A new technique for solving the freezing problem in the complex Langevin simulation of 4D SU(2) gauge theory with a theta term

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Gauge theory with a θ term

☆ θ term: topological property of the gauge theory, nonperturbative

$$S_\theta = -i\theta Q = -\frac{i\theta}{32\pi^2}\int d^4x \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\left[F_{\mu\nu}F_{\rho\sigma}\right]$$

- strong CP problem of QCD
  The experimental bound of θ is extremely small: $|\theta| < 10^{-10}$

- nontrivial phase structure of 4D SU(2) YM around $\theta = \pi$
  predicted by the 't Hooft anomaly matching

Numerical study of the θ term

Monte Carlo simulation of the lattice gauge theory for $\theta \neq 0$

- θ term is purely imaginary → the action is complex

  → sign problem

- various approaches…
  - Complex Langevin method [L. Bongiovanni, G. Aarts, E. Seiler, D. Sexty (2014)]
  - Density of state method [C. Gattringer, O. Orasch (2020)] (talk at LATTICE 2021)
  - Tensor renormalization group [Y. Kuramashi, Y. Yoshimura (2020)]
  - Lefschetz thimble method
  - etc…

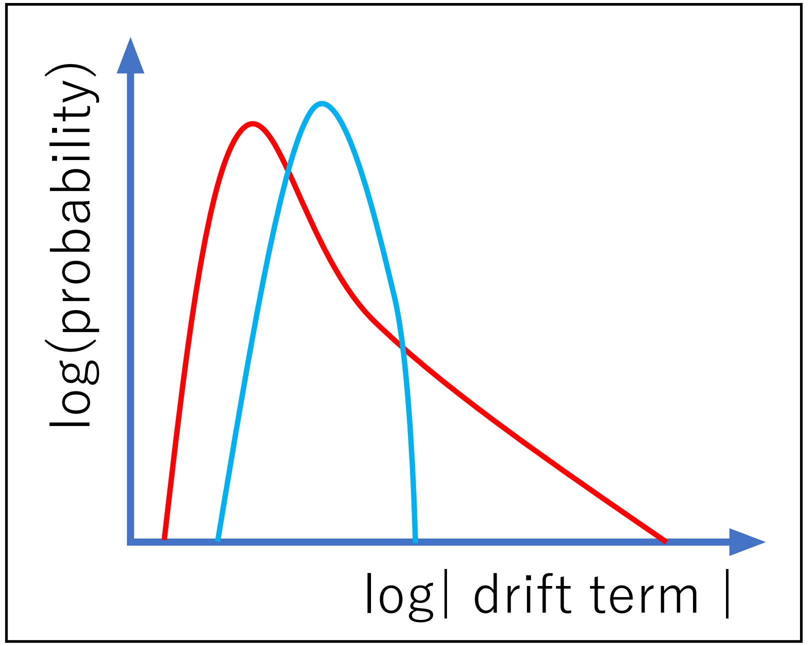
Complex Langevin method

complex Langevin method (CLM) [G. Parisi (1983)] [J. R. Klauder (1983)]

- Langevin equation: fictitious time evolution of dynamical variables
- real variable → complex variable

$$\frac{dz(t)}{dt} = \underbrace{-\frac{\partial S(t)}{\partial z}}_{\text{drift term}} + \underbrace{\eta(t)}_{\text{Gaussian noise}} \qquad x \longmapsto z = x + iy$$

- do not use "probability" → sign ~~problem~~
- condition required to be satisfied



The distribution of the drift term should falls off exponentially or faster.

[K. Nagata, J. Nishimura, S. Shimasaki (2016)]

CLM for the lattice gauge theory

- discretized complex Langevin equation for the link variables

$$U_{n,\mu}(t+\epsilon) = \exp\left[-i\epsilon \underbrace{D_{n,\mu} S(t)}_{\text{drift term}} + i\sqrt{\epsilon}\eta_{n,\mu}(t)\right] U_{n,\mu}(t)$$

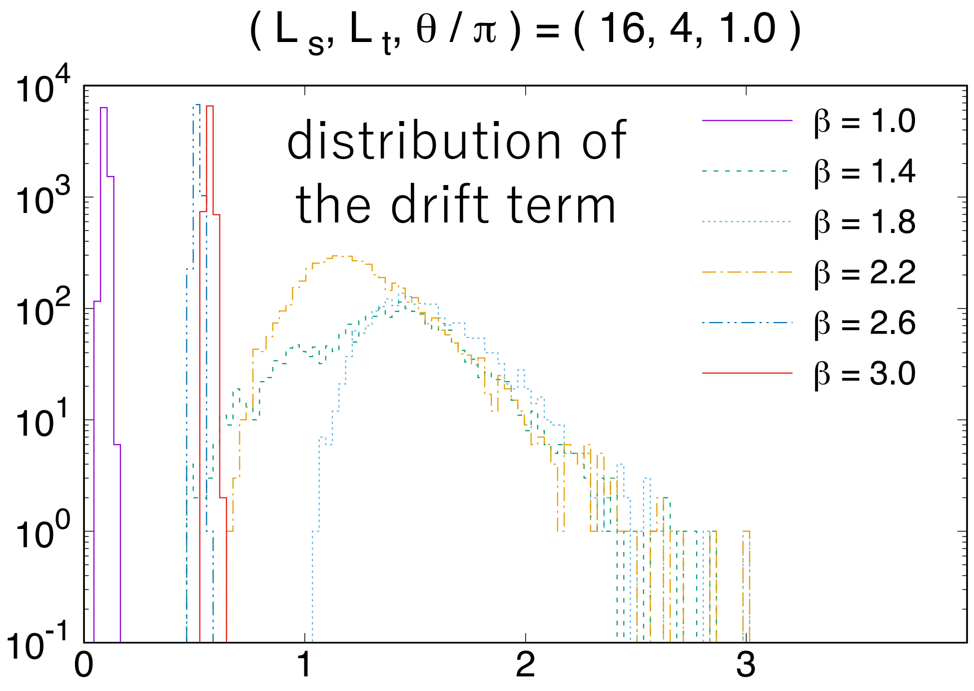
$$U_{n,\mu} \in \mathrm{SL}(2,\mathbb{C})$$

- gauge group is extended: $\mathrm{SU}(2) \to \mathrm{SL}(2,\mathbb{C})$ $\boxed{U_{n,\mu}^{\dagger} \to U_{n,\mu}^{-1}}$
- drift term and observables have to respect holomorphicity
- gauge cooling
  - gauge transformation to keep the link variable close to unitary
  - doesn't affect gauge invariant observables

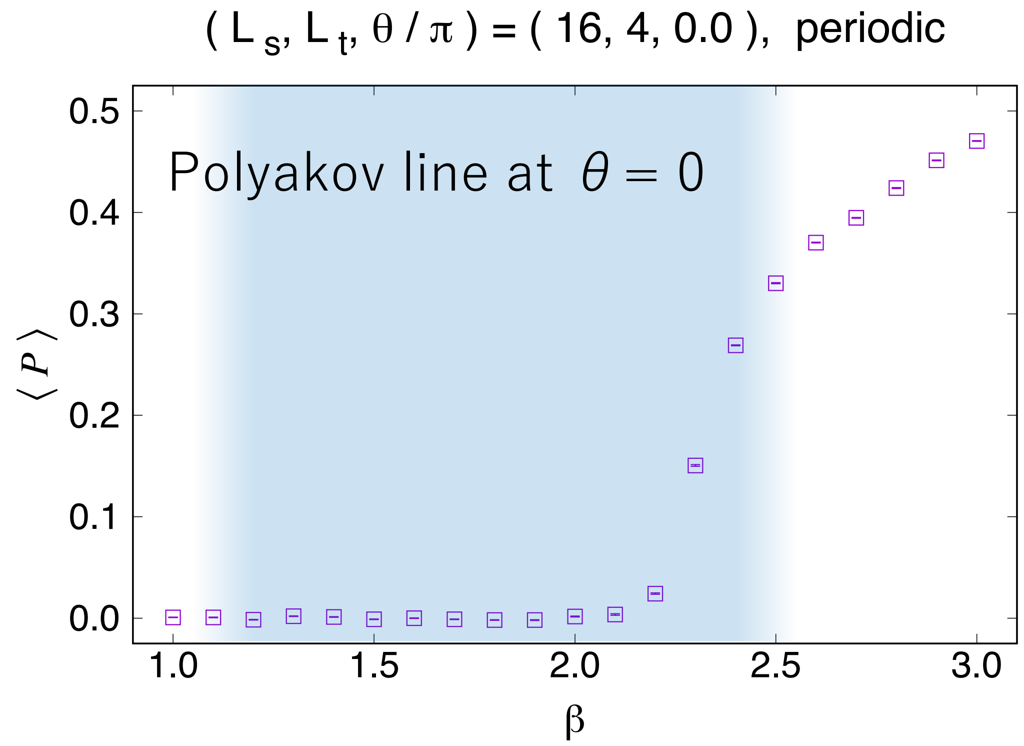
[E. Seiler, D. Sexty, I.-O. Stamatescu (2013)] [K. Nagata, J. Nishimura, S. Shimasaki (2016)]

Convergence of CLM

- The condition for the correct convergence is satisfied around $\beta < 1.1$ and $\beta > 2.4$.



$$u = \frac{1}{\sqrt{2}} \max_{n,\mu} \left\| \left(D^a_{n,\mu} S \right) t^a \right\|$$



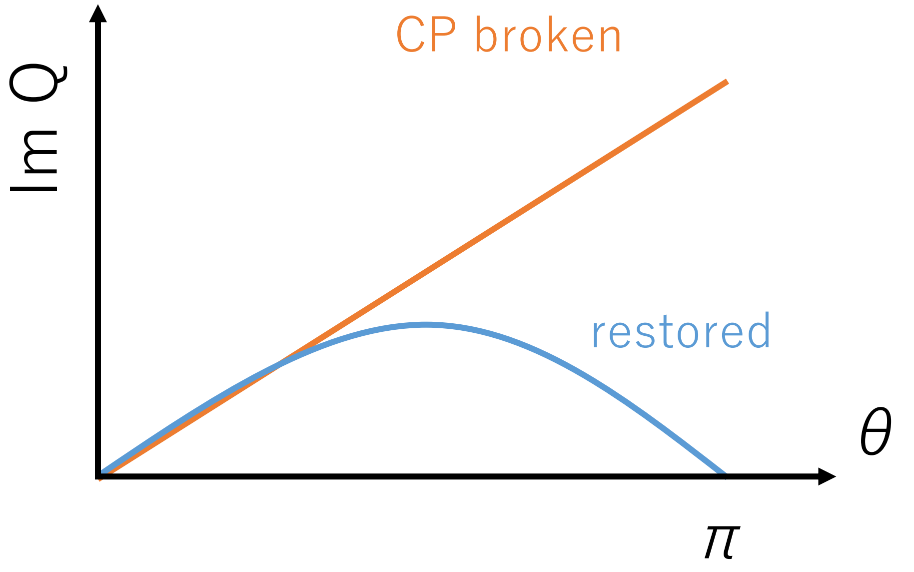
CP symmetry at $\theta = \pi$

- In the high temperatures region,
  CP is expected to be restored at $\theta = \pi$.
- The topological charge is CP odd.
  $\rightarrow$ $\langle Q \rangle = 0$ if CP is restored
- dilute instanton gas approximation

$$\mathrm{Im}\langle Q\rangle \simeq \chi \sin\theta$$

$$\chi = \frac{1}{V}\left(\langle Q^2\rangle - \langle Q\rangle^2\right) = -\frac{1}{V}\frac{\partial^2}{\partial\theta^2}\log Z$$

$$\langle Q\rangle = -i\frac{1}{Z}\frac{\partial Z}{\partial\theta}$$



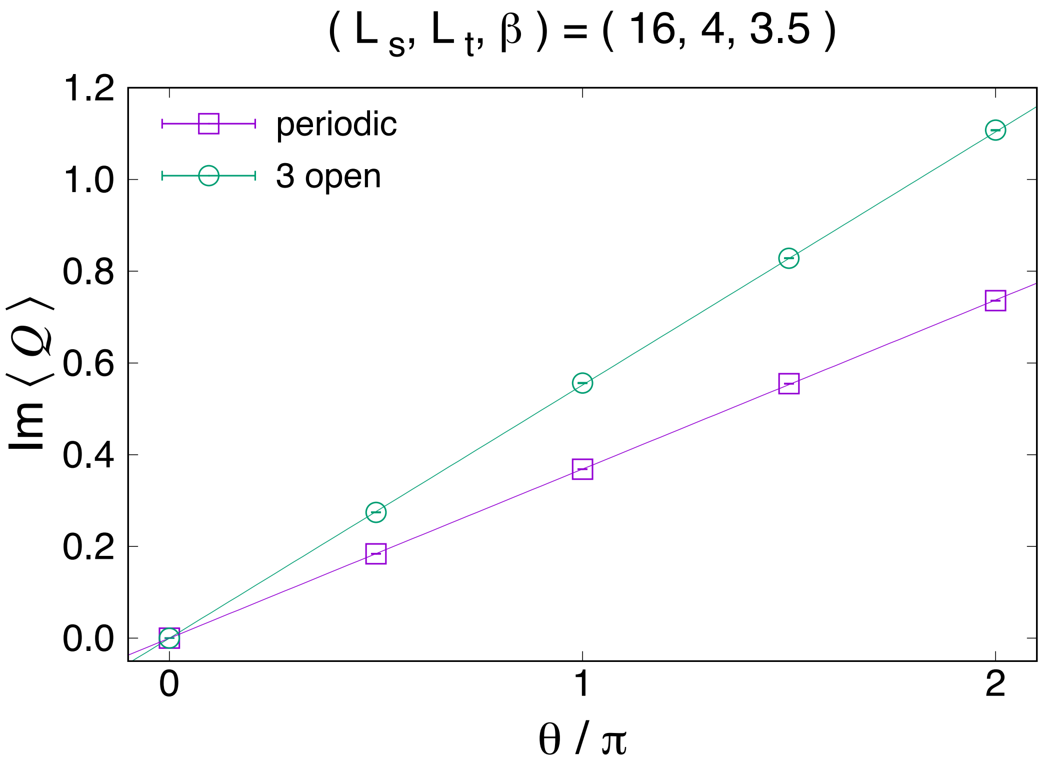
Expectation value of Q

We found $\mathrm{Im}\,Q \propto \theta$ for large β with periodic b.c. and open b.c.

- 2π periodicity is absent in both cases.
- open b.c. : Q is not an integer
- periodic b.c. : topology freezing

  → large β is difficult

Topology freezing does not occur for relatively small β ...



Stout smearing

- The topological charge is contaminated by UV fluctuation on the coarse lattice.

→ Recover the topological property by smoothing the gauge field

☆ stout smearing [C. Morningstar, M. Peardon (2004)]

$$U_{n,\mu}^{(k+1)} = e^{iY_{n,\mu}} U_{n,\mu}^{(k)} \qquad iY_{n,\mu} = -\frac{1}{2}\left(J_{n,\mu} - \frac{1}{N}\mathrm{Tr}\left[J_{n,\mu}\right]\right)$$

$$J_{n,\mu} = \sum_{\nu(\neq\mu)} \rho_{\mu\nu}\left[U_{n,\mu}\left(\quad + \quad \right) - \left(\quad + \quad \right) U_{n,\mu}^{-1}\right]$$

$\rho_{\mu\nu}$: step size for smearing

Stout smearing for CLM

- The topological charge is calculated from smeared configuration.

  $$S = S_g[U] - i\theta Q[U^{(n_\rho)}]$$ n_ρ : # of steps in smearing

- We use stout smearing also for calculation of the drift term so that the dynamics can reflect the topological property.

  $$-i\theta D^a_{n,\mu} Q[U^{(n_\rho)}(U)]$$

- The link variable remains in SL(N,C).
- The drift term is holomorphic.

  ※ Implement stout smearing to CLM for finite density QCD
  [D. Sexty (2019)]

Effect of smearing

- distribution of Q

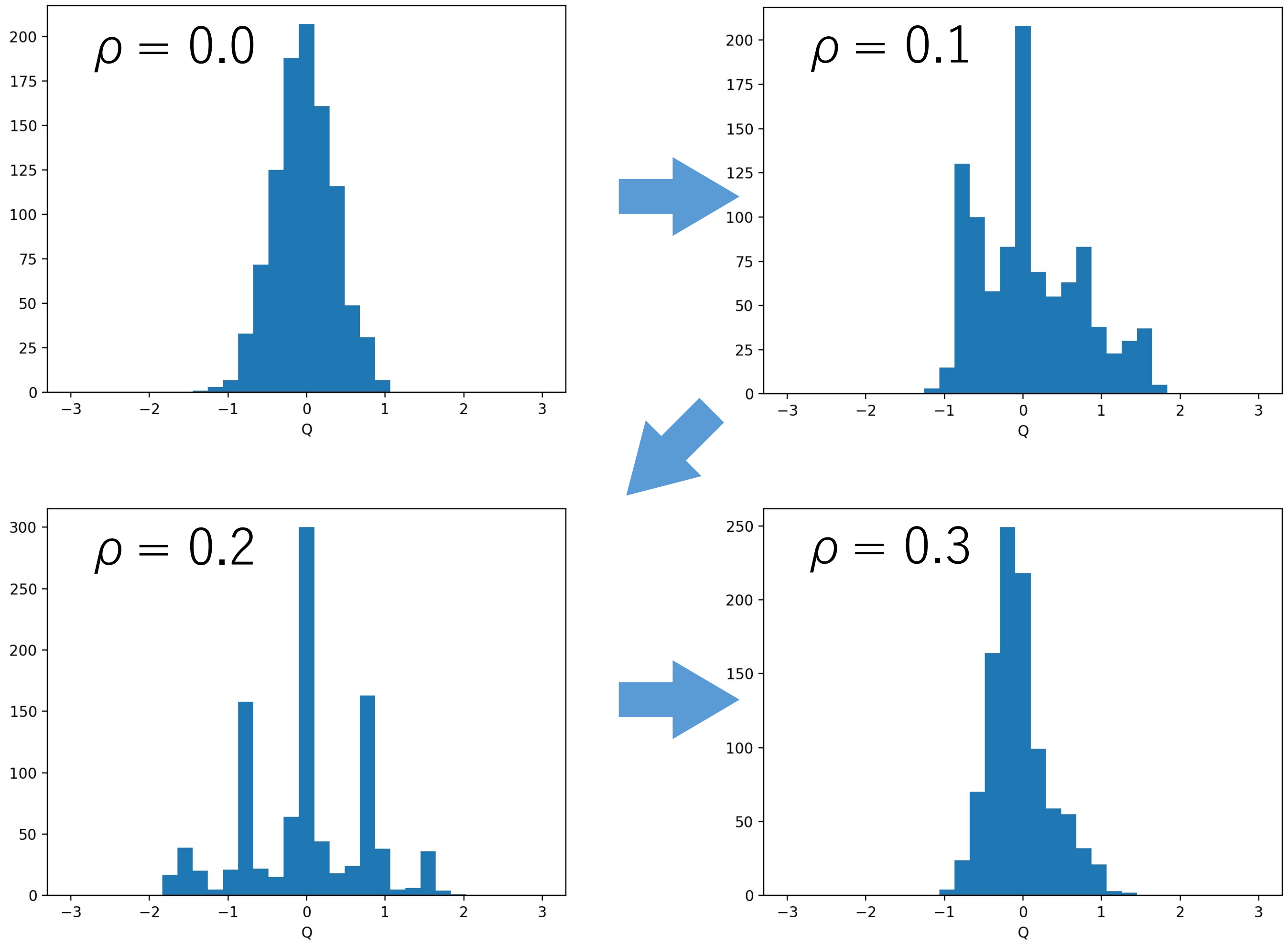
  $L = 8,\ \beta = 2.4,\ \theta = 0.0$

  # of steps : $n_\rho = 20$

  step size : $\rho_{\mu\nu} = \rho$

- Q approaches an integer for specific combinations of (ρ, n_ρ).

$\rightarrow$ sensitive to the parameters



Behavior of Q

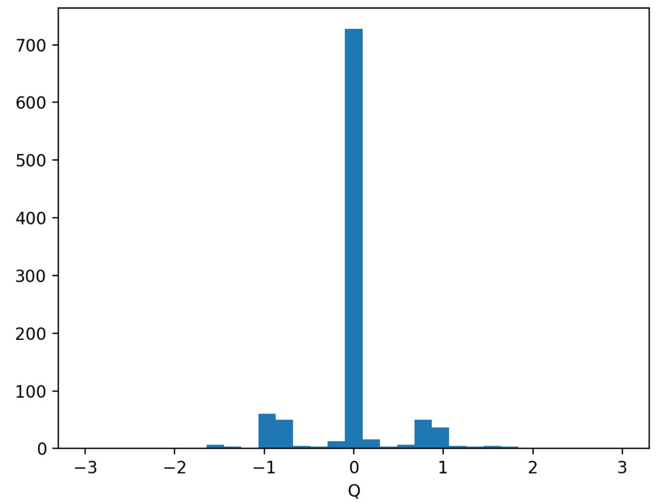
Langevin time evolution of
the topological charge for $\theta \neq 0$

- ImQ ~ 0 unless ReQ changes
- Contribution of Q ≠ 0 sectors
  → nontrivial theta dependence
- Jump of ReQ → singular drift

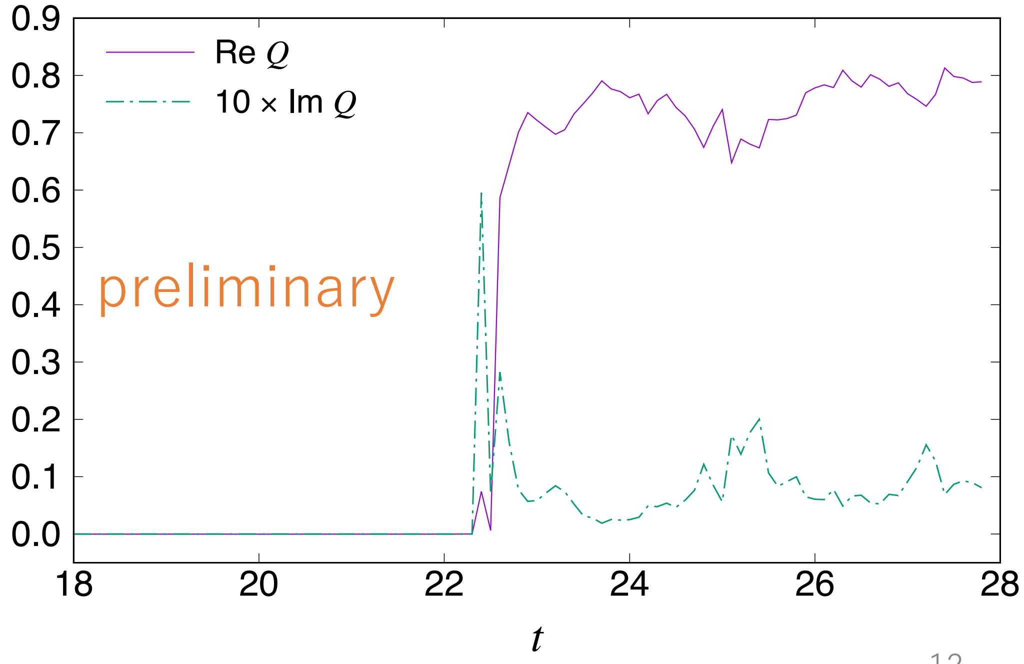
$$D^a_{n,\mu} Q[U^{(n_\rho)}(U)] \gg 1$$

- Further tuning of (ρ , n_ρ)
  is necessary.

$\rho = 0.1$
$n_\rho = 60$



(L, β, θ) = (8, 2.4, π / 4)



Summary

- We use the complex Langevin method to simulate 4D SU(2) gauge theory with a θ term, avoiding the sign problem.
- CLM works for $\beta < 1.1$ and $\beta > 2.4$, but 2π periodicity of θ cannot be observed.
- We implement stout smearing to calculation of Q and the drift term of CLM. → Q approaches an integer.
- We need to tune the parameters of smearing to stabilize the behavior of the drift term.

Thank you!