Nucleon Form Factors from the Feynman-Hellmann Method in Lattice QCD

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SPECIAL RESEARCH CENTRE FOR THE SUBATOMIC STRUCTURE OF MATTER

Structure of the nucleons

- How is the charge distributed inside the nucleons?
- How does this distribution change at smaller scales?

- Need to calculate matrix elements: $\langle P(p')| \mathcal{J}(0) |P(p)\rangle$

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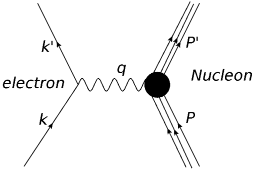
- Need to calculate matrix elements: $\langle P(p')|\,\mathcal{J}(0)\,|P(p)\rangle$
- matrix elements can be parameterised by two form factors: F_1, F_2



$$\langle P'|J^\nu|P\rangle = \bar{u}(p')\left[\gamma^\mu F_1(Q^2) + i\sigma^{\mu\nu}\frac{q_\nu}{2M}F_2(Q^2)\right]u(p)$$

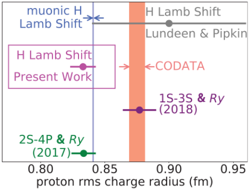
Often rewrite the matrix element in terms of G_E and G_M, as they are related to the charge and magnetisation distribution

Experimental results

- quantity of highest interest is the momentum dependence at various Q^2 values.
- At Q^2 close to 0, this dependence gives the charge radius.
- Recoil polarisation experiments have shown ineresting large Q^2 behaviour

Ratio $\frac{G_E}{G_M}$ crossing over zero?

- Conflict with older Rosenbluth separation data
  $\rightarrow$ more data is required



[Bezginov,2019]

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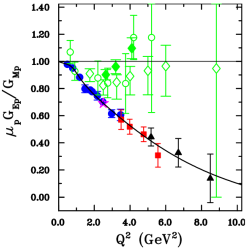
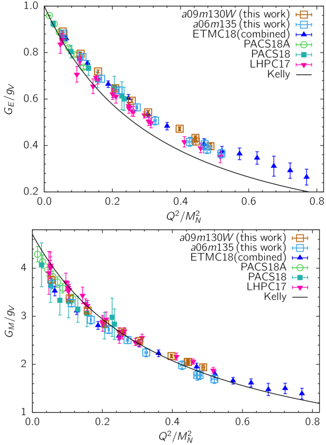


[JLab,2015]

Lattice results

- High quality data for low Q^2
- Fewer calculations at high Q^2 difficulties with signal-to-noise ratio and controlling excited states.
- Expensive to calculate high enough statistics at high momenta

Can the Feynman-Hellmann approach deliver a viable alternative at high Q^2?



[Jang et al,2020]

Feynman-Hellmann theorem $\frac{\partial E_\psi}{\partial \lambda} = \langle \psi | \frac{\partial H}{\partial \lambda} | \psi \rangle$

Consider the forward case for the proton:

- Insert a new term into the Lagrangian

$$\mathcal{L}(x) \to \mathcal{L}(x) + \lambda \mathcal{O}(x)$$

- Calculate the energy of the proton with this new term:

$$G(\lambda, \mathbf{p}, t) \xrightarrow{t \gg 0} A(\lambda) e^{-E(\lambda, \mathbf{p}) t}$$

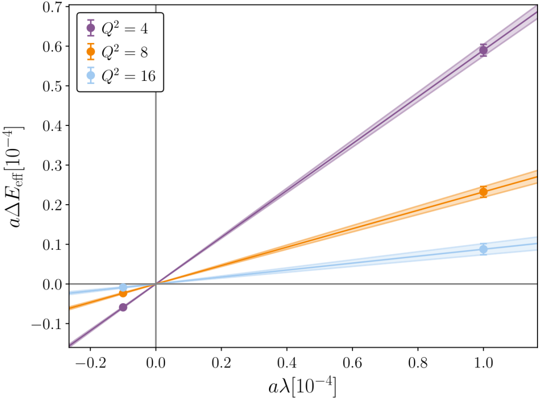
- Feynman-Hellmann theorem relates the energy shift to the matrix element:

$$\left. \frac{\partial E(\lambda, \mathbf{p})}{\partial \lambda} \right|_{\lambda=0} = \frac{1}{2E} \langle P(\mathbf{p}) | \mathcal{O}(0) | P(\mathbf{p}) \rangle$$

How does the energy shift dE behave at small λ?

Feynman-Hellmann theorem $\frac{\partial E_\psi}{\partial \lambda} = \langle \psi | \frac{\partial H}{\partial \lambda} | \psi \rangle$

- At $\lambda = 10^{-4}, -10^{-5}$ the energy shift behaves linearly with λ
- Can now calculate matrix elements by using only lattice two-point functions.



[Chambers 2017]

Feynman-Hellmann theorem $\frac{\partial E_\psi}{\partial \lambda} = \langle \psi | \frac{\partial H}{\partial \lambda} | \psi \rangle$

Electromagnetic form factors at large momentum

- Insert a term with vector current into the Lagrangian

$$\mathcal{L}(x) \rightarrow \mathcal{L}(x) + 2\lambda cos(\mathbf{q} \cdot \mathbf{x})\bar{q}(x)\gamma_\mu q(x)$$

- Requirement: Breit frame is necessary here.

  Since we want to access high Q^2, setting $\mathbf{p} = -\mathbf{p}'$ will satisfy this and keep noise to a minimum

- Using the temporal current: $\left.\frac{dE_p(\mathbf{p},\gamma_4)}{d\lambda}\right|_{\lambda=0} = \frac{m_p}{E_p(\mathbf{p})} G_{E,p}(Q^2)$

- Using the spatial current: $\left.\frac{dE_p(\mathbf{p},\sigma,\gamma_2)}{d\lambda}\right|_{\lambda=0} = \frac{[\mathbf{q}\times\hat{\mathbf{e}}]_2}{2E_p(\mathbf{p})} G_{M,p}(Q^2)$

Ratio of two-point functions

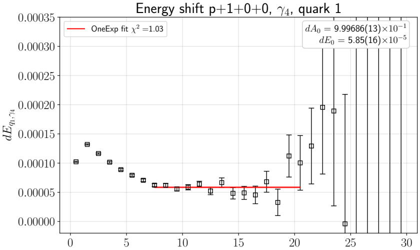
How do we best extract the shift in the energy ?

Construct a ratio of forwards and backwards propagating states with opposite parity projections which gives the energy shift in the large time limit.

$$R_{E,p}(\lambda, \pm\mathbf{p}, t) \equiv \left| \frac{G^{+}(\lambda, \pm\mathbf{p}, t)}{G^{+}(0, \pm\mathbf{p}, t)} \frac{G^{-}(0, \pm\mathbf{p}, -t)}{G^{-}(\lambda, \pm\mathbf{p}, -t)} \right|^{\frac{1}{2}}$$

$$\xrightarrow{\text{large t}} B(\lambda) e^{\Delta E(\lambda) t}$$

Good fit for low momentum transfer and heavy quark masses.



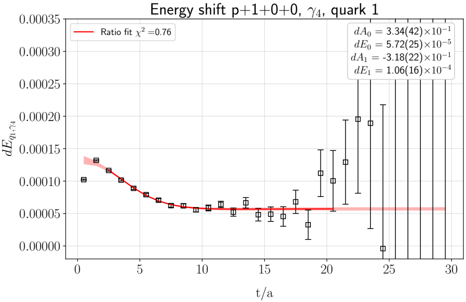
Excited states

Include the excited states in the fitting function to be able to include earlier timeslices in the fitting range.

$$R_{E,p}(\lambda, \pm\mathbf{p}, t) \equiv \left| \frac{G^{+}(\lambda, \pm\mathbf{p}, t)}{G^{+}(0, \pm\mathbf{p}, t)} \frac{G^{-}(0, \pm\mathbf{p}, -t)}{G^{-}(\lambda, \pm\mathbf{p}, -t)} \right|$$

$$\rightarrow \left| \frac{(A_0 + \Delta A_0)e^{-(E_0+\Delta E_0)t} + (A_1 + \Delta A_1)e^{-(E_1+\Delta E_1)t}}{(A_0 - \Delta A_0)e^{-(E_0-\Delta E_0)t} + (A_1 - \Delta A_1)e^{-(E_1-\Delta E_1)t}} \right|$$

Need 4 parameters from the fit to the ratio: $\Delta A_0, \Delta A_1, \Delta E_0, \Delta E_1$



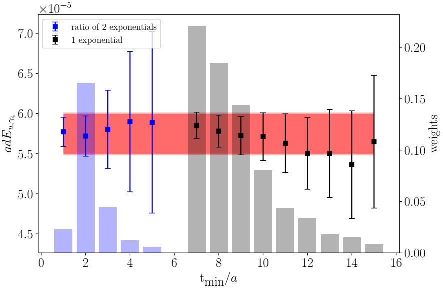
Weighted average of fits $N_{conf} = 1500, \quad 32^3 \times 64, \quad m_\pi \approx 470 MeV$

- Include fits with several different t_{min} values and give them weighting based on [NPL/QCDSF, PRD 2020]

$$\bar{E}_0 = \sum_{f=1}^{N} w^f E_0^f$$

$$p_f = \frac{\Gamma(N_{dof}/2, \chi_f^2/2)}{\Gamma(N_{dof}/2)}$$

$$\tilde{w}^f = \frac{p_f \left(\delta E_0^f\right)^{-2}}{\sum_{f'=1}^{N} p_{f'} \left(\delta E_0^{f'}\right)^{-2}}$$



p_f = p-value for the fit, w^f = weighting of the fit

Three-point functions

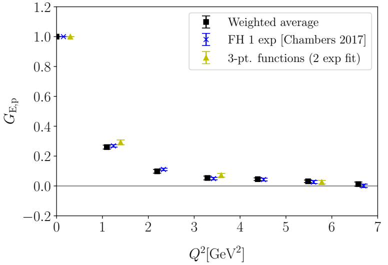
How does this method compare to the usual three-point functions?

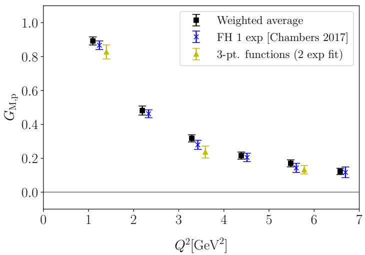
Recalculate the electromagnetic form factors of the nucleon with three-point functions on the same lattice and using the same breit frame momenta to facilitate an exact comparison.

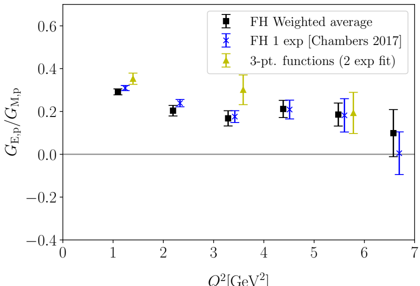
- UKQCD/QCDSF ensembles with $N_f = 2 + 1$
- $L^3 \times T = 32^3 \times 64, \;\; a = 0.074 \text{ fm}, \;\; m_\pi \approx 470 MeV$
- source-sink separations: $0.74 \text{ fm}, \;\; 0.96 \text{ fm}, \;\; 1.18 \text{ fm}$
- Same Breit frame momenta as the Feynman-Hellmann calculation
- Use a two-exponential fit to the three source-sink separations

Proton Form Factors

$N_{conf} = 1500, \quad 32^3 \times 64, \quad m_\pi \approx 470 MeV$







Flavour symmetry breaking

- We use the flavour breaking expansion analysis from [Bickerton et al, PRD 2020] to extrapolate our results to the physical quark masses.
- Calculate seven D_i quantities and five F_i from the form factors and fit to these as functions of the δm_l.

$$\bar{m} = \frac{1}{3}(m_u + m_d + m_s), \qquad \delta m \equiv m_q - \bar{m}$$

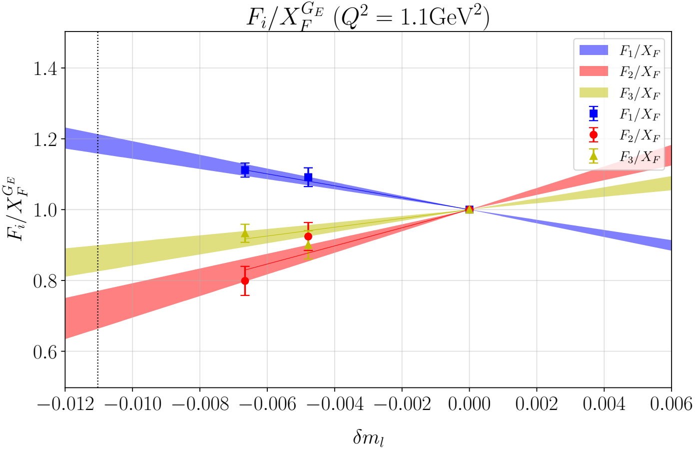
$$D_1 = -(A_{\bar{N}\eta N} + A_{\bar{\Xi}\eta\Xi})$$
$$D_2 = A_{\bar{\Sigma}\eta\Sigma}$$
$$D_3 = -A_{\bar{\Lambda}\eta\Lambda}$$

$$F_1 = \frac{1}{\sqrt{3}}(A_{\bar{N}\eta N} - A_{\bar{\Xi}\eta\Xi})$$
$$F_2 = (A_{\bar{N}\eta N} + A_{\bar{\Xi}\pi\Xi})$$
$$F_3 = A_{\Sigma\pi\Sigma}$$

Flavour symmetry breaking

$$N_{conf} = 1500, \quad 32^3 \times 64, \quad m_\pi = 310 - 466 \text{ MeV}$$

$$X_D = \frac{1}{6}(D_1 + 2D_2 + 3D_4), \qquad X_F = \frac{1}{6}(3F_1 + F_2 + 2F_3)$$



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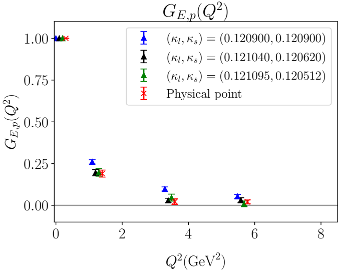
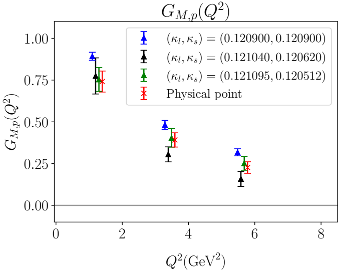
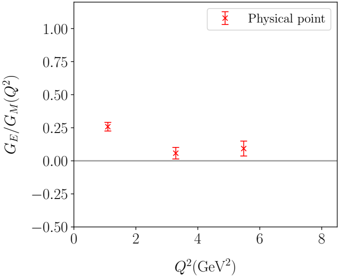
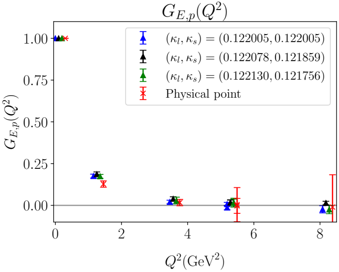


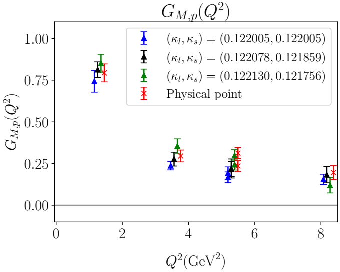


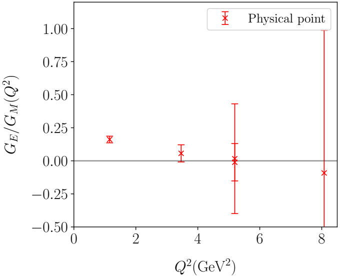


Flavour symmetry breaking

$N_{conf} \approx 800 - 1000, \quad 48^3 \times 96, \quad m_\pi = 310 - 466\ \mathrm{MeV}$







Conclusion

- The Feynman-Hellmann approach allows for the calculation of form factors at high momentum.
- Including the contributions from excited states in the fits makes the analysis more reliable across quark masses and momenta.

Next steps:

- Include more quark masses to further constrain the extrapolation to the physical point
- Increase N_{conf} for lighter quark masses to improve the uncertainties on the results
- Investigate the lattice spacing dependence of the results