Stable solvers for real-time Complex Langevin

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Arxiv: 2105.02735

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Introduction

arxiv: 2105.02735

- Real-time simulation (sign-problem)
- Complex Langevin equation
  (Stochastic Differential equation)

$$\frac{d\phi}{d\tau_L} = i\frac{\delta S[\phi]}{\delta\phi(x)} + \eta(x,\tau_L) \quad \text{with}$$

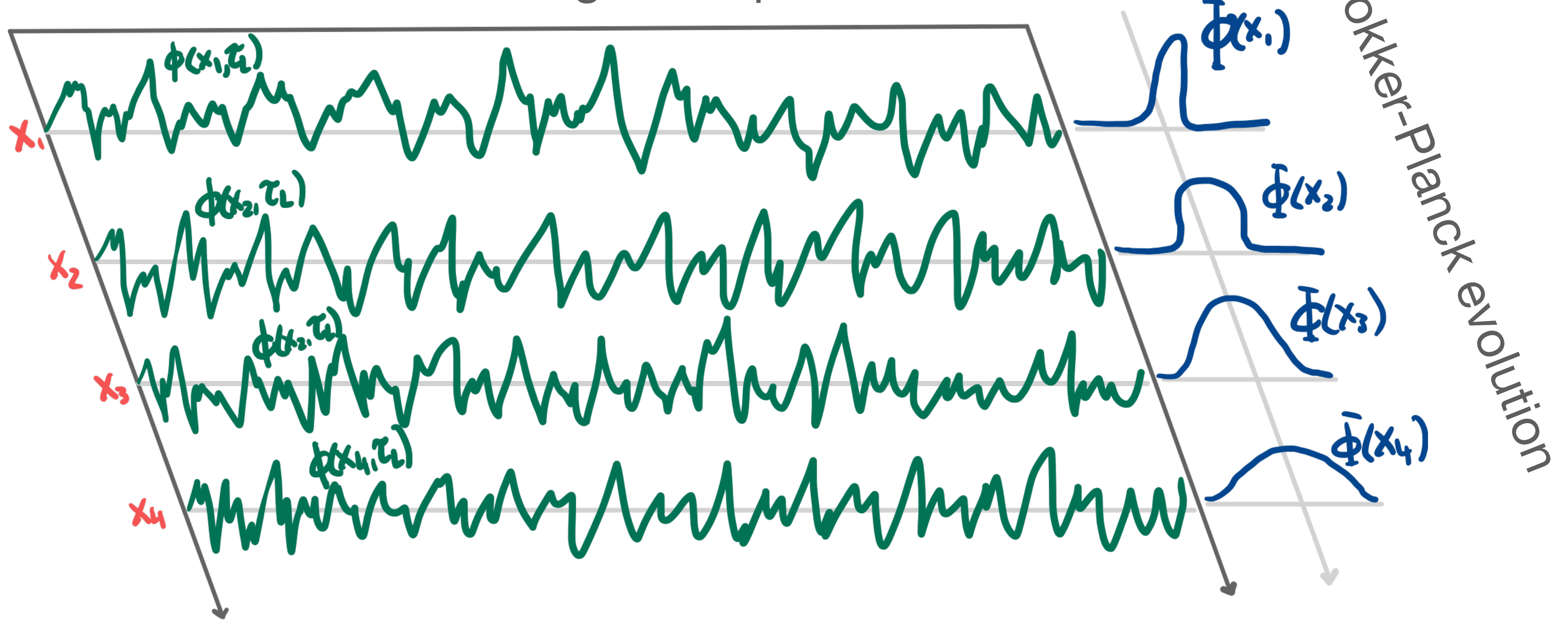
$$\langle \eta(x,\tau_L)\rangle = 0, \quad \langle \eta(x,\tau_L)\eta(x',\tau_L')\rangle = 2\delta(x-x')\delta(\tau_L-\tau_L')\,.$$

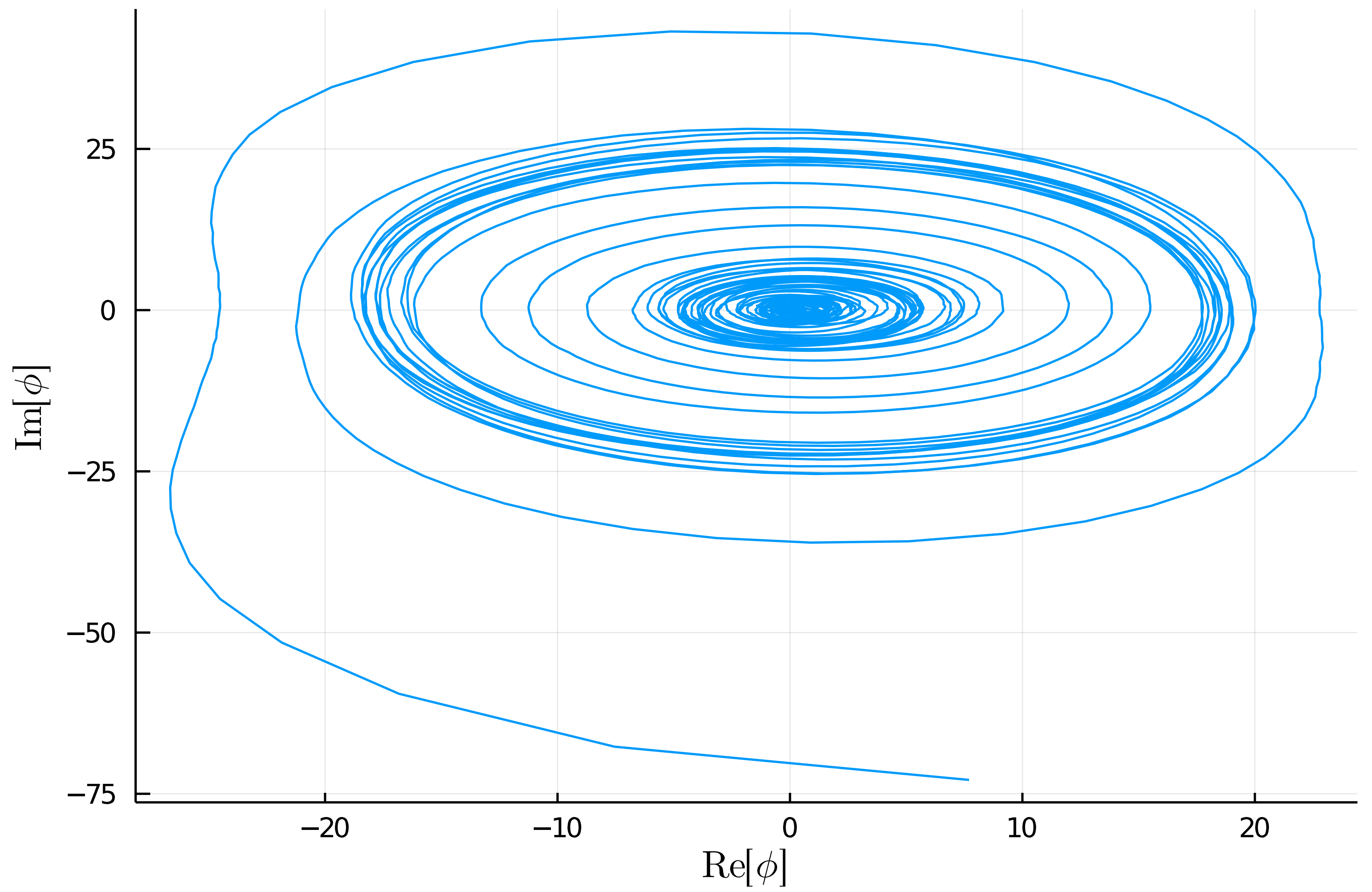
- Fokker-Planck equation

$$\frac{\partial}{\partial t}\Phi(x,t) = \sum_j \frac{\delta}{\delta\phi_j}\left[\frac{\delta}{\delta\phi_j} + \frac{\delta S[\phi]}{\delta\phi_j}\right]\Phi(x,t) = -H_{\mathrm{FP}}\Phi(x,t)$$

- Problems
  - Runaway solutions
  - Convergence to the wrong solution

Evolution of the Langevin equation





Simple overview of SDE solver

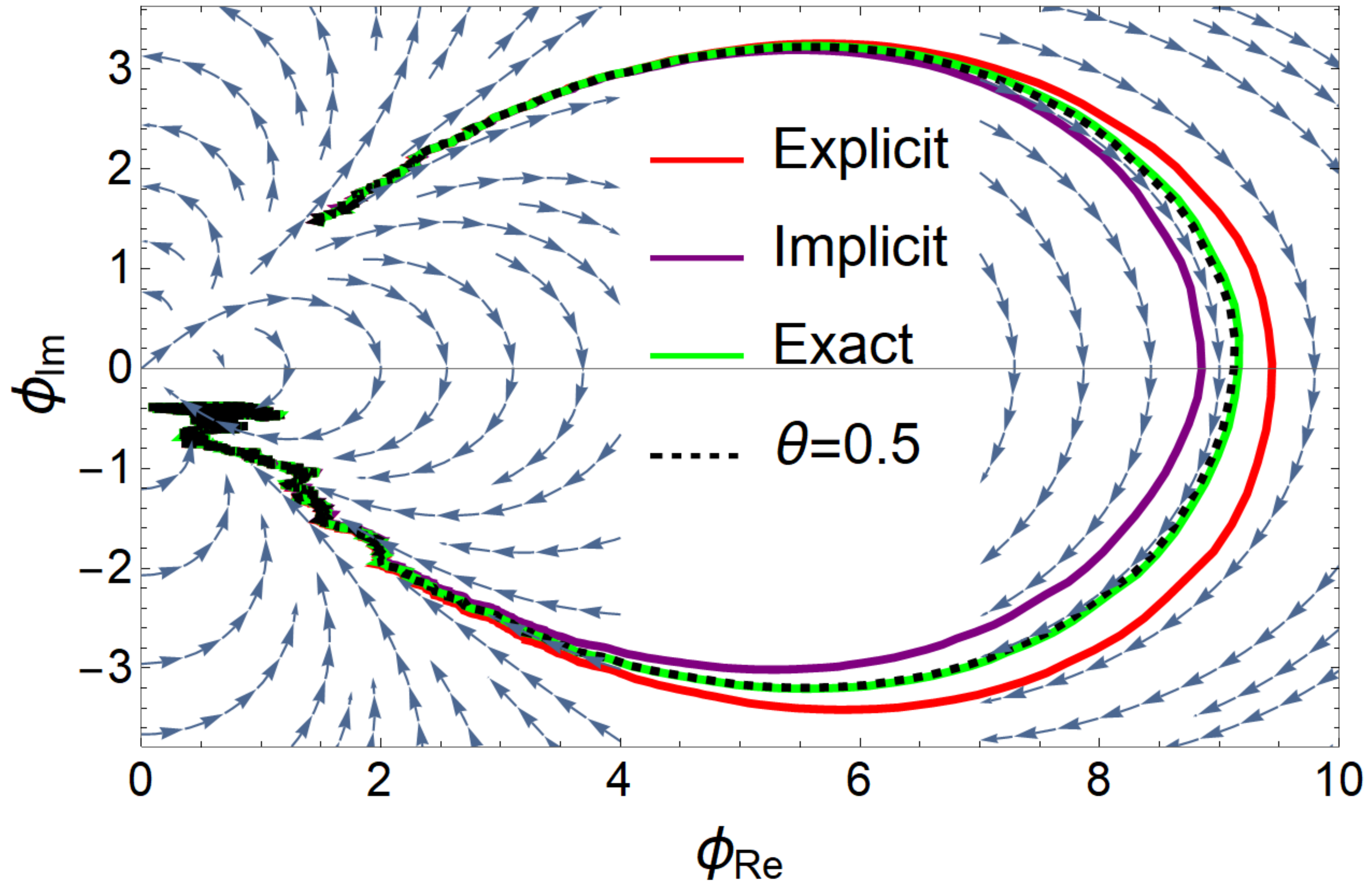
- General Euler-Maruyama Scheme:

$$\phi_j^{\lambda+1} = \phi_j^{\lambda} + i\epsilon_j \left[\theta \frac{\partial S^{\lambda+1}}{\partial \phi_j} + (1-\theta) \frac{\partial S^{\lambda}}{\partial \phi_j} \right] + \sqrt{\epsilon_j}\eta_j^{\lambda}$$

CLE:

$$\frac{d\phi}{d\tau_L} = i\frac{\delta S[\phi]}{\delta \phi(x)} + \eta(x, \tau_L)$$

  - Explicit ($\theta = 0.0$): Overshooting
  - Implicit ($\theta = 1.0$): Undershooting
  - Semi-implicit ($\theta = 0.5$): Stable and close to the exact solution
  - For all $\theta \geq 0.5$ we get rid of runaways (Unconditionally stable)

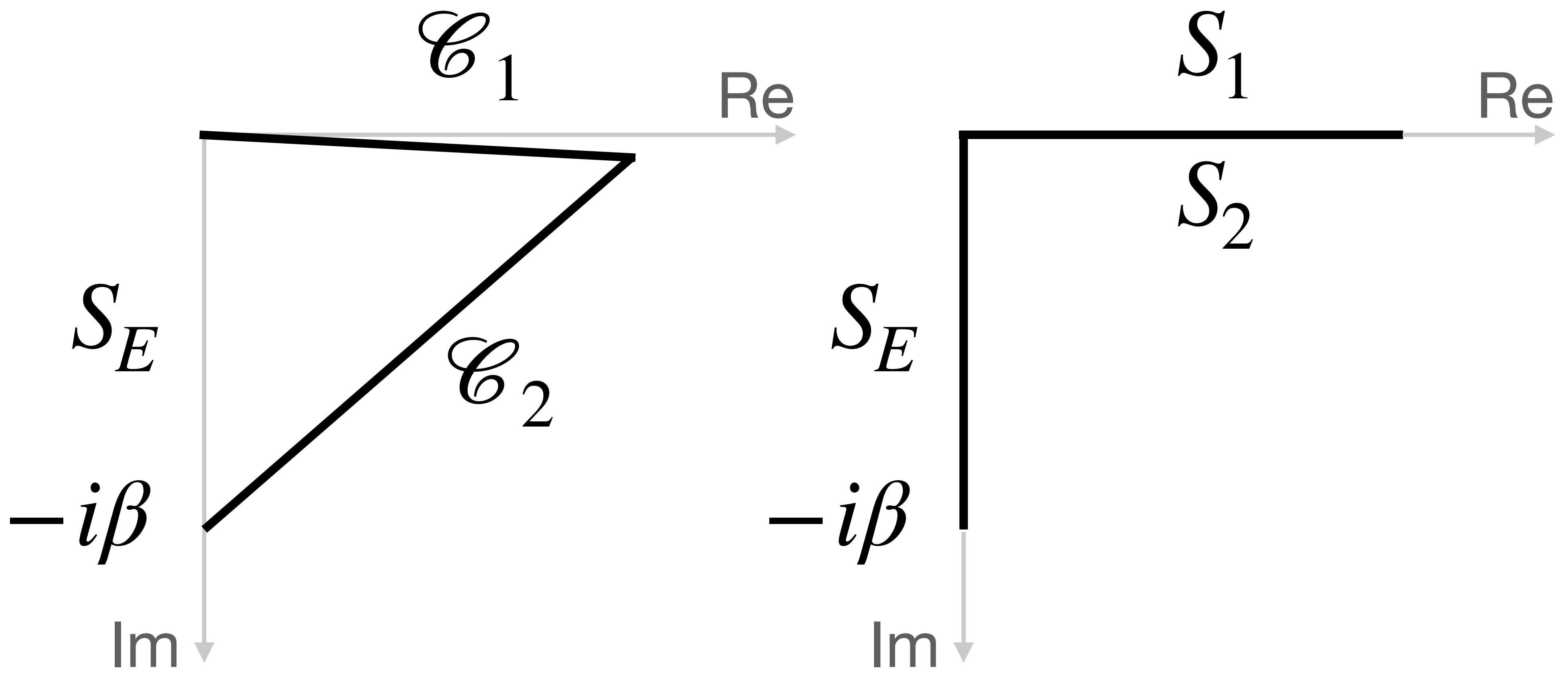


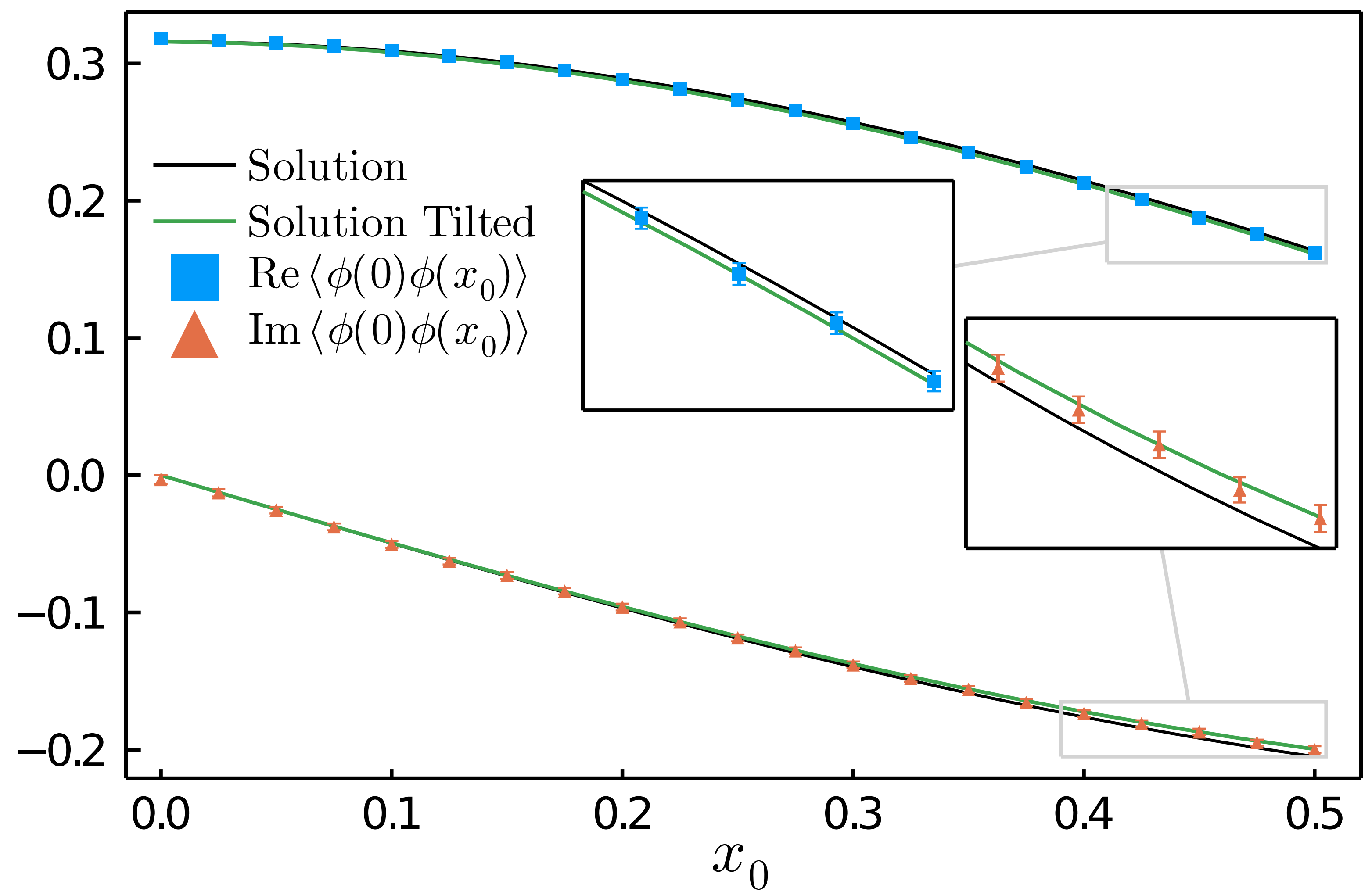
Simulations done with the DifferentialEquations.jl library in Julia

Regularising by tilting real-time contour

- Regularisation; Infinitesimal damping term:
  $\bar{S} = S + iR(\phi, \epsilon)$
  - Example: $R = \frac{1}{2}\epsilon\phi^2$
  - Tilted contour
- Problem: S_E not available, $\mathscr{C}_2 \neq S_2$ and $S_1 \not\approx \mathscr{C}_1$ for large real-times
- Strongly coupled quantum anharmonic oscillator with $m = 1, \lambda = 24$ on a tilted real-time contour

$$S = \int dx_0 \left\{ \frac{1}{2}\left(\frac{\partial\phi}{\partial x_0}\right)^2 - \frac{1}{2}m\phi^2 - \frac{\lambda}{4!}\phi^4 \right\}$$





Contour:
$\beta = 1$, $x_0^{\text{max}} = 0.5$,
tilting 0.01β and 64 points
Solver: Euler-Mayruyama with $\theta = 0.5$

Regularisation via an implicit scheme

- Free theory effective action $S_\theta = S_{\text{explicit}} + \frac{i\epsilon}{2}\theta \sum_j \left(\frac{\partial S}{\partial \phi_j}\right)^2$

General EM scheme:

$$\phi_j^{\lambda+1} = \phi_j^\lambda + i\epsilon_j \left[\theta \frac{\partial S^{\lambda+1}}{\partial \phi_j} + (1-\theta)\frac{\partial S^\lambda}{\partial \phi_j}\right] + \sqrt{\epsilon_j}\eta_j^\lambda$$

- How to regulate: combination between setting Langevin step-size ϵ and implicitness factor θ

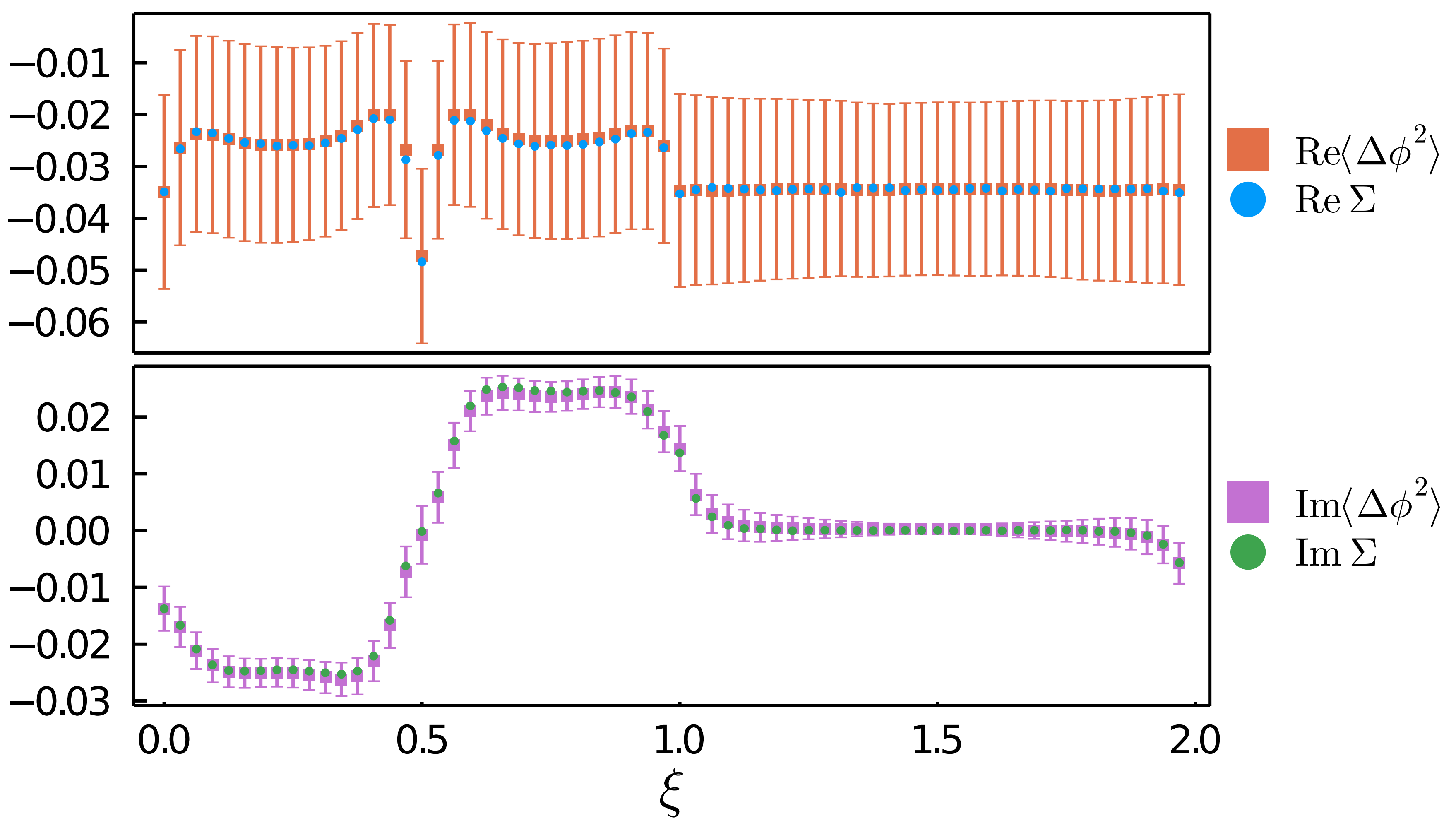
- Following (A. Kronfeld, P.T.P.S. 111, 1993) we take an educated guess on the form of the discrete Fokker-Planck induced by the finite step CLE.

$$0 = \nabla_j \left[\left(-i\frac{\partial \bar{S}}{\partial \phi_j} + \nabla_j\right)\mathscr{P}\right], \quad \bar{S} = S + \frac{\epsilon}{2}\sum_k \left\{\left(\frac{\partial^2 S}{\partial \phi_k^2}\right) + i\left(\frac{1}{2}-\theta\right)\left(\frac{\partial S}{\partial \phi_k}\right)^2\right\}.$$

- Field redefinition to undo finite ϵ: $\tilde{\phi}_j = \phi_j - \frac{i\epsilon}{2}\left(\frac{1}{2}-\theta\right)\frac{\partial S}{\partial \phi_j}$

$$\left\langle \tilde{\phi}_j^2 \right\rangle = \left\langle \phi_j^2 \right\rangle - \underbrace{\epsilon\left(\frac{1}{2}-\theta\right)\left\langle \phi_j\left(i\frac{\partial S}{\partial \phi_j}\right)\right\rangle}_{\Sigma} - \frac{\epsilon^2}{4}\left(\frac{1}{2}-\theta\right)^2\left\langle\left(\frac{\partial S}{\partial \phi_j}\right)^2\right\rangle$$

$\langle \Delta\phi^2 \rangle = \langle \phi^2 \rangle_{\text{CL}} - \langle \phi^2 \rangle_{\text{QM}}$

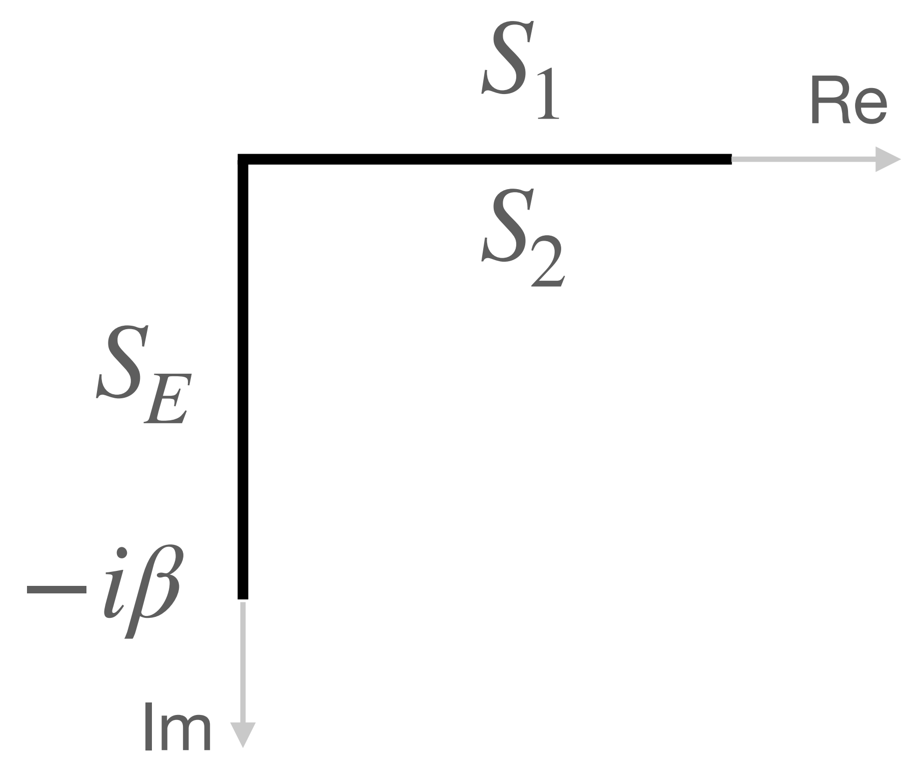


Free theory (QM) correction

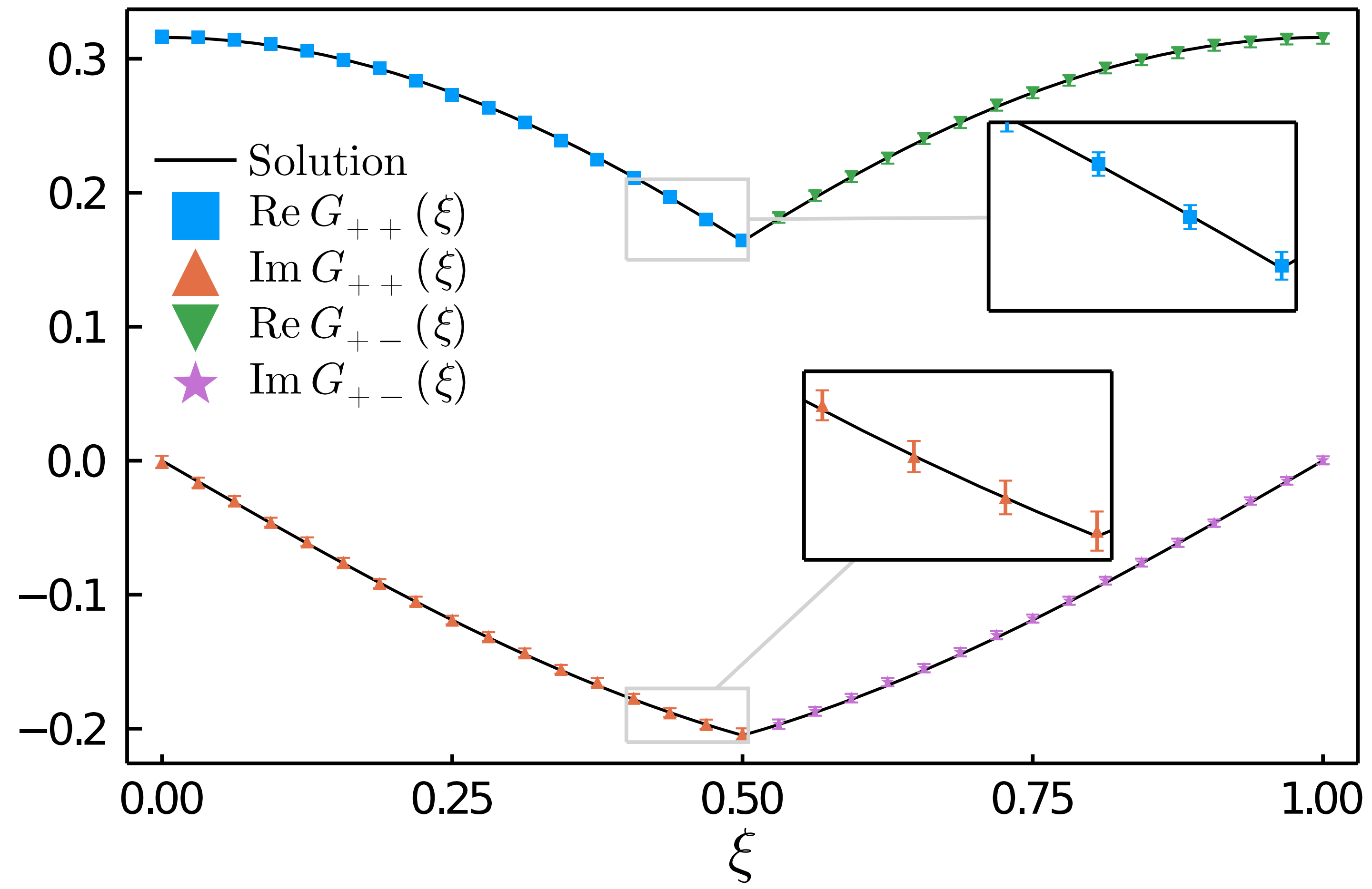
Dynamics in thermal equilibrium

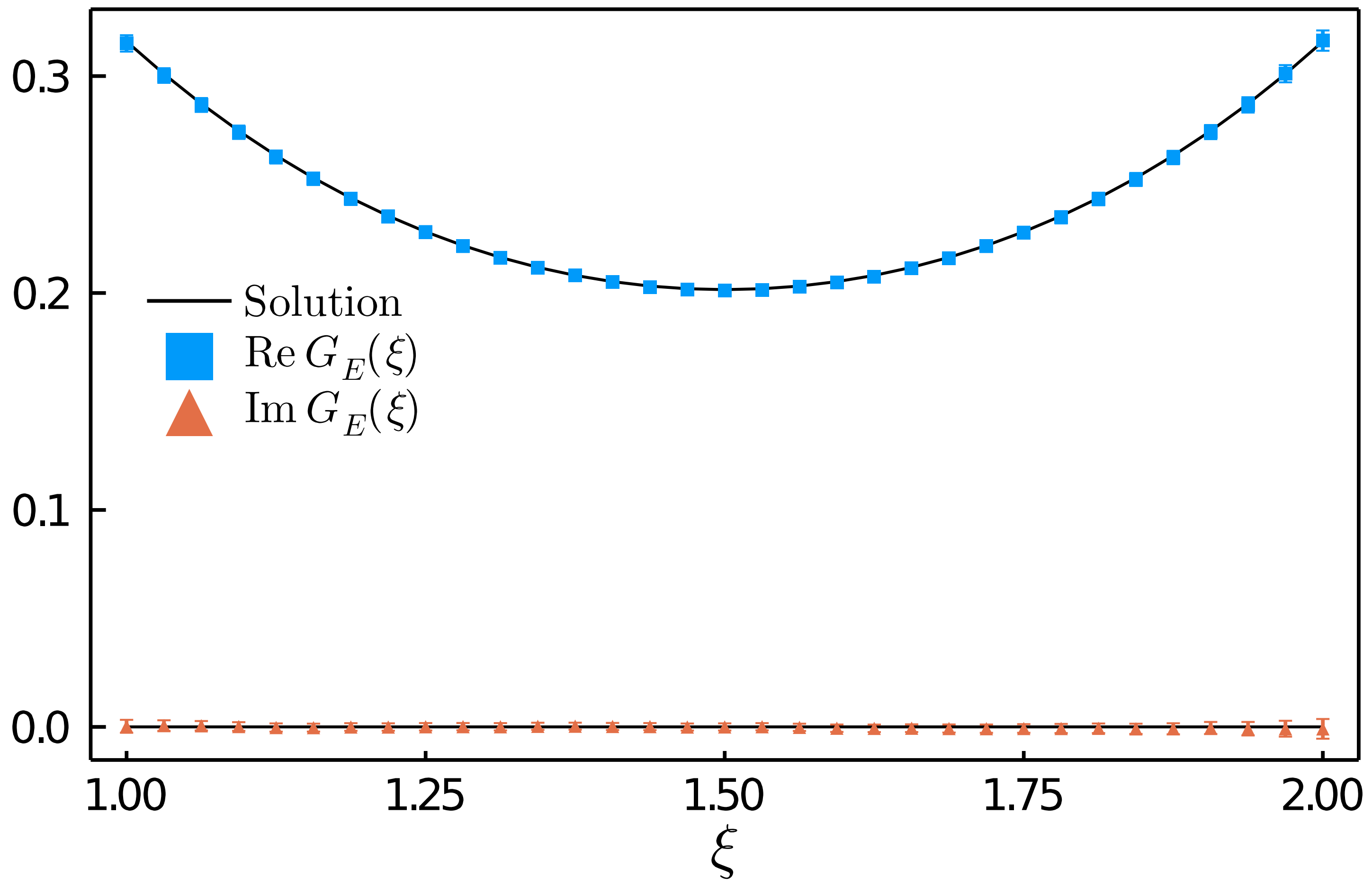
- Regulator: $\theta = 0.6$, Contour: $\beta = 1.0$, $x_0^{\text{max}} = 0.5$
- Parameters $m = 1$, $\lambda = 24$ (same as in Berges, Borsanyi, Sexty, Stamatescu, 2007)

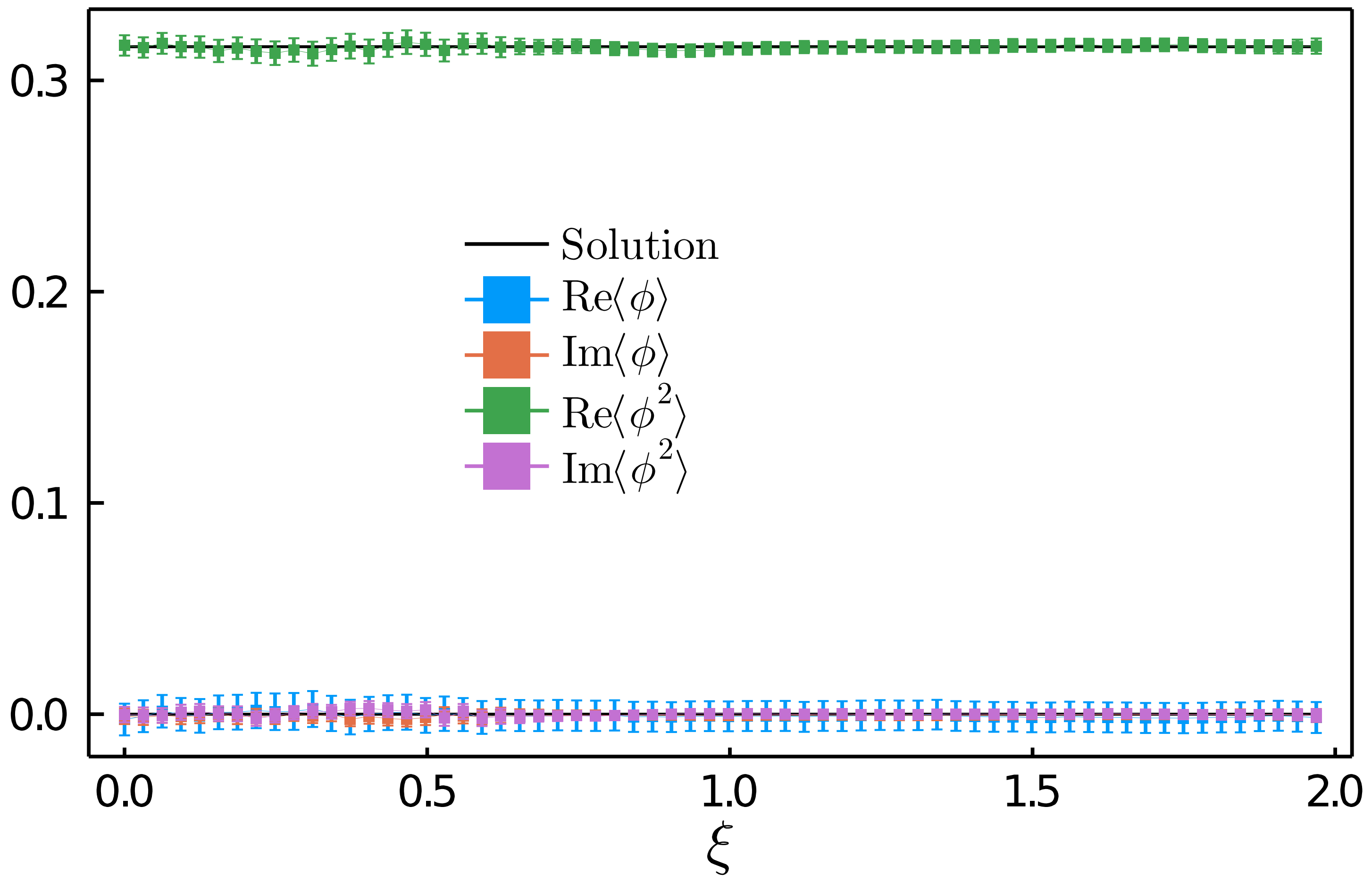
$$S = \int dx_0 \left\{ \frac{1}{2} \left(\frac{\partial \phi}{\partial x_0} \right)^2 - \frac{1}{2} m \phi^2 - \frac{\lambda}{4!} \phi^4 \right\}$$



- $G_{++}(\xi) = \langle \phi(0)\phi(\xi) \rangle - \langle \phi(0) \rangle \langle \phi(\xi) \rangle$ for $\xi \leq 0.5$
- $G_E(\xi) = \langle \phi(0)\phi(\xi) \rangle - \langle \phi(0) \rangle \langle \phi(\xi) \rangle$ for $\xi \geq 1$

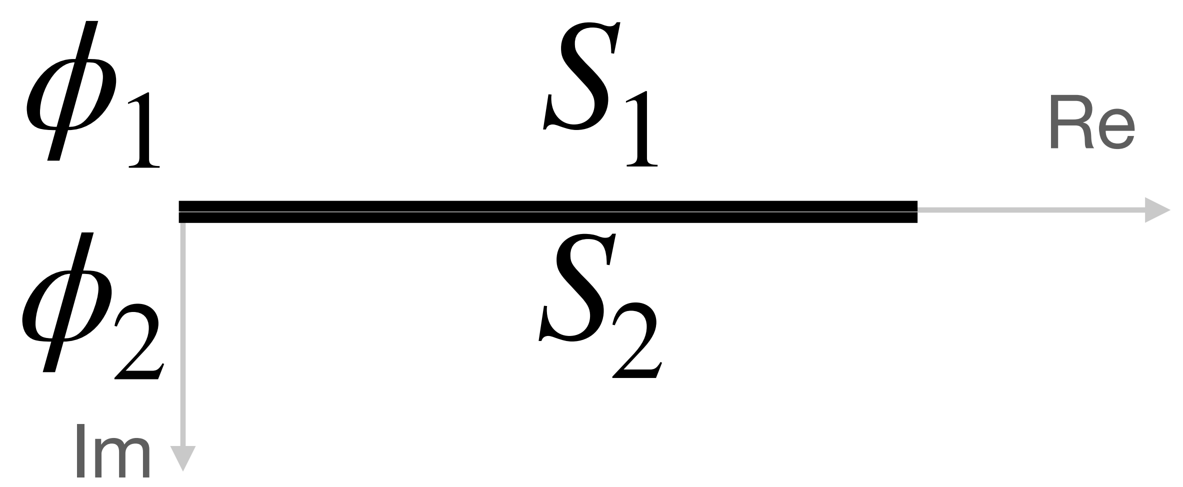






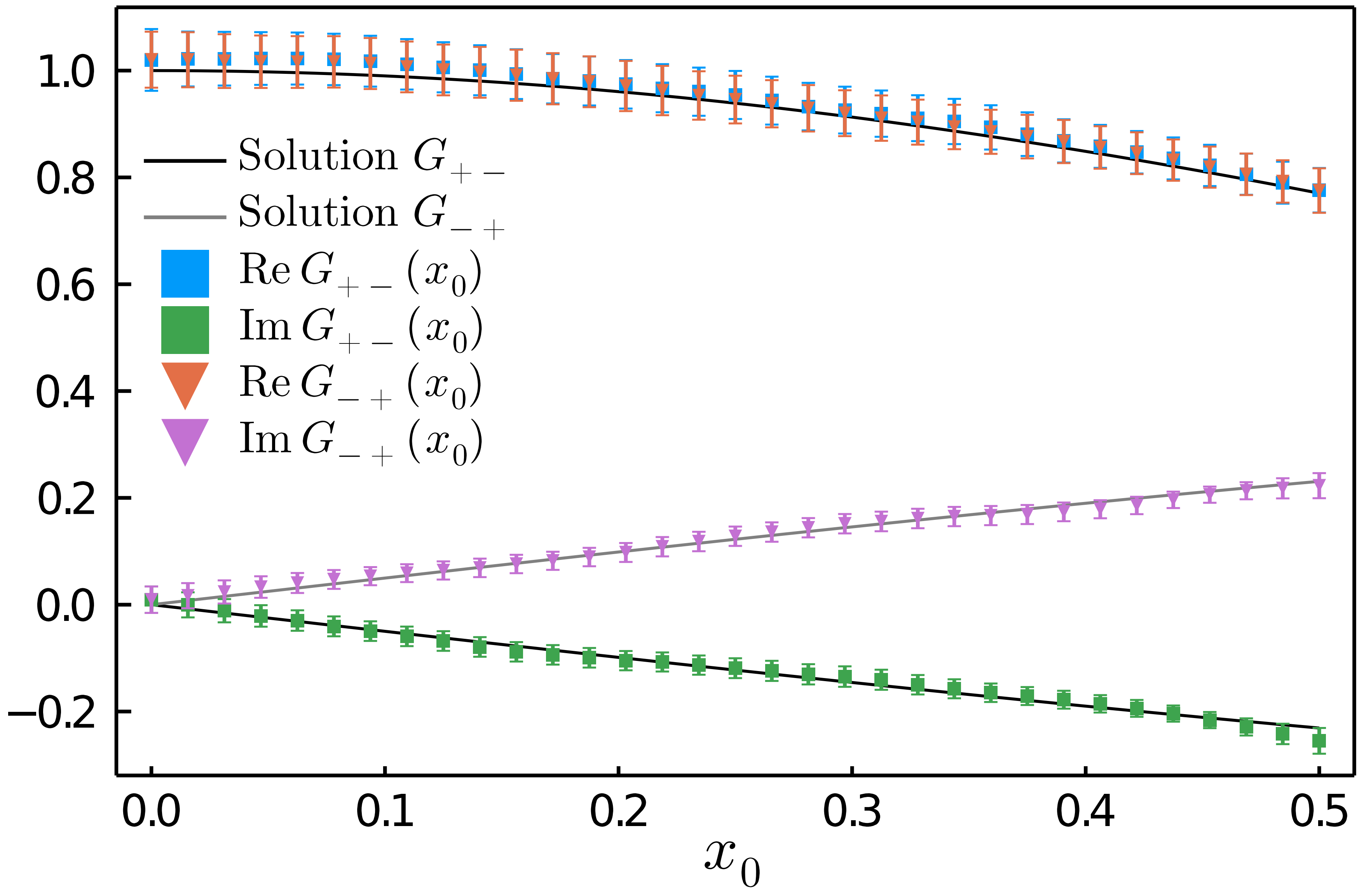
Non-Equilibrium dynamics

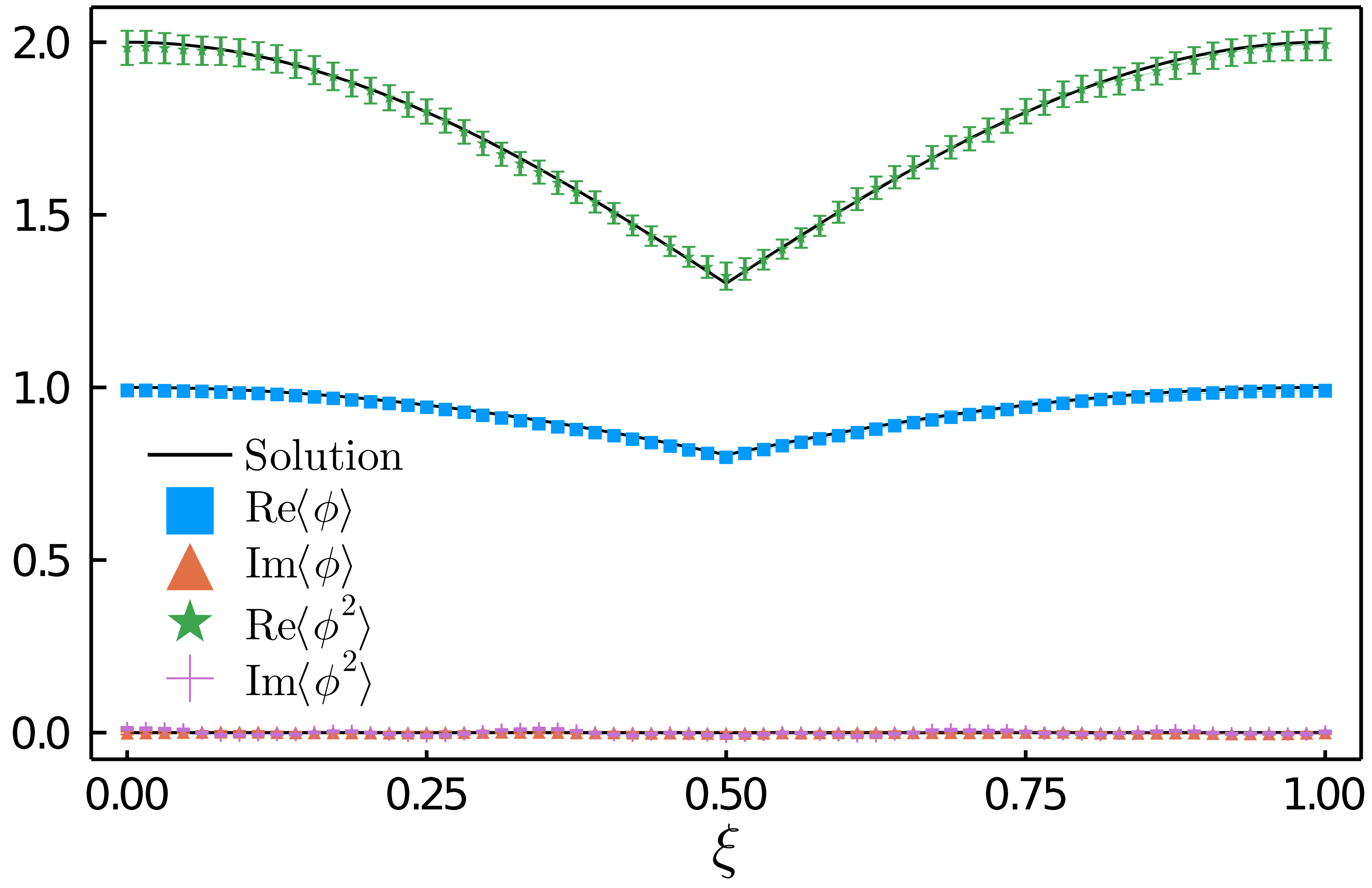
- Gaussian initial density matrix with
  $\langle\phi_0\rangle = 1,\ \langle\dot{\phi}_0\rangle = 0,\ \langle\phi_0\phi_0\rangle = 1,\ \langle\dot{\phi}_0\dot{\phi}_0\rangle = \frac{1}{4}$
  (Berges, Borsanyi, Sexty, Stamatescu, 2007)



- Small coupling $\lambda = 1$ and regulator $\theta = 0.6$

- Full access to G_{+-} and $G_{-+}(x_0) = \langle\phi_2\, \phi(x_0)\rangle - \langle\phi_2\rangle\langle\phi(x_0)\rangle$





Summary and outlook

- Implicit scheme
  - Unconditionally stable: No runaway solutions
  - Acts as a regulator in ϕ^4 theory (quantum mechanics)
  - Full access to the canonical Schwinger-Keldysh contour
- Distinguish between the breakdown of the numerics and the CLE itself
- Implicit solver for Gauge theories are work in progress
- Code can be found at CLSolvers.jl (zenodo.4888823)