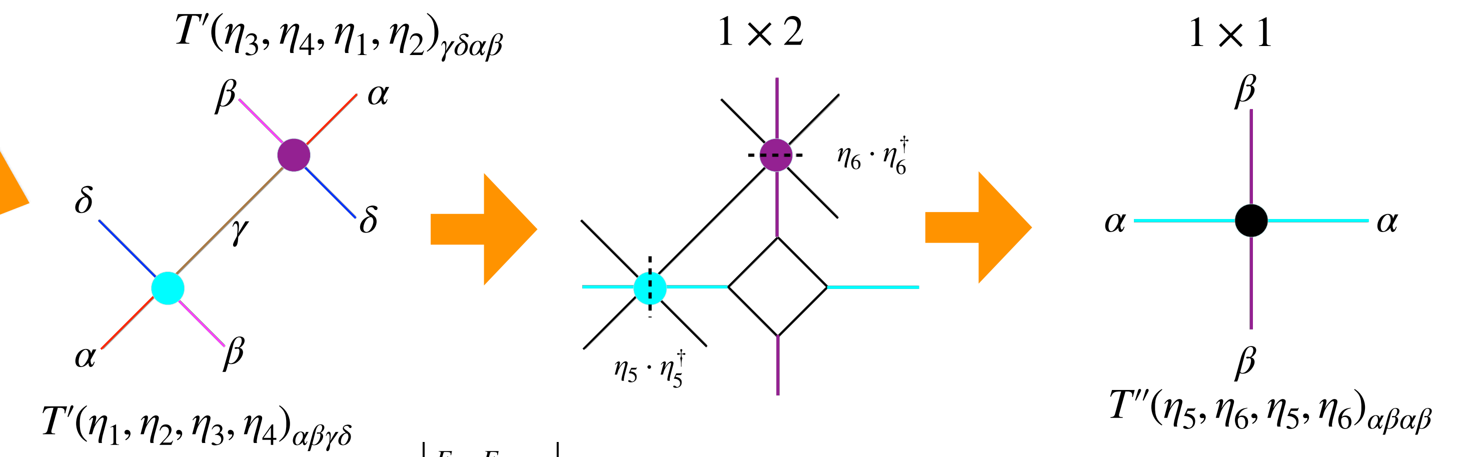
(Average in the last RG step)



While there is a deviation due to noise contamination (multiple use of common noise vectors in RG processes), we have better accuracy than the original TRG.

5. Independent noise

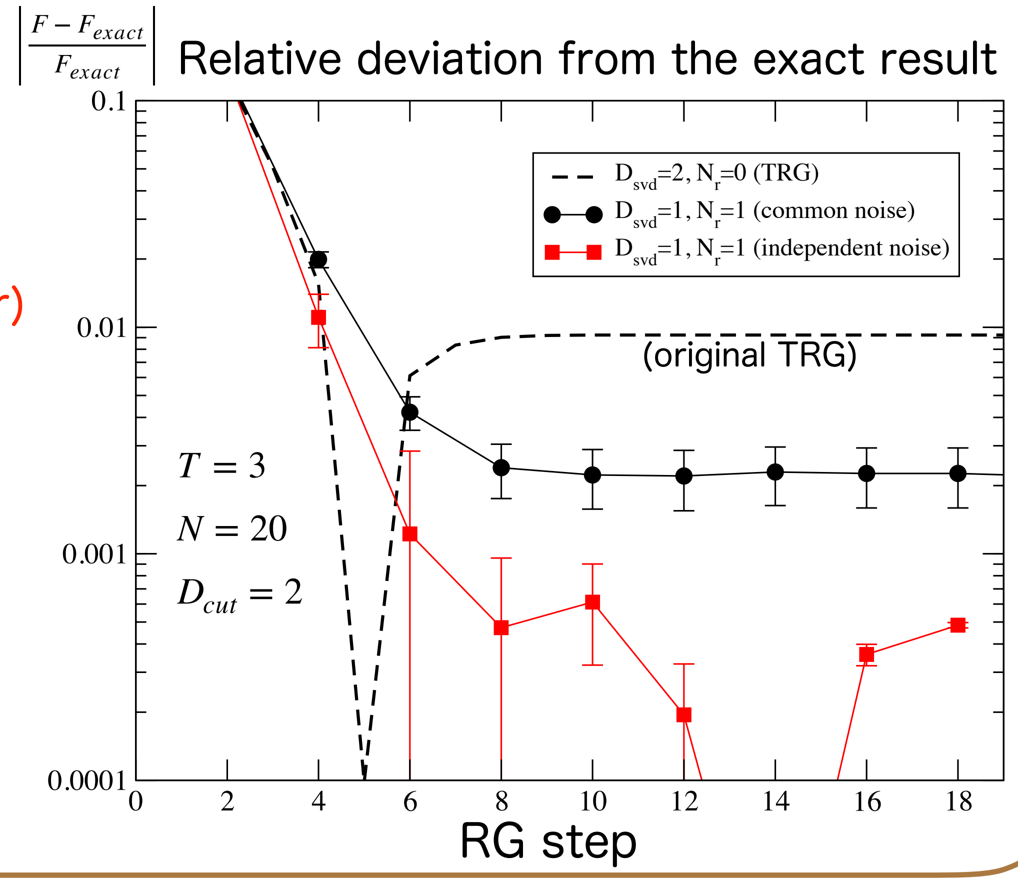
To avoid the noise contamination, use different noises in sites. $(\eta_1 \neq \eta_3,\ \ \eta_2 \neq \eta_4)$



Independent noise
→ no contamination
(No systematic error)

Exact in $N \to \infty$
even with finite D_{cut}

Better accuracy even with a small statistics (N=20)



6. Summary

We have proposed new low-rank approximation methods for TRG by utilizing a random noise vector in combining with the singular value decompositions, which enables to calculate the tensors statistically.
Optimization of singular values and noise spaces → a better result.
Independent noise → no systematic error, exact with finite D_{cut}.
Applicable to other networks with higher dimensions as long as SVD used.

Comparison of error and cost estimations

	Systematic error	Cost
Original	truncation	$\mathcal{O}(D^6\log V)$
Common noise	noise contamination	$\mathcal{O}(D_{cut}^6\log V)$, $(D_{cut}\sim D)$
Independent noise	—	$\mathcal{O}(D_{cut}^6 V)$, $(D_{cut}\ll D)$