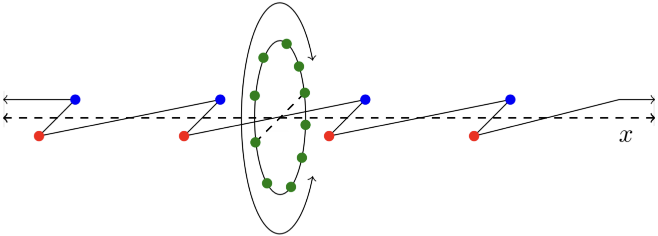
Lattice July 2021

Quantum Algorithms for Simulating the Lattice Schwinger Model

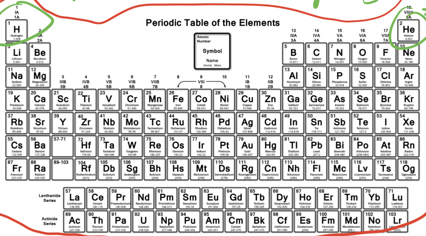


Alexander F. Shaw   University of Maryland - College Park

Based on work of same title in Quantum 4, 306 (2020)
by AF Shaw, P Lougovski, JR Stryker and N Wiebe

Why **Quantum** Simulation?

Elements we can simulate from First Principles



What we'd like to simulate

*[Yamazaki et. al (2010) PhysRevD.81.111504]

- Using fundamental models to calculate physics at longer length scales is hard.

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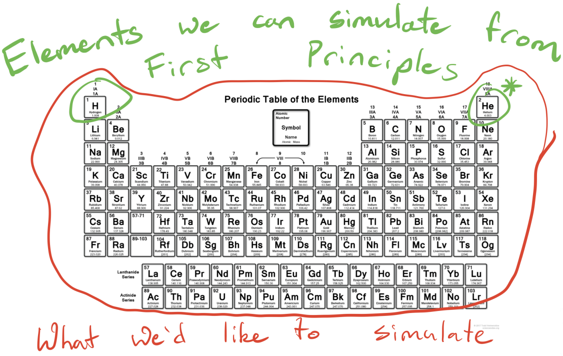

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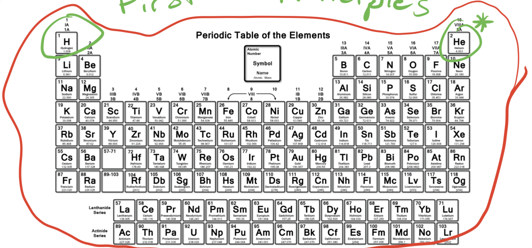

What we'd like to simulate

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- Using fundamental models to calculate physics at longer length scales is hard.
- Access different physics (real-time dynamics)
- Classical methods have been very successful, but hit roadblocks. (exponential resource scalings, sign problems)

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	Cheap	Expensive
Near-term (NISQ)	Single-qubit rotations	Entangling gates (CNOT)
Far-term (Fault-Tolerant)	Clifford + CNOT gates	T-gates

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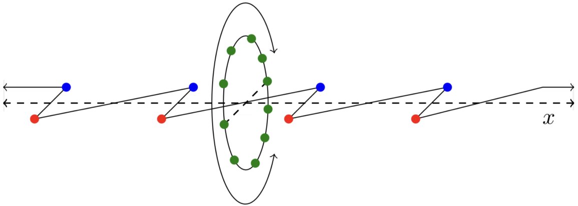
The Lattice Schwinger Model*



odd sites - electrons

even sites - positrons $\underset{\text{continuum limit}}{\Longrightarrow}$ QED in (1+1)D

link sites - electric field

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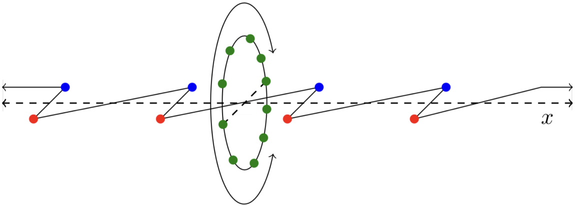

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Simplest model that replicates features of QCD (confinement, spontaneous breaking of chiral symmetry).

*[Kogut, Susskind (1975) 10.1103/PhysRevD.11.395]

Computing time evolution

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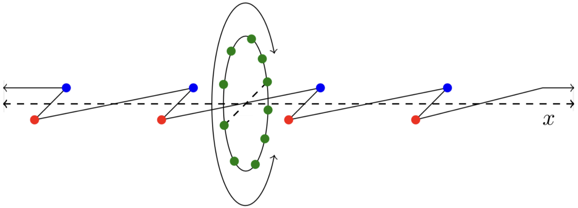

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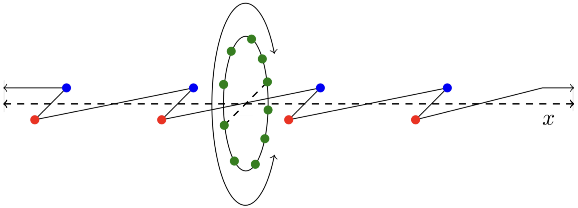

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computational basis	occupation basis
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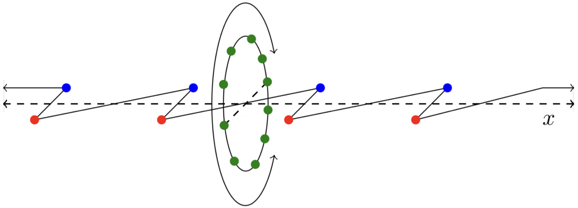

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The value of the electric field is represented as a binary integer in a qubit register, one for each link.

Computing time evolution

Decompose the Schwinger Model Hamiltonian

$$\begin{aligned} H_{\text{Schwinger}} &= \sum_r \hat{E}_r^2 + \mu \sum_r (-)^r \hat{\psi}_r^\dagger \hat{\psi}_r + x \sum_r \left[\hat{U}_r \hat{\psi}_r^\dagger \hat{\psi}_{r+1} - \hat{U}_r^\dagger \hat{\psi}_r \hat{\psi}_{r+1}^\dagger \right] \\ &= \underbrace{H_{\text{E-field}} + H_{\text{mass}}}_{\text{diagonal in computational basis}} + \underbrace{H_{\text{interaction}}}_{\substack{\text{linear combination of} \\ \text{tensor product of pauli matrices}}} \\ &= \sum_{j=1}^{m} H_j \quad \leftarrow \quad \text{Simulatable!} \end{aligned}$$

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$$V_{\text{trotter step}}(t) := \prod_{j=1}^{m} e^{-iH_j t/2} \prod_{j=m}^{1} e^{-iH_j t/2}$$

$$\left\| e^{-iH_{\text{Schwinger}} t} - V_{\text{trotter step}}(t) \right\| \leq \frac{1}{12} \sum_{x, y>x, z>x} \| [[H_x, H_y], H_z] \| t^3$$

$$+ \frac{1}{24} \sum_{x, y>x} \| [[H_x, H_y], H_x] \| t^3 \quad \doteq \quad \delta(\text{lattice parameters}, \text{evolution time}).$$

Error bound from [Childs et al. (2021) 10.1103/PhysRevX.11.011020]

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- **Upper bound** on sufficient quantum-computational resources required for an arbitrary simulation:

$$\text{COST}(\text{total evolution time}, \text{error in final state}, \text{lattice parameters}) \in \widetilde{O}\left(\frac{N^{3/2} T^{3/2} \Lambda x^{1/2}}{\delta^{1/2}}\right)$$

where N = lattice size, T = evolution time, Λ = E-field cutoff, and $x = 1/(ag)^2$ (a lattice spacing, g coupling constant)

Data and Conclusion (Room for Improvement)

We compile these results further with **estimation of mean positron/electron density** :

Near-term (NISQ) Simulation (no ancilla)

	$\delta_g = 10^{-3}$		$\delta_g = 10^{-4}$		$\delta_g = 10^{-5}$		$\delta_g = 10^{-6}$		$\delta_g = 10^{-7}$	
	$\tilde{\epsilon}^2$	CNOT	$\tilde{\epsilon}^2$	CNOT	$\tilde{\epsilon}^2$	CNOT	$\tilde{\epsilon}^2$	CNOT	$\tilde{\epsilon}^2$	CNOT
$x = 10^{-2}$	—	7.3e4	—	1.6e5	—	3.4e5	—	7.3e5	5.6e-2	1.6e6
$x = 10^{-1}$	—	1.6e4	—	3.5e4	—	7.5e4	5.9e-2	1.6e5	2.7e-3	3.5e5
$x = 1$	—	4.6e3	—	9.9e3	1.0e-1	2.1e4	4.7e-3	4.6e4	2.2e-4	9.9e4
$x = 10^2$	—	2.8e3	8.3e-1	6.1e3	3.8e-2	1.3e4	1.8e-3	2.8e4	8.2e-5	6.0e4

$x = (ag)^{-2}$ with a lattice spacing and g coupling constant
δ_g = error in CNOT channel
$\tilde{\epsilon}^2$ = worst case mean square error in mean positron density

Data and Conclusion (Room for Improvement)

Far-term Simulation (N is size of lattice, Λ is electric cutoff)

Upper Bounds on T-gate Cost of Specific Simulations ($\mu = 1,\ \tilde{\epsilon}^2 = 0.1$)				
	Short Time ($T = 10/x$)		Long Time ($T = 1000/x$)	
	Sampling	Estimating	Sampling	Estimating
$N = 4,\ \Lambda = 2$				
Strong Coupling ($x = 0.1$)	$6.5 \cdot 10^{7}$	$2.4 \cdot 10^{11}$	$8.8 \cdot 10^{10}$	$3.3 \cdot 10^{14}$
Weak Coupling ($x = 10$)	$5.0 \cdot 10^{6}$	$1.8 \cdot 10^{10}$	$7.0 \cdot 10^{9}$	$2.6 \cdot 10^{13}$
$N = 16,\ \Lambda = 2$				
Strong Coupling ($x = 0.1$)	$7.2 \cdot 10^{8}$	$2.5 \cdot 10^{12}$	$9.4 \cdot 10^{11}$	$3.3 \cdot 10^{15}$
Weak Coupling ($x = 10$)	$5.6 \cdot 10^{7}$	$1.9 \cdot 10^{11}$	$7.6 \cdot 10^{10}$	$2.7 \cdot 10^{14}$
$N = 16,\ \Lambda = 4$				
Strong Coupling ($x = 0.1$)	$1.9 \cdot 10^{9}$	$6.3 \cdot 10^{12}$	$2.3 \cdot 10^{12}$	$8.1 \cdot 10^{15}$
Weak Coupling ($x = 10$)	$9.6 \cdot 10^{7}$	$3.2 \cdot 10^{11}$	$1.2 \cdot 10^{11}$	$4.2 \cdot 10^{14}$
$N = 64,\ \Lambda = 2$				
Strong Coupling ($x = 0.1$)	$6.6 \cdot 10^{9}$	$2.1 \cdot 10^{13}$	$8.5 \cdot 10^{12}$	$2.9 \cdot 10^{16}$
Weak Coupling ($x = 10$)	$5.2 \cdot 10^{8}$	$1.6 \cdot 10^{12}$	$6.9 \cdot 10^{11}$	$2.3 \cdot 10^{15}$
$N = 64,\ \Lambda = 4$				
Strong Coupling ($x = 0.1$)	$1.7 \cdot 10^{10}$	$5.4 \cdot 10^{13}$	$2.0 \cdot 10^{13}$	$6.9 \cdot 10^{16}$
Weak Coupling ($x = 10$)	$8.7 \cdot 10^{8}$	$2.7 \cdot 10^{12}$	$1.1 \cdot 10^{12}$	$3.6 \cdot 10^{15}$
$N = 64,\ \Lambda = 8$				
Strong Coupling ($x = 0.1$)	$4.5 \cdot 10^{10}$	$1.5 \cdot 10^{14}$	$5.3 \cdot 10^{13}$	$1.8 \cdot 10^{17}$
Weak Coupling ($x = 10$)	$1.5 \cdot 10^{9}$	$4.6 \cdot 10^{12}$	$1.7 \cdot 10^{12}$	$5.8 \cdot 10^{15}$

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- How do other algorithms (LCU, Qubitization) compare?
- Could other formulations of the model be beneficial? (SU(2) - Quantum Link Model, Loop-String-Hadron)
- How can we best realize gauge invariance during simulation?

Our work is a **benchmark** to aid in answering these questions.

LCU - [Childs, Wiebe (2012) 10.26421/QIC12.11-12]
Qubitization - [Low, Chuang (2019) 10.22331/q-2019-07-12-163]
Quantum Link Model - [Chandrasekharan, Wiese (1996) 10.1016/S0550-3213(97)80041-7]
Loop-String-Hadron - [Raychowdhury, Stryker (2019) 10.1103/PhysRevD.101.114502]

Preliminary Results - SU(2):

Using abelian gauge invariant hopping decomposition* - Discussed in J. Stryker's talk **this evening, 10:45pm EST**:

$\mathrm{T}\left(\frac{1}{a^2 g^2}, \Lambda, \frac{2m}{g^2 a}, N, T\right) =$ Trotter Steps to reach error 10^{-3}.

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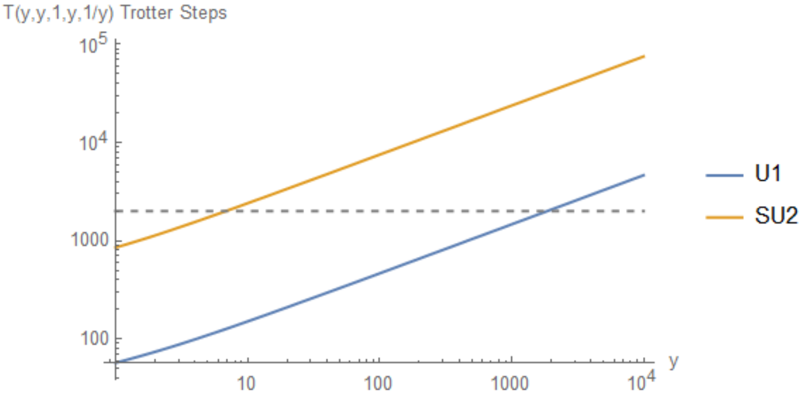
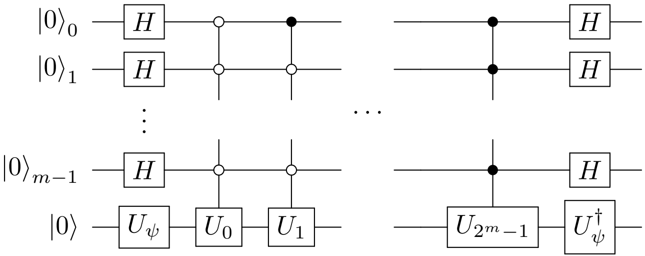


*[J. Stryker (2021) arXiv:2105.11548]

Using amplitude estimation to estimate $\langle \hat{O} \rangle$



If $\hat{O} = \sum_{a=0}^{2^m-1} \hat{U}_a$, $\hat{U}_a$ unitary, then above circuit ends up in:

Prob(measure $|00...0\rangle$) $= C\langle \hat{O} \rangle$, C known constant.