Hitting two birds with four stones: A numerical and theoretical study of multilevel performance for two-point correlator calculations

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Who am I?

- ▷ Supervised by **Andreas Jüttner** and **Kostas Skenderis**
- ▷ Part of a broader collaboration **LatCos** looking into **Holographic Cosmology** on a lattice

- Guido Cossu (Edinburgh, now Braid Technologies)
- Luigi Del Debbio (Edinburgh)
- Elizabeth Dobson (Edinburgh, now University of Graz)
- Elizabeth Gould (Southampton, now in Queen's University, Canada)
- Ben Kitching-Morley (Southampton)
- Andreas Jüttner (CERN, Southampton)
- Joseph K.L. Lee (Edinburgh)
- Valentin Nourry (Edinburgh and Southampton, now Université de Paris)
- Antonin Portelli (Edinburgh)
- Henrique Bergallo Rocha (Edinburgh)
- Kostas Skenderis (Southampton)

LatCos Research and talks

- LatCos research motivated by holographic models of the early universe (McFadden and Skenderis, 2010)
- Recent works include (Cossu et al., 2021) and (Del Debbio et al., 2021)
- Other LatCos talks:
  - 28/7 Wed 14:00 (Andreas Jüttner)
  - 28/7 Wed 14:15 (Henrique Bergallo Rocha)
  - 28/7 Wed 14:30 (Joseph Lee)
  - 29/7 Thur 13:15 (Luigi Del Debbio)

Background

- ▷ Multilevel was investigated to improve performance of our simulations
- ▷ Multilevel is extremely useful in lattice gauge theory (∼200 papers)
- ▷ There is active research in extending multilevel
  - In QCD using approximate locality of the action (Cè, Giusti, and Schaefer, 2017)
  - For reducing autocorrelation times at criticality (Jansen, Müller, and Scheichl, 2020)
  - In theories where symmetries can be used to better leverage multilevel (Della Morte and Giusti, 2011)
  - In $g_\mu - 2$ (Dalla Brida et al., 2021)

This talk in one slide

1. The 2D-Ising model has been used as a toy model
2. Multilevel is a technique involving a decomposition of the path integral (Parisi, Petronzio, and Rapuano, 1983, Lüscher and Weisz, 2001)
3. Multilevel can reduce statistical error of two-point functions
4. Performance is better when the correlation length is small compared to the lattice size
5. We predicted this performance theoretically in a model independent way

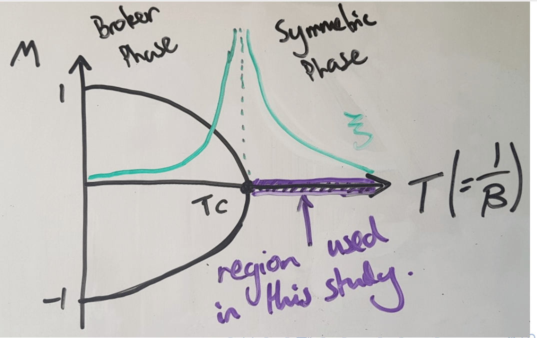
Exploring multilevel: The Ising Model

- Use Ising model → simple, but captures much of the physics
- The fields in this model can be "up" or "down", e.g. $\phi_i \in \{-1, 1\}$

Ising model partition function

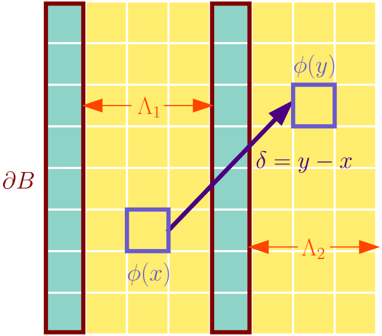
$$Z = \int \mathcal{D}\phi exp\left(-\beta \int d^{dim}xJ \sum_{(i,j)n.n.} \phi_i\phi_j + B\sum_i \phi_i\right)$$

- Taken $J = 1$, $B = 0$ and $dim = 2$ in this project.
- Critical point known exactly $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$ (Onsager, 1944)



What is Multilevel?

First proposed in (Parisi, Petronzio, and Rapuano, 1983) and extended in (Lüscher and Weisz, 2001) as a solution to the **Signal-to-noise problem**



boundary layer(s) ∂B and **sub-regions** separated by the boundary(s) Λ_r.

We use N **boundary configurations** labelled by $i \in [1, 2, ..., N]$ and M **sub-lattice configurations** labelled by $j_r \in [1, 2, ..., M]$ for region Λ_r per boundary configuration.

Factorization of the Path Integral

$$\int_{x\in\Lambda} \mathcal{D}\phi(x) e^{-S[\Lambda;\partial B]} = \int_{x\in\partial B} \mathcal{D}\phi(x) e^{-S[\partial B]} \prod_r \int_{x\in\Lambda_r} \mathcal{D}\phi(x) e^{-S[\Lambda_r;\partial B]}$$

Building multilevel correlators

Use **slice-coordinates** from now on.

$$\Phi_i(x) = \frac{1}{M} \sum_{j_r=1}^{M} \phi_{ij_r}(x)$$

$\phi_{ij_r(x)}$ is distributed with some distribution $D_{x,i}$, with a mean $\mu_{\phi(x),i}$ and a standard deviation $\sigma_{\phi(x),i}$ which **depend on the boundary** ∂B_i

Two-point estimator $C_{\delta,x}$ at site x

$$c_2(\delta; x)_i = \Phi_i(x)\Phi_i(y); \qquad C_{\delta,x} = \frac{1}{N} \sum_i c_2(\delta; x)_i$$

where $x \in \Lambda_1$, $y \in \Lambda_2$, $\delta = y - x$

As $N \to \infty$, $\frac{1}{N} \sum_i \mu_{\phi(x),i} \to \bar{\phi}$ and $C_{\delta,x} \to C_{2,phys}$ (Multilevel is **unbiased**)

Correlator Errors

Standard deviation of estimator depends on proximity of x to boundary.

Stanard Deviation of estimator $\sigma^2_{c_{\delta,x}}$ at x

$$\sigma^2_{c_{\delta,x}} = \frac{1}{N} Var(\mu_{\phi(x)\phi(y)}) + \frac{1}{NM} \left(E(\mu^2_{\phi(x)}\sigma^2_{\phi(y)} + \mu^2_{\phi(y)}\sigma^2_{\phi(x)}) \right) + \frac{1}{NM^2} E(\sigma^2_{\phi(x)}\sigma^2_{\phi(y)}).$$

where Var and E at boundary level, μ and σ at the sub-lattice level.

See also (García Vera, 2017).

Weighted average of two-point correlators

$$C_2(\delta) = \sum_x W(x) c_{\delta,x} \sim N(C_{2,real}, \mathbf{W} \cdot Cov \cdot \mathbf{W})$$

where $\sum_x W(x) = 1$

Best and Worst Case Scenarios

We compare to a **computationally equivalent** single-level algorithm, with $N \times M$ configurations.

Single Level Scaling

$$\sigma_s \propto \frac{1}{\sqrt{NM}}$$

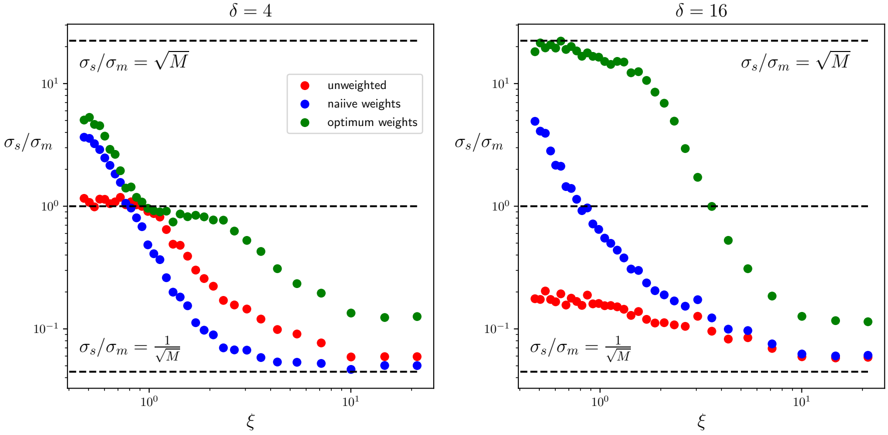
Performance of the multilevel algorithm depends on the correlation of the field insertions to the boundary.

Multilevel Scaling

$$\sigma_m \propto \frac{1}{\sqrt{NM^2}} \approx \frac{\sigma_s}{\sqrt{M}} \qquad (\textit{Best Case})$$

$$\sigma_m \propto \frac{1}{\sqrt{N}} \approx \sqrt{M}\sigma_s \qquad (\textit{Worst Case})$$

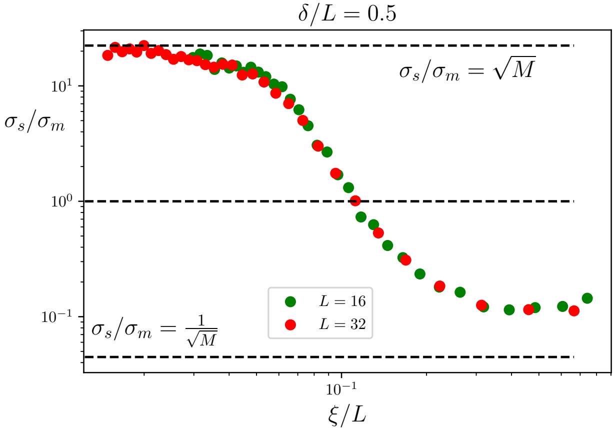
Numerical Results $L = 32$, $N = 500$, $M = 500$



- Multilevel is more effective when the correlation length, ξ, is shorter.
- Multilevel is more effective for the longest two-point separations, δ.
- Signal-to-noise is worst for large δ so this works well

Scaling

Hypothesis: Performance of the algorithm is invarient under a rescaling of the system ($L \to \alpha L$, $\Delta\delta \to \alpha\Delta\delta$, $\xi \to \alpha\xi$)



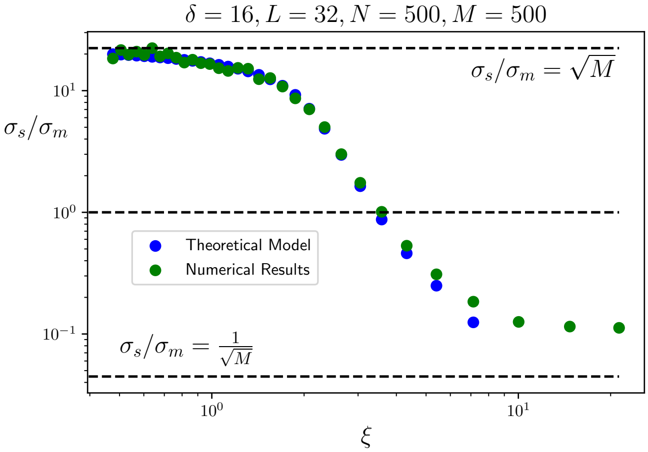
A Theoretical Model

Can we make a prediction of these curves without even running a simulation?

The Model

- Field insertions are random variables correlated to each other
- Correlation between field insertions given by two-point function of slice-coordinates - e.g. $\langle \phi(x)\phi(x+\delta)\rangle = Ae^{\frac{-\delta}{\xi}}$.
- Field insertions in different sub-regions correlated indirectly through the boundary
- This is a leading order approximation
- Implemented in Python

Model Results



- Excellent agreement between model and observed data including at the crossover point
- Large ξ outside the range of validity of the model

Conclusions and Outlook

- Multilevel performance has been explored for a model system and predicted in a model independent way
- The algorithm performs excellently for long-distance correlators and when the correlation length is small
- This performance was predicted well by modelling fields as statistically correlated random variables
- This project could be extended by investigation of more complicated spectra and by comparison to multilevel studies in QCD, and other theories where multilevel is being applied.

Thank you for listening!