The qqbar potential from Wilson loop and the qqbar potential from NBS wave function

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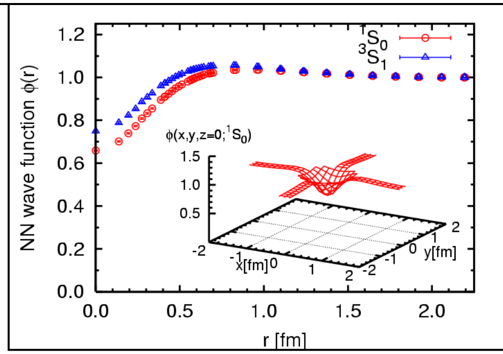
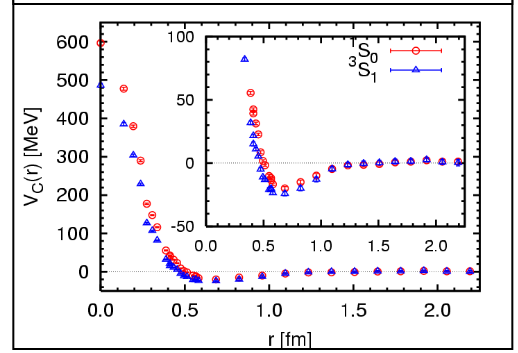
HAL QCD method for hadron-hadron potentials

LQCD ⇒ equal-time NBS wave func. ⇒ Schrödinger eq. ⇒ potential

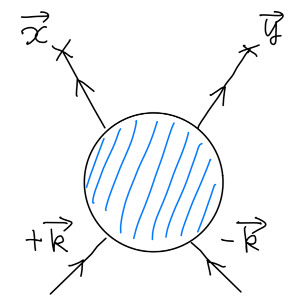




(1) **Equal-time NBS**(Nambu-Bethe-Salpeter) wave function from Lattice QCD

$$\psi(\mathbf{x}-\mathbf{y}) \equiv \langle 0 \,|\, N(\mathbf{x}) N(\mathbf{y}) \,|\, N(+k) N(-k), \mathrm{in} \rangle$$

$$\simeq e^{i\delta(k)} \frac{\sin(k\,|\mathbf{x}-\mathbf{y}|+\delta(k))}{k\,|\mathbf{x}-\mathbf{y}|} + \cdots \quad \text{for } |\mathbf{x}-\mathbf{y}| \to \text{large}$$



The same functional form as scattering wave functions of non-rela Q.M. at long distance.

(2) The potential $\hat{V}$ is defined by inversely solving Schrödinger eq.

$$\left(E_n - \hat{H}_0\right) \psi_n(\mathbf{r}) = \hat{V} \psi_n(\mathbf{r})$$

is faithful to the scattering phase shift $\delta(k)$

HAL QCD method has also been applied to qqbar potential

✦ Equal-time qqbar NBS wave func.(＝NBS amplitude) → **qqbar potential $\hat{V}$**

$$\psi(\mathbf{x}-\mathbf{y}) \equiv \langle 0 | \bar{q}_c(\mathbf{x}) q_c(\mathbf{y}) | \psi \rangle \quad \Rightarrow \quad \left(E + \frac{\nabla^2}{m_q} \right) \psi(\mathbf{r}) = \hat{V} \psi(\mathbf{r})$$

- ✓ Y.Ikeda, H.Iida,
  arXiv:1011.2866; arXiv:1102.2097; PTP128,No5,941(2012).
- ✓ T.Kawanai, S.Sasaki,
  PRL107,091601(2011); PRD89,054507(2014); PRD92,094503(2015).
- ✓ K.Nochi, T.Kawanai, S.Sasaki, PRD94,114514(2016)

✦ This method needs
a bit different formalism from the conventional HAL QCD method.

- ✓ Reproduction of **scattering phase** is not usable for the justification of $\hat{V}$.
  (Because of **color confinement**, no asymptotic states exist for an isolated quark)
- ✓ Only the **reproduction of hadron spectrum** can be used for the justification.
  →
  We will referred to this method as "**NBS amplitude method**".

NBS amplitude method for qqbar potential has a remarkable property

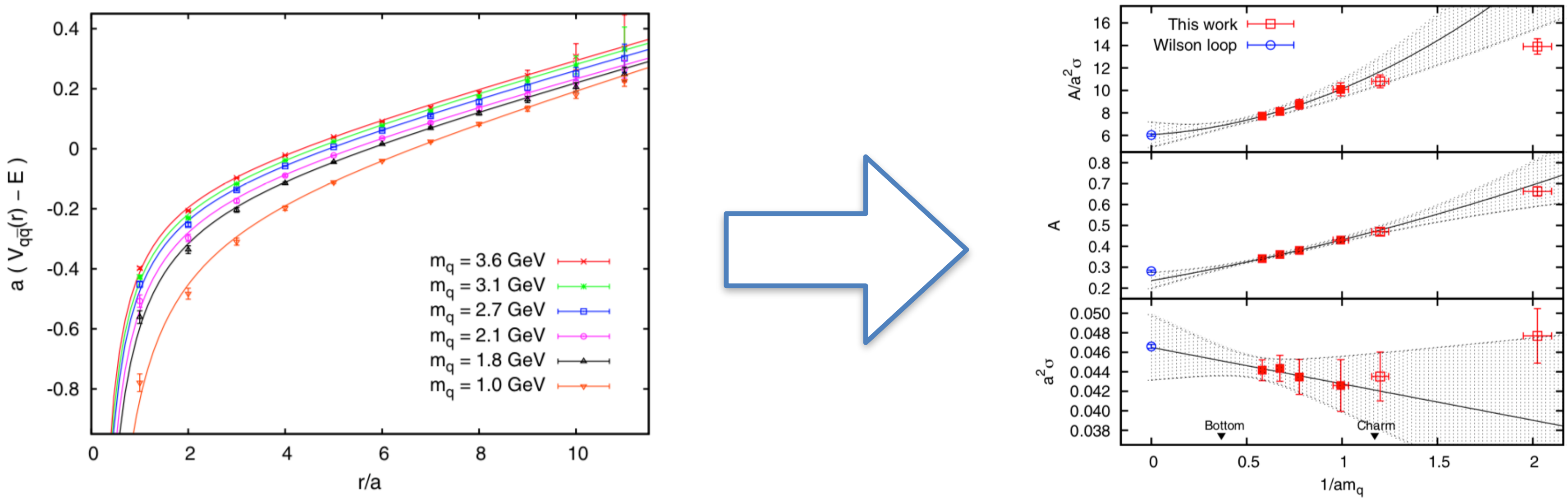
✦ It is reported that agreement between

-- qqbar potential from equal-time NBS wave func.

-- **static qqbar potential** from Wilson loop

is very good in the heavy quark mass limit.

$$V_0(r) = -\frac{A}{r} + \sigma r + \text{const.}$$



from Kawanai-Sasaki,PRL107

✦ However,

it is not mentioned how these two potentials can agree to each other.

✦ In this talk,

we consider the relation between these two qqbar potentials.

Main Part

qqbar potential from NBS wave function

[Kawanai-Sasaki,PRL107]

✦ Equal-time qqbar NBS wave func. (in **Coulomb gauge**)

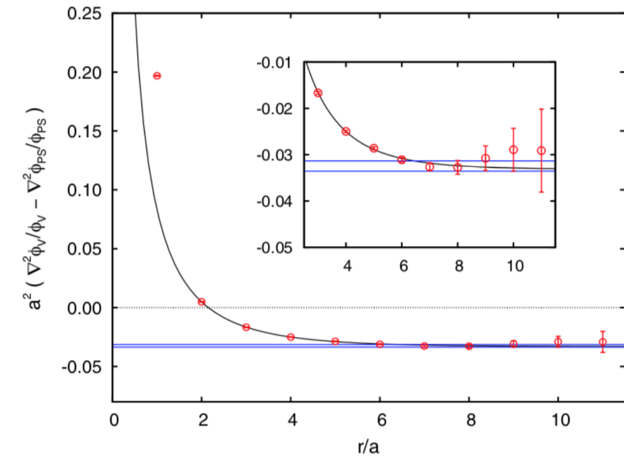
$$\psi_n(\mathbf{x}-\mathbf{y}) \equiv \langle 0 | \bar{q}_c(\mathbf{x}) q_c(\mathbf{y}) | n \rangle$$

✦ Schrödinger eq. (using PS and V states)

$$\left(E_n + \frac{\nabla^2}{m_q} \right) \psi_n(\mathbf{r}) = \hat{V} \psi_n(\mathbf{r})$$

qqbar mass

$$E_n \equiv M_n - 2m_q$$



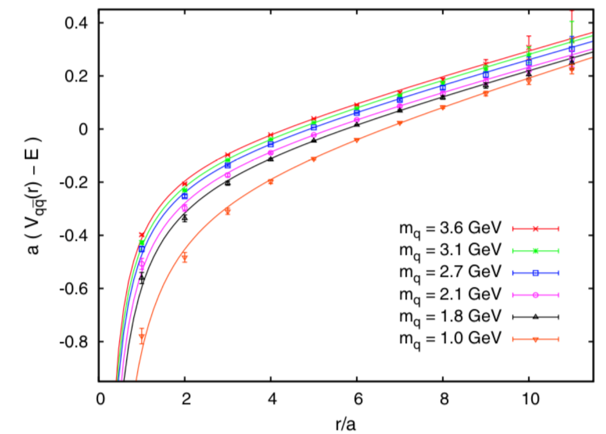
✓ Derivative expansion

$$\hat{V} \equiv V_0(r) + V_\sigma(r)\mathbf{s}_1 \cdot \mathbf{s}_2 + \cdots$$

✓ quark mass

Naïve procedure does not work due to confinement

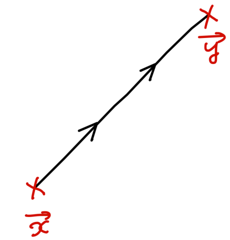
Kawanai-Sasaki's proposal:



$$V_\sigma(r) = \frac{1}{m_q} \left(\frac{\nabla^2 \psi_{\mathrm{V}}(r)}{\psi_{\mathrm{V}}(r)} - \frac{\nabla^2 \psi_{\mathrm{PS}}(r)}{\psi_{\mathrm{PS}}(r)} \right) + M_{\mathrm{V}} - M_{\mathrm{PS}} \to 0 \quad \textbf{as } r \to \infty$$

$$\boxed{m_q \equiv \frac{1}{M_{\mathrm{V}} - M_{\mathrm{PS}}} \lim_{r\to\infty} \left(\frac{\nabla^2 \psi_{\mathrm{PS}}(\mathbf{r})}{\psi_{\mathrm{PS}}(\mathbf{r})} - \frac{\nabla^2 \psi_{\mathrm{V}}(\mathbf{r})}{\psi_{\mathrm{V}}(\mathbf{r})} \right)}$$

We temporarily include Wilson line for simple argument



- Equal-time qqbar NBS wave func. (gauge invariant NBS wave func.)

$$\psi_n(\mathbf{x}-\mathbf{y}) \equiv \langle 0 | \bar{q}_c(\mathbf{x}) U_{cc'}(\mathbf{x},\mathbf{y}) q_{c'}(\mathbf{y}) | n \rangle$$

is extracted from (gauge invariant) R-correlator(=modified 4-point corr.) as

$$R(\mathbf{x}-\mathbf{y},t;\mathcal{J}) \equiv e^{2m_q} \langle 0 | T\{\bar{q}(\mathbf{x},t) U(\mathbf{x},t;\mathbf{y},t) q(\mathbf{y},t) \cdot \mathcal{J}(t=0)\} | 0 \rangle$$

$$= \sum_n \psi_n(\mathbf{x}-\mathbf{y}) a_n \exp\left[-t(M_n - 2m_q)\right] \quad \text{for } t>0$$

where $\mathcal{J}$ is a (gauge invariant) wall source

$$\mathcal{J}(t) \equiv \frac{1}{V^2} \sum_{\mathbf{x}',\mathbf{y}'} \bar{q}(\mathbf{y}',t) U(\mathbf{y}',t;\mathbf{x}',t) q(\mathbf{x}',t)$$

- We require that $\psi_n(\mathbf{x})$ satisfy Schrödinger eq.

$$\left(E_n + \frac{\nabla^2}{m_q}\right)\psi_n(\mathbf{r}) = \hat{V}\psi_n(\mathbf{r})$$

then R-corr. satisfies the time-dependent Schroedinger eq. as

$$\left(-\frac{\partial}{\partial t} + \frac{\nabla^2}{m_q}\right) R(\mathbf{r},t;\mathcal{J}) = \hat{V} R(\mathbf{r},t;\mathcal{J})$$

which can be used to determine $\hat{V}$

Large m_q limit

- Time-dependent Schrödinger eq. (large m_q limit)

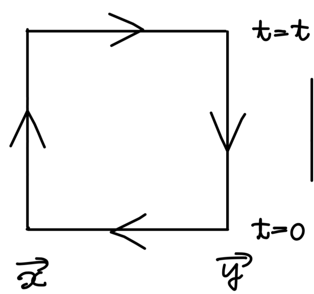
$$\left(-\frac{\partial}{\partial t}+\cancel{\frac{\nabla^2}{m_q}}\right)R(\mathbf{r},t;\mathcal{J}) = \hat{V}R(\mathbf{r},t;\mathcal{J})$$

$$\boxed{-\frac{\partial}{\partial t}R(\mathbf{r},t;\mathcal{J}) = \hat{V}R(\mathbf{r},t;\mathcal{J})}$$

- quark propagator (large m_q limit)

$$S_q(\mathbf{x},t;\mathbf{x}',0)/e^{-m_q t} = \delta^3(x-x')U(\mathbf{x},t;\mathbf{x}',0)\frac{1+\gamma_0}{2}+\cdots$$

- R-correlator (large m_q limit)

$R(\mathbf{r},t;\mathcal{J}) =$ $\langle 0|$  $|0\rangle$

$$= \sum_n a_n \exp\left[-t\tilde{V}_n(\mathbf{r})\right]$$

$$\boxed{-\frac{\partial}{\partial t}R(\mathbf{r},t;\mathcal{J}) \simeq \tilde{V}_0(\mathbf{r})R(\mathbf{r},t;\mathcal{J}) \quad \text{as} \quad t \to \text{large}}$$

qqbar potential
from NBS wave func.

=

qqbar potential
from Wilson loop

NBS amplitude mothod with Coulomb gauge

NBS wave function with Coulomb gauge (w/o Wilson line)

$$\psi_n(\mathbf{x}-\mathbf{y}) \equiv \langle 0 | \bar{q}_c(\mathbf{x}) q_c(\mathbf{y}) | n \rangle$$

✦ Gauge fixing matrix

$$\boxed{A_i^U \equiv -iU\left(\partial_i + iA_i\right)U^\dagger}$$

✓ $U[A](\mathbf{x},t) \in \mathrm{SU}(3)$ is defined to minimize

$$W_A[U] \equiv \int d^3x \, \mathrm{Tr}\left(A_i^U(\mathbf{x},t)^2\right)$$

✓ $U[A]$ is a matrix which transforms $A_i \to A_i^{\mathrm{fix}}$

$$\mathbf{A} \mapsto \mathbf{A}^{U[A]} = \mathbf{A}^{\mathrm{fix}} \quad \text{with} \quad \nabla \cdot \mathbf{A}^{\mathrm{fix}} = 0$$

✦ Comment

✓ In QED, analytic expression is available

$$U[A](\mathbf{x},t) = \exp\left[i \triangle^{-1} \nabla \cdot \mathbf{A}(\mathbf{x},t)\right]$$

In QCD, it is obtained only through numerical calculation

✓ For $A = A^{\mathrm{fix}}$ which satisfies the gauge fixing condition,

$$U[A^{\mathrm{fix}}](\mathbf{x},t) = 1$$

Gauge fixing matrix

✦ $U[A]$ is determined up to a constant $\Omega_0 \in \mathrm{SU}(3)$ from left.

$$\Omega_0 \cdot U[A]$$

✦ Under local gauge transformation $\Omega(\mathbf{x}) \in \mathrm{SU}(3)$:

$$q(\mathbf{x}) \mapsto \Omega(\mathbf{x}) q(\mathbf{x})$$

$$A_i(\mathbf{x}) \mapsto A_i^{\Omega}(\mathbf{x}) \equiv -i\Omega(\mathbf{x})\left(\partial_i + iA_i(\mathbf{x})\right)\Omega^\dagger(\mathbf{x})$$

$U[A]$ transforms as

$$U[A](\mathbf{x}) \mapsto U[A^{\Omega}](\mathbf{x}) = U[A](\mathbf{x}) \cdot \Omega^\dagger(\mathbf{x})$$

✦ The following combination is local gauge invariant,

and global SU(3) ambiguity cancels out:

$$\bar{q}(\mathbf{y})U^\dagger[A](\mathbf{y}) \cdot U[A](\mathbf{x})q(\mathbf{x})$$

$$\mapsto \bar{q}(\mathbf{y})\Omega^\dagger(\mathbf{y}) \cdot \Omega(\mathbf{y})U^\dagger(\mathbf{y})\Omega_0^\dagger \cdot \Omega_0 U[A](\mathbf{x})\Omega^\dagger(\mathbf{x}) \cdot \Omega(\mathbf{x})q(\mathbf{x})$$

$$= \bar{q}(\mathbf{y})U^\dagger[A](\mathbf{y}) \cdot U[A](\mathbf{x})q(\mathbf{x})$$

R-correlators with Coulomb gauge

- Gauge invariant R-correlator

$$\tilde{R}(\mathbf{x}-\mathbf{y},t;\tilde{\mathcal{J}})$$
$$\equiv e^{2m_q t}\left\langle 0 \left| T\left\{\bar{q}(\mathbf{x},t)U[A]^{\dagger}(\mathbf{x},t)\cdot U[A](\mathbf{y},t)q(\mathbf{y},t)\cdot\tilde{\mathcal{J}}(t=0)\right\}\right| 0\right\rangle$$

Gauge invariant wall source

$$\tilde{\mathcal{J}} \equiv \frac{1}{V^2}\int d^3x' \int d^3y'\, \bar{q}(\mathbf{y}')U[A]^{\dagger}(\mathbf{y}')\cdot U[A](\mathbf{x}')q(\mathbf{x}')$$

- Coulomb gauge R-corr.

$$R(\mathbf{x}-\mathbf{y},t,\mathcal{J}) \equiv e^{2m_q t}\left\langle 0 \left| T\left\{\bar{q}(\mathbf{x},t)q(\mathbf{y},t)\cdot\mathcal{J}\right\}\right| 0\right\rangle$$

$$\mathcal{J} \equiv \frac{1}{V^2}\int d^3x' \int d^3y'\, \bar{q}(\mathbf{y}')\cdot q(\mathbf{x}')$$

If Coulomb gauge fixing is employed

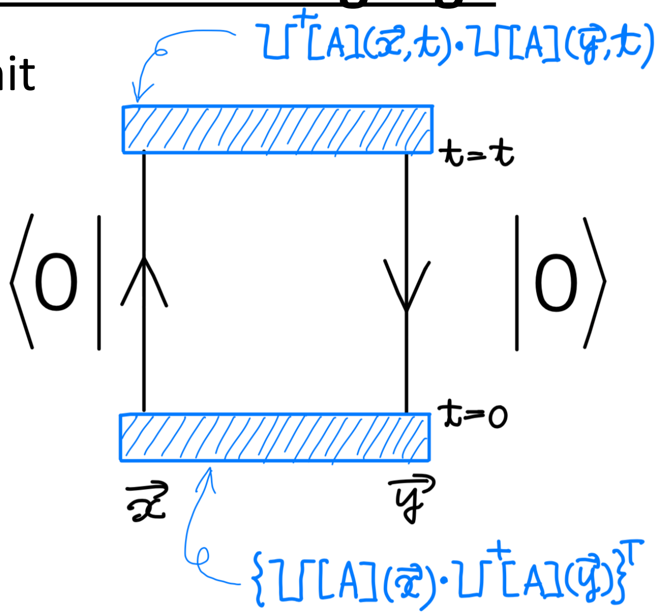
➜ $U[A^{\mathrm{fix}}]=1$ ➜ $\tilde{R}(\mathbf{r},t;\tilde{\mathcal{J}})$ reduces to $R(\mathbf{r},t;\mathcal{J})$

➜$\tilde{R}$ is a gauge invariant generalization of the Coulomb gauge R-correlator.

R-correlators with Coulomb gauge

✦ In large m_q limit

$\tilde{R}(\mathbf{r}, t; \mathcal{J}) =$



Note:

$U^\dagger[A](\mathbf{x}, t) \cdot U[A](\mathbf{y}, t)$

is a functional of $\mathbf{A}(x)$ contained in a single time-slice t.

➜We can regard it as "fat" Wilson loop.

✓ Wilson loop with "fat" source and "fat" sink ➜ "fat" Wilson loop

$$= \sum_n b_n \exp\left[-t\tilde{V}_n(r)\right]$$

✓ Overlap with each intermediate state changes.

✓ Energy eigenvalues $\tilde{V}_n(r)$ do not change.

$$-\frac{\partial}{\partial t}\tilde{R}(\mathbf{r}, t; \mathcal{J}) \simeq \tilde{V}_0(\mathbf{r})\tilde{R}(\mathbf{r}, t; \mathcal{J}) \quad \text{as } t \to \text{large}$$

qqbar potential from NBS wave func.
=
qqbar potential from Wilson loop

Summary

✦ There are several activities to obtain qqbar potential from the equal-time NBS wave function (HAL QCD method $\simeq$NBS amplitude method).

-- Ikeda-Iida
-- Kawanai-Sasaki
-- Nochi-Kawanai-Sasaki

✓ Remarkable property
agreement with the static quark potential is numerically shown very good in the heavy quark mass limit.

✓ However, it has not mentioned how these two potentials can agree.

✦ In this talk, we considered how the two potentials can agree in large m limit

✓ qqbar potential from NBS wave function

✓ qqbar potential from Wilson loop