Investigating a Renormalization Group Multigrid Approach for Domain Wall Fermions

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The ensembles used in this work were produced by Jiqun Tu.

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Motivation

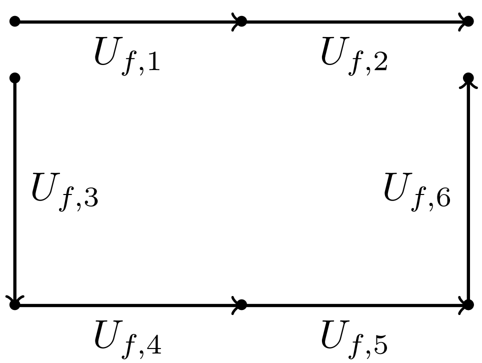
- RBC-UKQCD ensemble generation using CG in production
  - Shamir Domain wall and Mobius Domain Wall fermions used.
  - Many optimizations employed: Hasenbusch masses, force gradient integrator, optimized code, ...
- In HMC/RHMC, no effective multigrid solver for (M)DWF to date
  - Setup time for Hierarchically Deflated CG (Boyle) can be amortized during a trajectory, but no net speed-up (Boyle, McGlynn)
  - In the next talk, Peter Boyle will talk about the general status and report on his efforts to combine multigrid ideas and domain decomposition within HMC.
- Given a fine ensemble, can a coarse ensemble with 2× the lattice spacing and the same long distance physics, be used as a low-setup time preconditioner for fermion solves on the fine ensemble?
  - Good a^2 scaling of MDWF + Iwasaki + DSDR ensembles intriguing
  - Jiqun Tu generated such matched ensembles for other purposes [Lattice 2017, 10.1051/epjconf/201817502006, EPJ Web Conf. 175 (2018) 02006]

Matched Fine and Coarse Ensembles

	fine $\langle O \rangle_f$	coarse $\langle O \rangle_c$	blocked coarse $\langle O \rangle_c^b$
size	$24^3 \times 64 \times 12$	$12^3 \times 64 \times 12$	$12^3 \times 64 \times 12$
β	1.943	1.633	-
am_l	0.000787	0.008521	0.007494
am_h	0.019896	0.065073	0.064150
a^{-1}(GeV)	2.001(18)	1.015(16)	1.010(16)
am_{res}	0.004522(12)	0.007439(86)	0.00847(21)
m_π(MeV)	300(3)	307(5)	308(8)
m_K(MeV)	491(5)	506(8)	507(11)
m_Ω(MeV)	1557(71)	1652(27)	1685(52)
f_π(MeV)	138(2)	147(2)	151(3)
f_K(MeV)	155(2)	166(3)	169(4)

Blocked coarse ensemble generated from the fine ensemble with an APE style RG-blocking

$C = \sum_\mu$



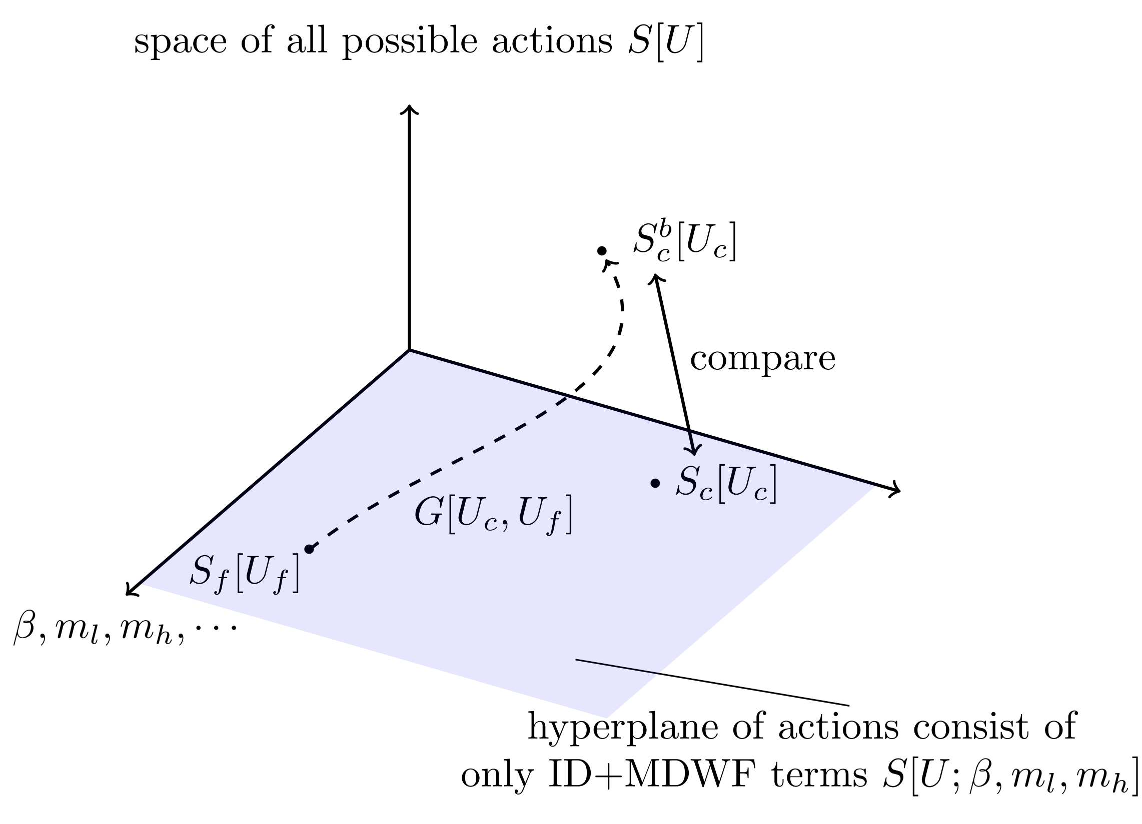
$\rightarrow$

$$g_b[U_f] = \mathcal{P}[(1-\alpha)U_{f,1}U_{f,2} + \alpha C/6]$$

Jiqun Tu

Rationale

- Small O(a^2) scaling violations for coarse (1 GeV) and fine (2 GeV) ensembles
  - * This implies they lie on essentially the same RG trajectory.
- Simple RG blocking creates a coarse lattice from a given fine lattice
  - * Blocking fast to do numerically
- Since fine and blocked-coarse lattice have approximatey identical physcs, can the blocked lattice be used as a preconditioner for fine lattice DWF solves?



Preconditioned CG

Preconditioned CG algorithm

Result: solution for $Ax = b$

$r_0 = b - Ax_0$

$z_0 = M^{-1} r_0$

$p_0 = z_0$

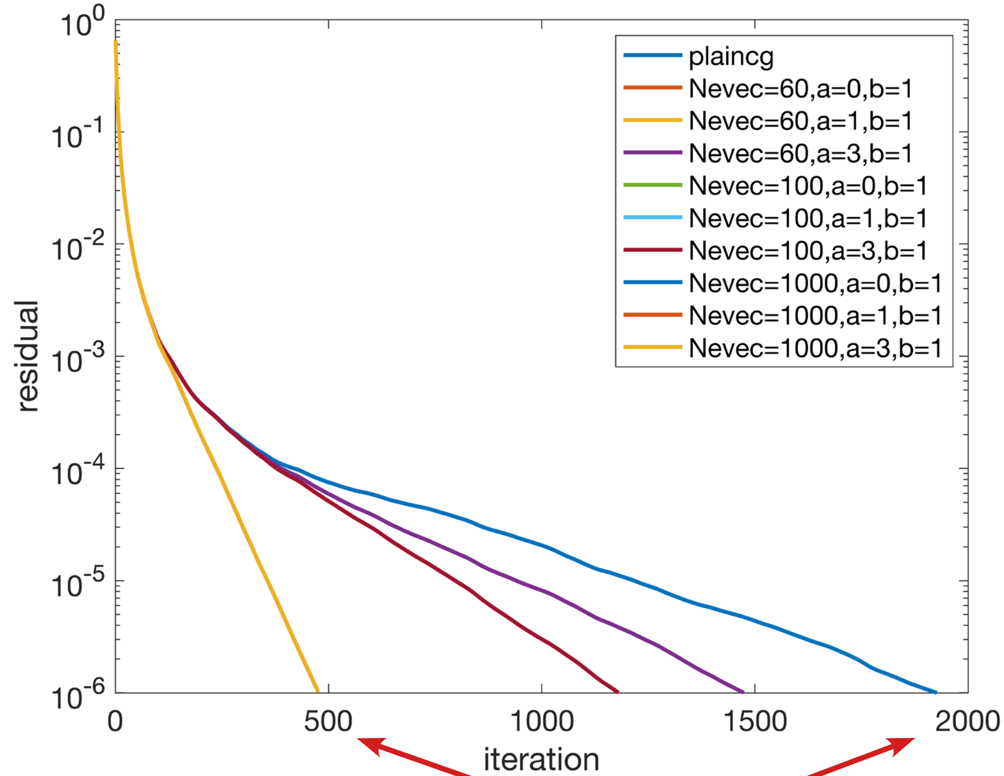
$k = 0$

while $|r_{k+1}| > \epsilon$ **do**

$$\alpha_k = \frac{\langle r_k | z_k \rangle}{\langle p_k | A | p_k \rangle}$$
$$x_{k+1} = x_k + \alpha_k p_k$$
$$r_{k+1} = r_k - \alpha_k A p_k$$
$$z_{k+1} = M^{-1} r_{k+1}$$
$$\beta_k = \frac{\langle z_{k+1} | r_{k+1} \rangle}{\langle z_k | r_k \rangle}$$
$$p_{k+1} = z_{k+1} + \beta_k p_k$$
$$k = k + 1$$

end

Results for fine lattice CG solve with mass = m_s



4× fewer iterations even with quark mass of m_s

- Working with $A = D^\dagger D$
- Precondition with fine lattice eigenvectors

$$M^{-1} = a + b\left(1 + \sum_i^N |v_{h,i}><v_{h,i}|\left(\frac{1}{\lambda_{h,i}} - 1\right)\right)$$

Coarse Eigenvector Preconditioning

Want to change from fine eigenvector $v_{h,i}$ preconditioner

$$M^{-1} = a + b(1 + \sum_{i}^{N} |v_{h,i} >< v_{h,i}|(\frac{1}{\lambda_{h,i}} - 1))$$

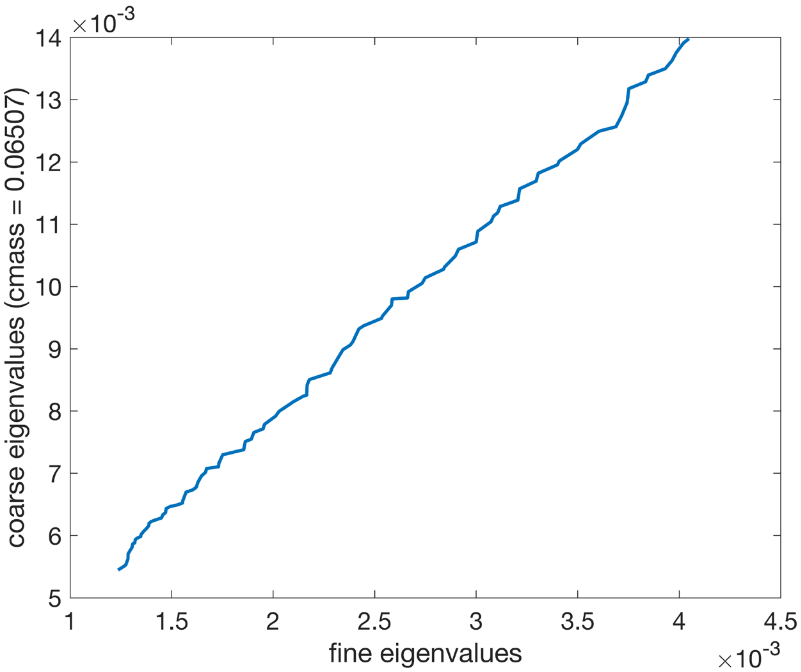
to one based on coarse eigenvectors $v_{2h,i}$ from blocked coarse lattice

$$M^{-1} = 1 + bPI(\sum_{i}^{N} |\psi_{2h,i} >< \psi_{2h,i}|\frac{1}{\lambda_{2h,i}})RP$$

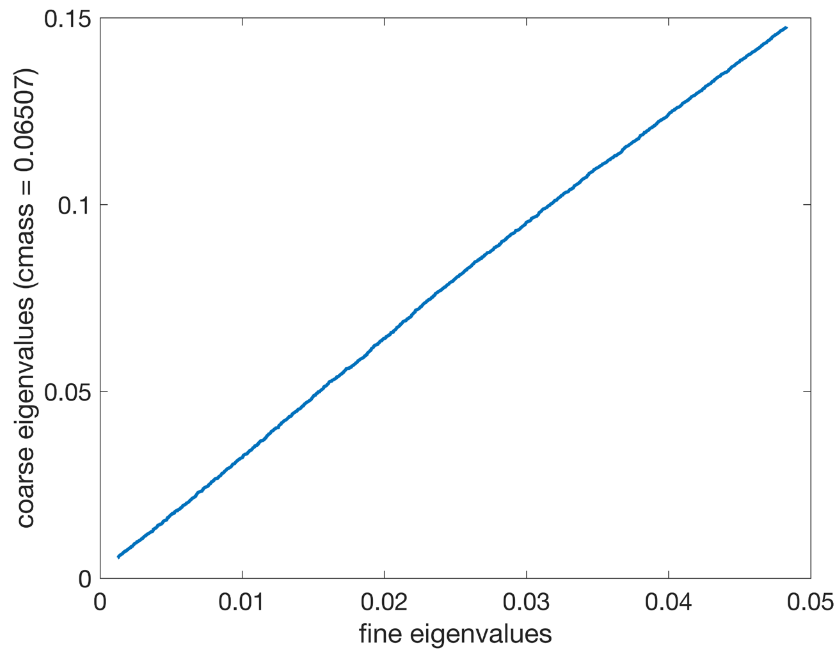
Work in Landau gauge

* R is a restriction operator, I is an interpolation or prolongation operator
* P is a smoother or filter

Comparing Eigenvalue Spectrum



First 100 eigenvalues

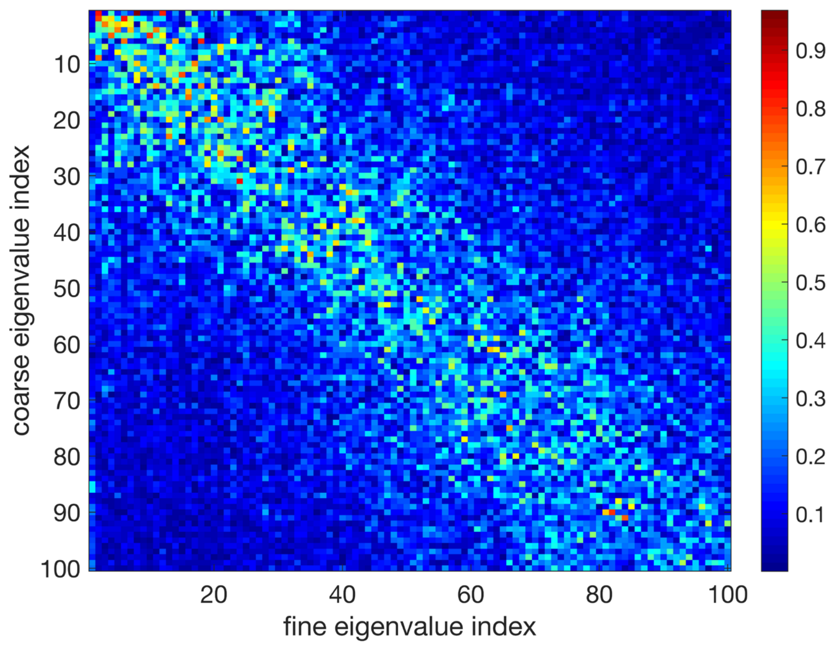


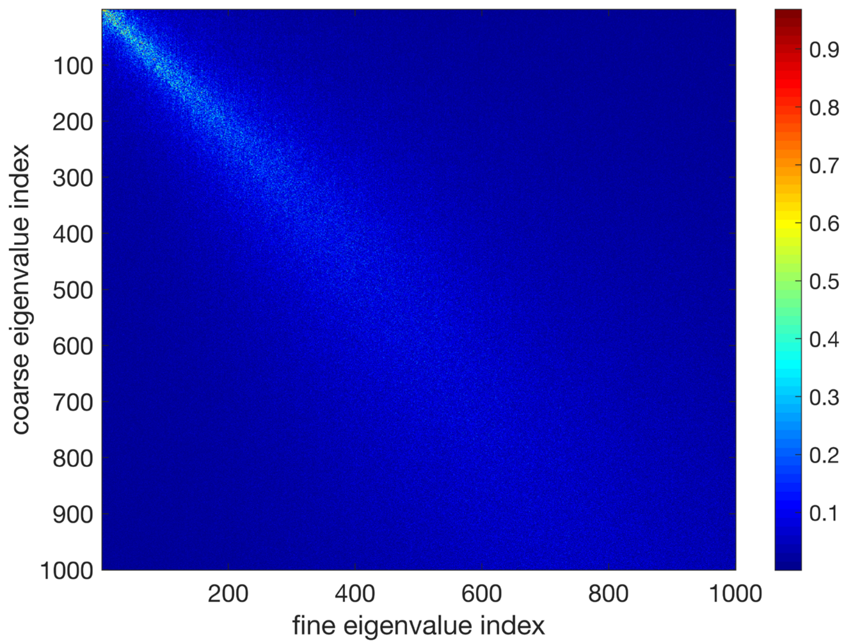
First 1000 eigenvalues

Eigenvalue densities are very similar and eigenvalues differ by 3×

Comparing Eigenvectors

Calculate magnitude of inner product of fine eigenvector $v_{h,i}$ with interpolated blocked coarse eigenector $Iv_{2h,j}$





Individual eigenvectors are not in one-to-one correspondence.

Compare Low Mode Subspace

Defining the coarse and fine lattice inverses as

$$S_c = I \sum_i^N |\psi_{2h,i}\rangle\langle\psi_{2h,i}| \frac{1}{\lambda_{2h,i}} R$$

$$S_f = \sum_i^N |\psi_{h,i}\rangle\langle\psi_{h,i}| \frac{1}{\lambda_{h,i}}$$

we calculate

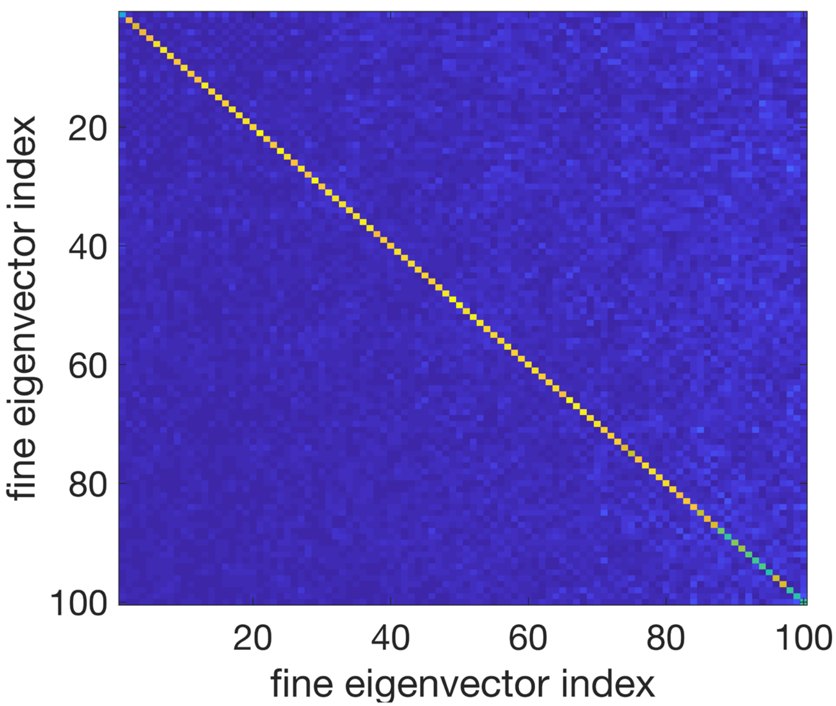
$$X = I \sum_i^N |\psi_{2h,i}\rangle\langle\psi_{2h,i}| \frac{1}{\lambda_{2h,i}} R \sum_j^N |\psi_{h,j}\rangle\langle\psi_{h,j}| \lambda_{h,j}$$

If the subspaces were indentical and complete, we would find X = the identity

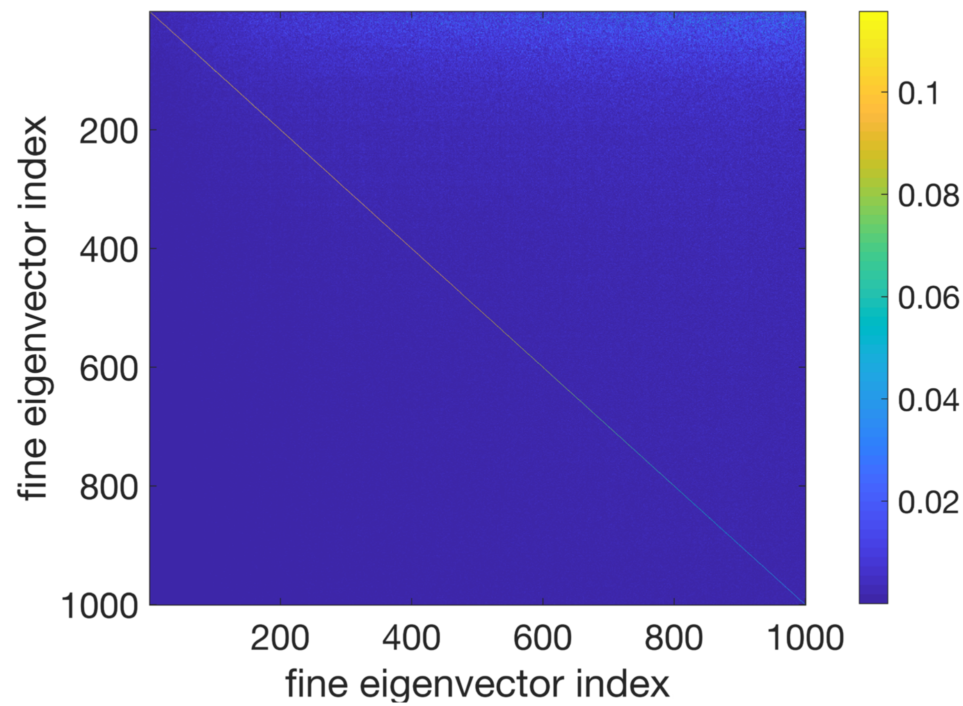
Comparison of Low Mode Subspaces

$$X = I \sum_{i}^{N} |\psi_{2h,i}\rangle\langle\psi_{2h,i}| \frac{1}{\lambda_{2h,i}} R \sum_{j}^{N} |\psi_{h,j}\rangle\langle\psi_{h,j}| \lambda_{h,j}$$

Plots of X show it is primarily diagonal



First 100 eigenvalues

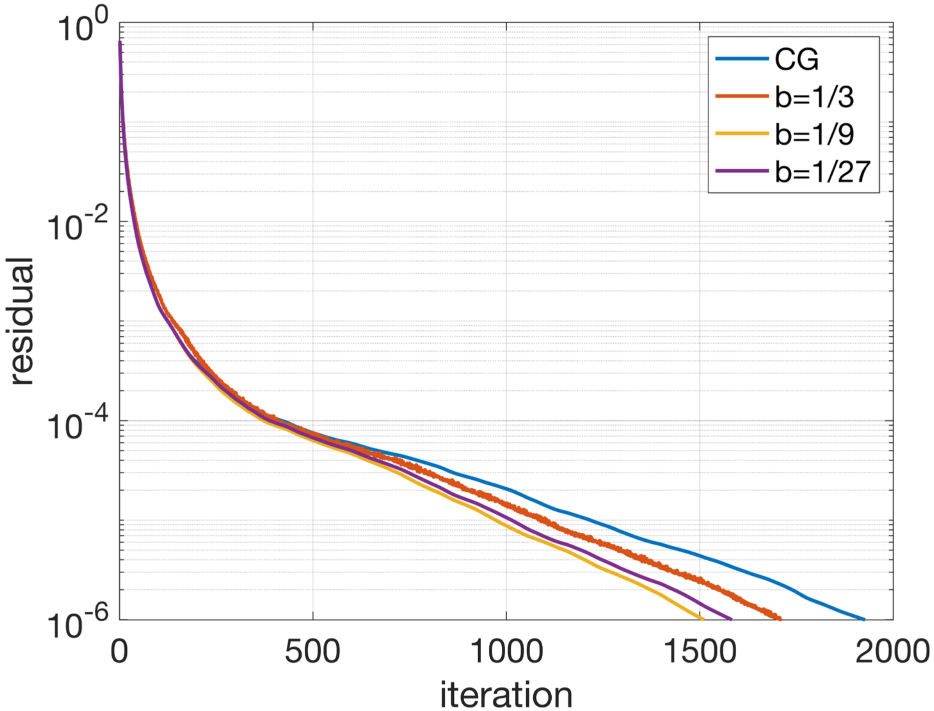


First 1000 eigenvalues

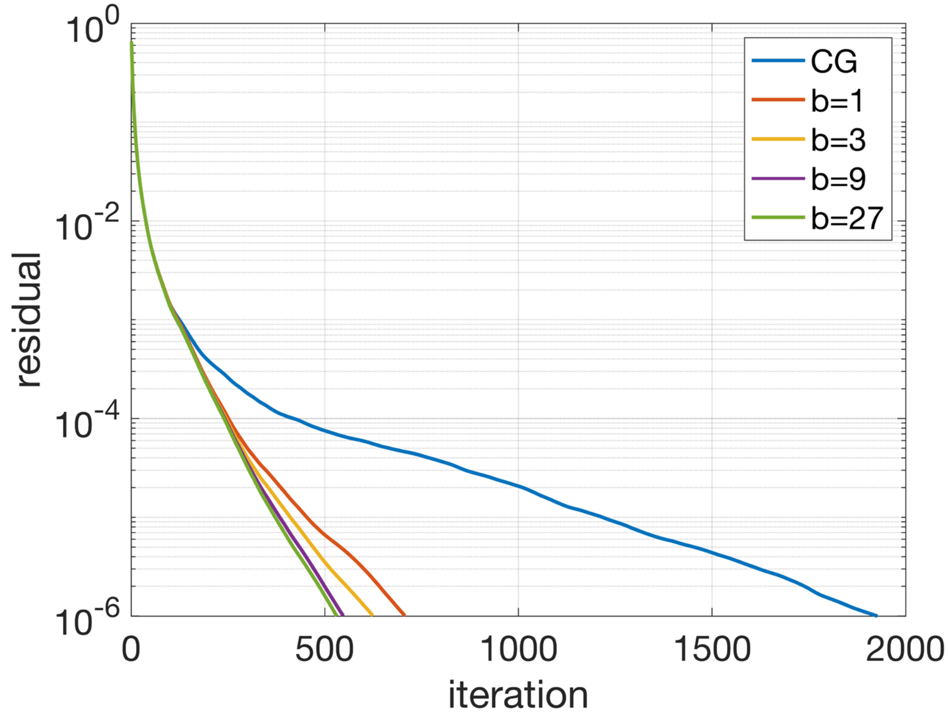
Individual eigenvectors are not in one-to-one correspondence, but the coarse low mode approximation to the inverse is quite similar to the high mode one.

Coarse Preconditioner

$$M^{-1} = 1 + bPI\left(\sum_{i}^{N} |\psi_{2h,i} >< \psi_{2h,i}| \frac{1}{\lambda_{2h,i}}\right)RP$$



1000 coarse eigenvectors
No filter, P = 1
Some improvement



1000 coarse eigenvectors
Use fine eigenvectors for filter
4× improvement

$$P = \sum_{i}^{N} |\psi_{h,i} >< \psi_{h,i}|$$

Exploring Various Filters/Smoothers

- Have tried Jacobi solver and Chebyshev polynomial filter as smoothers.
- Chebyshev polynomials work better in the smoother than Jacobi
- The total count of $D^\dagger D$ only modestly below the unpreconditioned case

Use these 5 steps as the preconditioner, setting u = 0 in step 1.

Jacobi and Chebyshev used for P

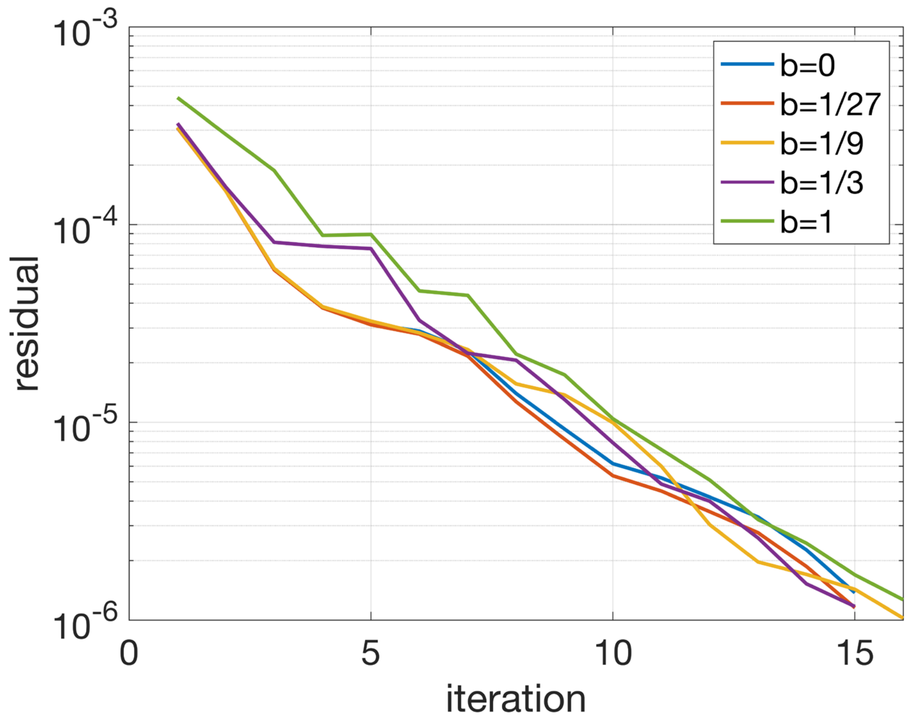
1) $u = P(r, u)$ (pre-smoothing)

2) $d = R(A_h u - r)$

3) $v = A_{2h}^{-1} d$

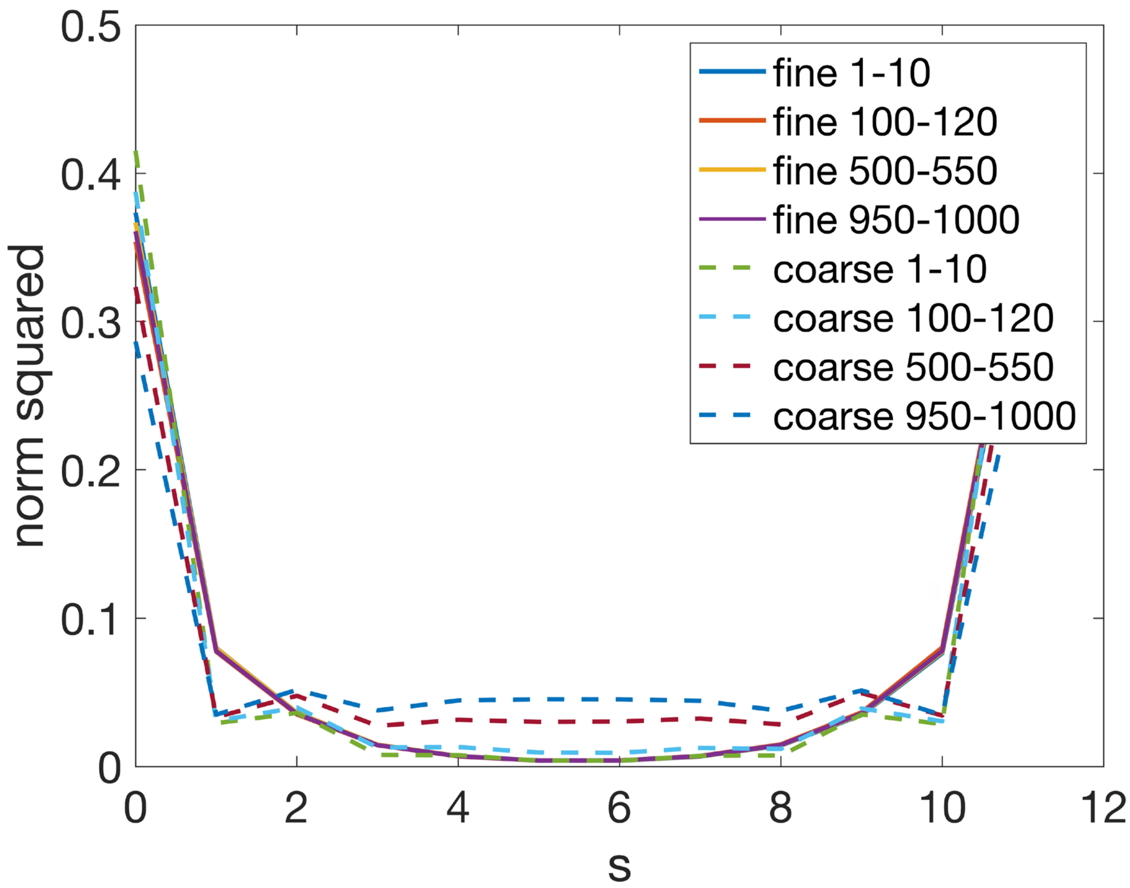
4) $u = u - bIv$

5) $u = P(r, u)$ (post-smoothing)



100 $D^\dagger D$ applications used in inner solver so total $D^\dagger D$ count is about 1500, similar to the 2000 iterations for CG.

Coarse and fine eigenmodes differ in 5th dimension



Summary

- APE-smearing style blocking produces a coarse lattice with 2× the lattice spacing
- Low mode subspace on coarse lattice can be prolongated to a good approximation to low mode subspace on fine latice.
- Using low mode subspace as a preconditioner for CG increases converence rate - better results possible for m_{ud} than for m_s, as used here.
- Have tried various high-mode filters/smoothers without much improvement.
- 5d structure of fine and coarse modes differs - perhaps a direction for improvement