Calculation of the Second Moment of the Pion Light-Cone Distribution Amplitude

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Outline

Light-Cone Distribution Amplitude

- LCDA $\varphi_\pi(\xi)$ defined via

$$\langle 0|\bar{d}(-z)\gamma_\mu\gamma_5\mathcal{W}[-z,z]u(z)|\pi^+(p)\rangle = ip_\mu f_\pi \int_{-1}^{1} d\xi\, e^{-i\xi p\cdot z}\varphi_\pi(\xi)$$

  - z represents light-like separation – not amenable to direct lattice calculation
- Represents amplitude for π transitioning into $q\bar{q}$ pair with momenta $(1+\xi)p/2$, $(1-\xi)p/2$
- QCD factorization theorems – many physical processes (EM form factor, $\gamma\gamma^* \to \pi^0$, etc.) depend on φ_π (times perturbative parts)

Lattice Determination of LCDA

- Our approach: expand LCDA into Mellin moments

$$\langle \xi^n \rangle = \int_{-1}^{1} d\xi \, \xi^n \varphi_\pi(\xi)$$

  - This talk: Computation of $\langle \xi^2 \rangle$
  - Next talk (Robert Perry): Exploratory computation of $\langle \xi^4 \rangle$
- Previous lattice calculations
  - Local matrix elements (give $\langle \xi^2 \rangle$, but higher moments suffer from power divergences)
  - Light-quark operator product expansion
  - Quasi-PDF and pseudo-PDF (determine $\varphi_\pi(\xi)$ without recourse to moments)
- Challenging but important problem – want multiple independent approaches

Heavy-Quark Operator Product Expansion (HOPE)

- Form hadronic tensor from flavor-changing axial currents:

$$U^{\mu\nu}(q,p) = \int d^4x\, e^{iq\cdot x} \langle 0|\mathcal{T}\left[A^\mu(x/2)A^\nu(-x/2)\right]|\pi^+(p)\rangle$$

$$A^\mu = \bar{\Psi}\gamma^\mu\gamma_5\psi + \bar{\psi}\gamma^\mu\gamma_5\Psi$$

where ψ is a light quark and Ψ is a heavy quark

- Hadronic tensor can be expanded in terms of moments

$$U^{\mu\nu}(p,q) = \frac{2if_\pi\varepsilon^{\mu\nu\rho\sigma}q_\rho p_\sigma}{\tilde{Q}^2} \sum_{\substack{n=0\\ \text{even}}}^{\infty} \frac{\tilde{\omega}^n}{2^n(n+1)} C_W^{(n)}(\tilde{Q}, m_\Psi, \mu)\langle\xi^n\rangle(\mu) + O\left(\frac{\Lambda_{\text{QCD}}}{\tilde{Q}}\right)$$

with $\tilde{\omega} = 2p\cdot q/\tilde{Q}^2$ and $\tilde{Q}^2 = -q^2 - m_\Psi^2$

- Heavy quark mass m_Ψ suppresses higher-twist effects

Hadronic Tensor

$$U^{\mu\nu}(q,p) = \int d^4x\, e^{iq\cdot x} \left\langle 0 \middle| \mathcal{T}\left[A^\mu\left(\frac{x}{2}\right) A^\nu\left(-\frac{x}{2}\right)\right] \middle| \pi^+(p) \right\rangle$$

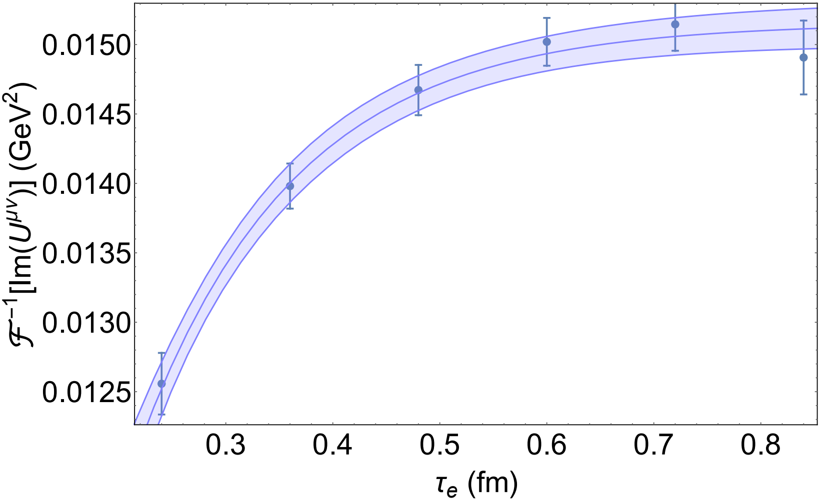
$$\int dq_4 e^{-iq_4\tau}\, U^{\mu\nu}(q,p) = \int d^3\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}} \left\langle 0 \middle| \mathcal{T}\left[A^\mu\left(\frac{\mathbf{x}}{2},\frac{\tau}{2}\right) A^\nu\left(-\frac{\mathbf{x}}{2},-\frac{\tau}{2}\right)\right] \middle| \pi^+(\mathbf{p}) \right\rangle$$

- Inverse FT of $U^{\mu\nu}$ calculable on lattice in terms of 2-point and 3-point functions

$$C_2(\tau) = \langle \mathcal{O}_\pi(\tau)\mathcal{O}_\pi^\dagger(0)\rangle$$
$$C_3(\tau_e,\tau_m) = \langle A^\mu(\tau_e)A^\nu(\tau_m)\mathcal{O}_\pi^\dagger(0)\rangle$$

- Isolation of ground state relies on sufficiently large separation between 0 and $\min\{\tau_e,\tau_m\}$

Excited States



- $\tau_m - \tau_e$ fixed at 0.06 fm
- Excited state contamination becomes $\sim 1\%$ by $\tau_e = 0.7$ fm

Heavy Quark Masses

- Hadronic tensor equals twist-2 OPE up to $O(\Lambda_{\mathrm{QCD}}/\tilde{Q}) = O(\Lambda_{\mathrm{QCD}}/m_\Psi)$ corrections
- Want $m_\Psi \gg \Lambda_{\mathrm{QCD}}$ but $am_\Psi < 1$
- Choose five m_Ψ values between 1.8 and 4.5 GeV in order to extrapolate to $m_\Psi \to \infty$ limit
- Requires fine lattices (spacings down to 0.04 fm)

Ensembles Used

$L^3 \times T$	a (fm)	N_{cfg}	N_{src}	N_Ψ	N_{prop}
$24^3 \times 48$	0.0813	650	3	1	39,000
$32^3 \times 64$	0.0600	450	10	3	270,000
$40^3 \times 80$	0.0502	250	6	4	120,000
$48^3 \times 96$	0.0407	341	10	5	341,000

- Quenched approximation with $m_\pi = 550$ MeV
  - Fine dynamical ensembles prohibitively expensive
  - Total compute time: $O(10^5)$ KNL node-hours
- Wilson-clover fermions with non-perturbatively tuned c_{SW}
- With clover term, results fully $O(a)$ improved
  - Axial current renormalizes multiplicatively: $A^\mu \to A^\mu Z_A(1 + \tilde{b}_A a\tilde{m}_q)$
  - This only affects overall normalization (not $\langle \xi^2 \rangle$)

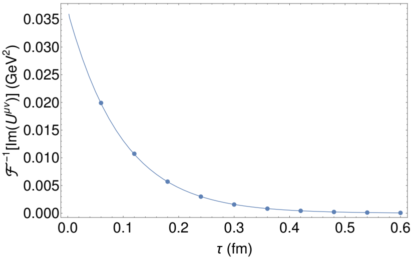
Choice of Kinematics

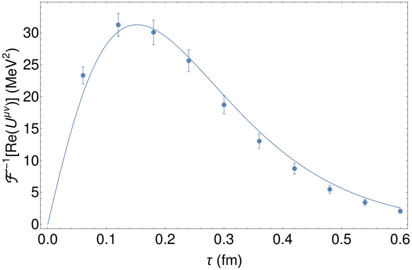
$$U^{\mu\nu}(p,q) = \frac{2if_\pi \varepsilon^{\mu\nu\rho\sigma} q_\rho p_\sigma}{\tilde{Q}^2} \sum_{\substack{n=0 \\ \text{even}}}^{\infty} \frac{\tilde{\omega}^n}{2^n(n+1)} C_W^{(n)}(\tilde{Q}, m_\Psi, \mu)\langle \xi^n \rangle(\mu) + O\left(\frac{\Lambda_{\mathrm{QCD}}}{\tilde{Q}}\right)$$

- Wilson coefficients $C_W^{(n)}(\mu = 2\text{ GeV})$ calculated to 1-loop (hep-lat/2103.09529)
- Fit parameters: $f_\pi, m_\Psi, \langle \xi^2 \rangle$
- Contribution of second moment $\langle \xi^2 \rangle$ suppressed by $\tilde{\omega}^2$
- At low momenta, $\tilde{\omega}/2 \lesssim 0.1$, so $\langle \xi^2 \rangle$ term is percent-level contribution
- For $\mu = 1$, $\nu = 2$, $p_3 = 0$, $\mathbf{p} \cdot \mathbf{q} \neq 0$
  - $\mathrm{Im}[U^{\mu\nu}]$ dominated by $\langle \xi^0 \rangle$ – fit f_π, m_Ψ
  - $\mathrm{Re}[U^{\mu\nu}]$ independent of $\langle \xi^0 \rangle$ (at tree level) – fit $\langle \xi^2 \rangle$
- Choose $\mathbf{p} = (1,0,0) = (0.64\text{ GeV}, 0, 0)$, $2\mathbf{q} = (1,0,2) = (0.64\text{ GeV}, 0, 1.28\text{ GeV})$

Fitting Hadronic Tensor

- Fit ratio of 2- and 3-point correlators to inverse FT of OPE

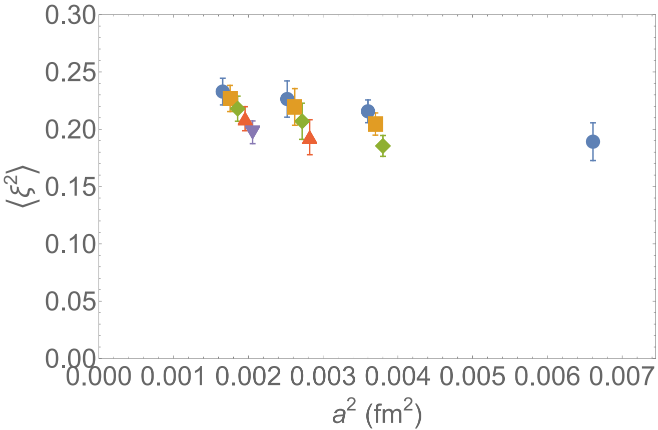




$$f_\pi = 141 \pm 2 \text{ MeV}$$
$$m_\Psi = 1.81 \pm 0.01 \text{ GeV}$$

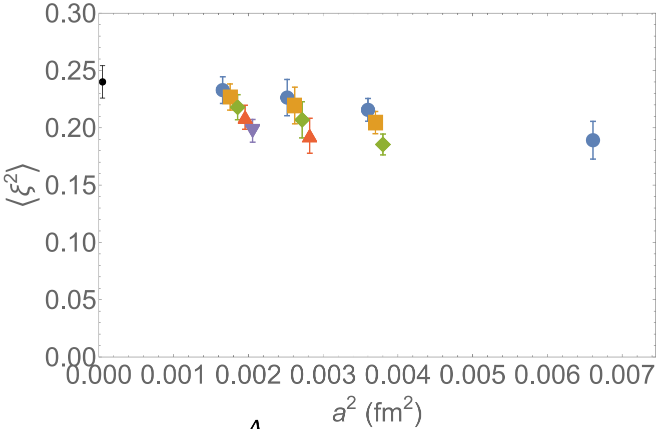
$$\langle \xi^2 \rangle = 0.216 \pm 0.010$$

Fits to Various Ensembles



Masses are (left to right) $\{1.8, 2.5, 3.3, 3.9, 4.5\}$ GeV

Continuum Extrapolation

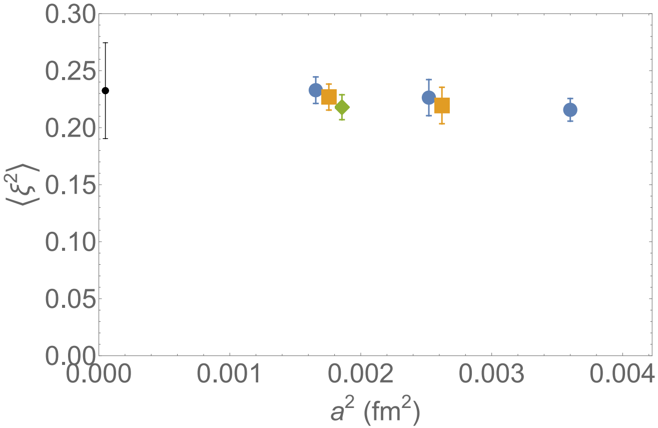


$$\text{data} = \langle \xi^2 \rangle + \frac{A}{m_\Psi} + Ba^2 + Ca^2 m_\Psi + Da^2 m_\Psi^2$$

Extrapolate away both discretization errors and twist-3 effects

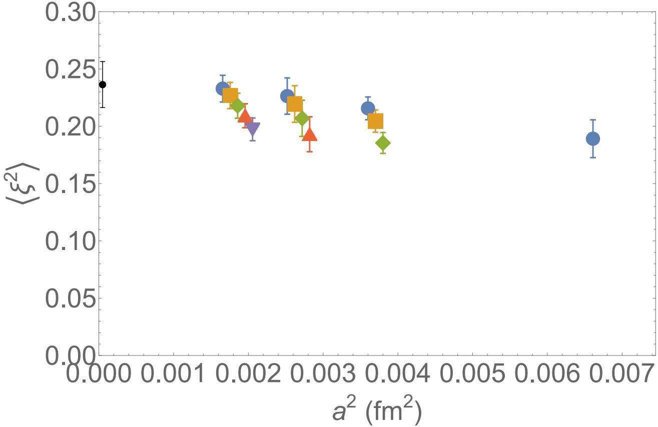
$$\langle \xi^2 \rangle = 0.240 \pm 0.014 \text{ (stat.)}$$

Uncertainty in Continuum Extrapolation



- Original fit restricted am_Ψ to < 1
- Could take a more conservative threshold, e.g. $am_\Psi < 0.7$
- Fit result: $\langle \xi^2 \rangle = 0.232 \pm 0.042$

Uncertainty in Higher-Twist Effects



- Could add twist-4 term to fit as well

$$\text{data} = \langle \xi^2 \rangle + \frac{A}{m_\Psi} + \frac{B}{m_\Psi^2} + Ca^2 + Da^2 m_\Psi + Ea^2 m_\Psi^2$$

- Fit result: $\langle \xi^2 \rangle = 0.236 \pm 0.020$

Uncertainty in Quenching

- Formally uncontrollable – cannot be reliably estimated
- One component of quenching error – change in α_s
- At $\mu = 2$ GeV, $\alpha_s(\text{quenched}) = 0.20$ but $\alpha_s(\text{dynamical}) = 0.29$
- Using dynamical α_s instead of quenched α_s gives handle on one piece of quenching error
- Fit result: $\langle \xi^2 \rangle = 0.219 \pm 0.013$

Remaining Uncertainties

- Excited state contamination: estimated at 1%
- Finite volume effects: $m_\pi L = 5.4 \Rightarrow \frac{1}{m_\pi L} e^{-m_\pi L} = 8 \times 10^{-4}$
- Unphysical pion mass ($m_\pi = 550$ MeV): Likely a $\sim 5\%$ error (V. M. Braun et al., hep-lat/1503.03656)
- Fit range: UV divergences at small τ from operator overlap/mixing
  - Excluding $\tau = 1$ as well gives discrepancy of ± 0.008
- Wilson coefficients: Performing fit at $\mu = 4$ GeV and running back to 2 GeV gives discrepancy of ± 0.002

Combined Uncertainty

$$\begin{aligned}
\langle \xi^2 \rangle = 0.240 &\pm 0.014 \text{ (statistical)} \\
&\pm 0.008 \text{ (continuum)} \\
&\pm 0.004 \text{ (higher twist)} \\
&\pm 0.002 \text{ (excited states)} \\
&\pm 0.0002 \text{ (finite volume)} \\
&\pm 0.014 \text{ (unphysically heavy pion)} \\
&\pm 0.008 \text{ (fit range)} \\
&\pm 0.002 \text{ (running coupling)} \\
\hline
\langle \xi^2 \rangle = 0.240 &\pm 0.023 \text{ (total, exc. quenching)} \\
&\pm 0.021 \text{ (quenching}^1\text{)}
\end{aligned}$$

[1] Rough estimate of quenching error

Comparison to Literature

Second Mellin moment of pion LCDA:

$$\langle \xi^2 \rangle = 0.240 \pm 0.023 \pm 0.021$$

Important but hard problem – want multiple approaches

- Del Debbio et al. (2002): $\langle \xi^2 \rangle = 0.280 \pm 0.051$ (quenched)
- Zhang et al. (2020): $\langle \xi^2 \rangle = 0.244 \pm 0.030$
- Bali et al. (2019): $\langle \xi^2 \rangle = 0.235 \pm 0.008$
- Arthur et al. (2011): $\langle \xi^2 \rangle = 0.28 \pm 0.02$

Overall, consistent with previous results

- Complementary check to other methods
- Potential for generalization to higher moments