Patterns of flavour symmetry breaking in hadron matrix elements involving *u*, *d* and *s* quarks

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[Monday 26/7/21 13:15 EST = 18:15 BST (Zoom)]

QCDSF strategy

[arXiv:1102.5300]

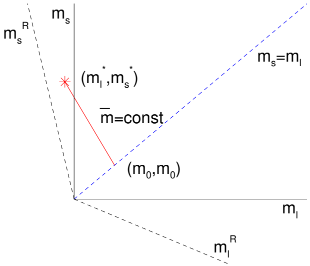
$2+1$ simulations: many paths to approach the physical point [$m_u = m_d \equiv m_l$ case]

QCDSF: extrapolate from a point on the $SU(3)_F$ flavour symmetry line to the physical point

$$(m_0, m_0) \longrightarrow (m_l^*, m_s^*)$$

Choice here: keep the singlet quark mass $\overline{m}$ constant

$$m_0 = \overline{m} \equiv \frac{1}{3}(2m_l + m_s)$$



- $SU(3)$ flavour symmetry breaking expansion for hadron masses
- Expansion in: $SU(3)$ flavour symmetric point $\delta m_q = 0$

$$\delta m_q = m_q - \overline{m}$$

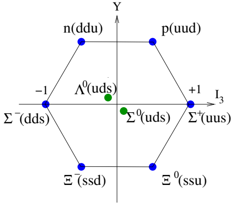
- expansion coefficients are functions of $\overline{m}$

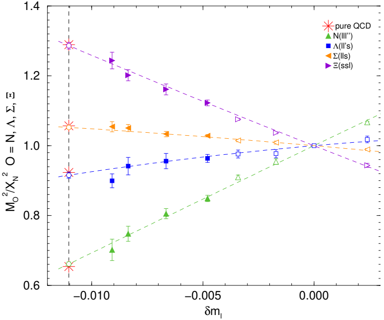
return

$SU(3)$ flavour symmetry breaking expansions: eg Baryon mass Spectrum

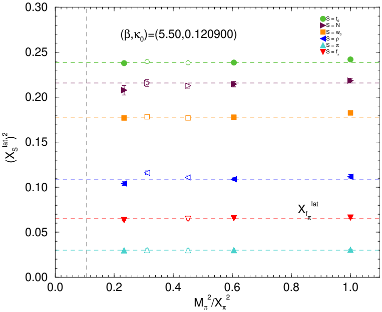
Constrained expansion known to $O(\delta m_l^3)$:

$$M_N = M_0 + 3A_1 \delta m_l + \ldots$$
$$M_\Lambda = M_0 + 3A_2 \delta m_l + \ldots$$
$$M_\Sigma = M_0 - 3A_2 \delta m_l + \ldots$$
$$M_\Xi = M_0 - 3(A_1 - A_2) \delta m_l + \ldots$$





‘Fan’ plot



Singlets: eg

$$X_N = (M_N + M_\Sigma + M_\Xi)/3 = M_0 + O(\delta m_l^2)$$

return

Now develop similar expansions for matrix elements

$$\langle B_i | J^{F_j} | B_k \rangle \equiv A_{\bar{B}_i F_j B_k}$$

where

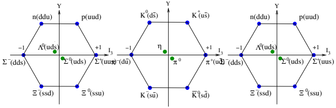
Index	Baryon (B)	Meson (F)	Current (J^{F})
1	n	K^0	$\bar{d}\gamma s$
2	p	K^+	$\bar{u}\gamma s$
3	Σ^-	π^-	$\bar{d}\gamma u$
4	Σ^0	π^0	$\frac{1}{\sqrt{2}}\left(\bar{u}\gamma u - \bar{d}\gamma d\right)$
5	Λ^0	η	$\frac{1}{\sqrt{6}}\left(\bar{u}\gamma u + \bar{d}\gamma d - 2\bar{s}\gamma s\right)$
6	Σ^+	π^+	$\bar{u}\gamma d$
7	Ξ^-	K^-	$\bar{s}\gamma u$
8	Ξ^0	$\bar{K}^0$	$\bar{s}\gamma d$
0		η'	$\frac{1}{\sqrt{3}}\left(\bar{u}\gamma u + \bar{d}\gamma d + \bar{s}\gamma s\right)$

[Convention $J^{\pi^+}|0\rangle = |\pi^+\rangle$]

Wigner-Eckart theorem $\Longrightarrow$ 'reduced' matrix element $\times$ CG coeff

$4 + 3 = 7$ diagonal elements $+$ (transition) elements between the same isospin multiplet

$A_{\bar{N}\eta N}$, $A_{\bar{\Sigma}\eta\Sigma}$, $A_{\bar{\Lambda}\eta\Lambda}$, $A_{\bar{\Xi}\eta\Xi}$ and $A_{\bar{N}\pi N}$, $A_{\bar{\Sigma}\pi\Sigma}$, $A_{\bar{\Xi}\pi\Xi}$



I		
0	$\langle n\vert J^{\eta}\vert n\rangle$	$A_{\bar{N}\eta N}$
0	$\langle p\vert J^{\eta}\vert p\rangle$	$A_{\bar{N}\eta N}$
0	$\langle\Sigma^{-}\vert J^{\eta}\vert\Sigma^{-}\rangle$	$A_{\bar{\Sigma}\eta\Sigma}$
0	$\langle\Sigma^{0}\vert J^{\eta}\vert\Sigma^{0}\rangle$	$A_{\bar{\Sigma}\eta\Sigma}$
0	$\langle\Sigma^{+}\vert J^{\eta}\vert\Sigma^{+}\rangle$	$A_{\bar{\Sigma}\eta\Sigma}$
0	$\langle\Lambda^{0}\vert J^{\eta}\vert\Lambda^{0}\rangle$	$A_{\bar{\Lambda}\eta\Lambda}$
0	$\langle\Xi^{-}\vert J^{\eta}\vert\Xi^{-}\rangle$	$A_{\bar{\Xi}\eta\Xi}$
0	$\langle\Xi^{0}\vert J^{\eta}\vert\Xi^{0}\rangle$	$A_{\bar{\Xi}\eta\Xi}$

For example

$$\langle p|J^{\pi^+}|n\rangle = \sqrt{2}\,A_{\bar{N}\pi N} = \sqrt{2}\,\langle p|J^{\pi^0}|p\rangle$$

giving

$$\langle p|\bar{u}\gamma d|n\rangle = \langle p|(\bar{u}\gamma u - \bar{d}\gamma d)|p\rangle$$

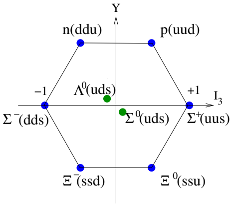
I		
1	$\langle n\vert J^{\pi^0}\vert n\rangle$	$-A_{\bar{N}\pi N}$
1	$\langle p\vert J^{\pi^0}\vert p\rangle$	$A_{\bar{N}\pi N}$
1	$\langle n\vert J^{\pi^-}\vert p\rangle$	$\sqrt{2}A_{\bar{N}\pi N}$
1	$\langle p\vert J^{\pi^+}\vert n\rangle$	$\sqrt{2}A_{\bar{N}\pi N}$
1	$\langle\Sigma^{-}\vert J^{\pi^0}\vert\Sigma^{-}\rangle$	$-A_{\bar{\Sigma}\pi\Sigma}$
1	$\langle\Sigma^{0}\vert J^{\pi^0}\vert\Sigma^{0}\rangle$	0
1	$\langle\Sigma^{+}\vert J^{\pi^0}\vert\Sigma^{+}\rangle$	$A_{\bar{\Sigma}\pi\Sigma}$
1	$\langle\Sigma^{-}\vert J^{\pi^-}\vert\Sigma^{0}\rangle$	$A_{\bar{\Sigma}\pi\Sigma}$
1	$\langle\Sigma^{0}\vert J^{\pi^-}\vert\Sigma^{+}\rangle$	$-A_{\bar{\Sigma}\pi\Sigma}$
1	$\langle\Sigma^{0}\vert J^{\pi^+}\vert\Sigma^{-}\rangle$	$A_{\bar{\Sigma}\pi\Sigma}$
1	$\langle\Sigma^{+}\vert J^{\pi^+}\vert\Sigma^{0}\rangle$	$-A_{\bar{\Sigma}\pi\Sigma}$
1	$\langle\Lambda^{0}\vert J^{\pi^0}\vert\Lambda^{0}\rangle$	0
1	$\langle\Xi^{-}\vert J^{\pi^0}\vert\Xi^{-}\rangle$	$-A_{\bar{\Xi}\pi\Xi}$
1	$\langle\Xi^{0}\vert J^{\pi^0}\vert\Xi^{0}\rangle$	$A_{\bar{\Xi}\pi\Xi}$
1	$\langle\Xi^{-}\vert J^{\pi^-}\vert\Xi^{0}\rangle$	$-\sqrt{2}A_{\bar{\Xi}\pi\Xi}$
1	$\langle\Xi^{0}\vert J^{\pi^+}\vert\Xi^{-}\rangle$	$-\sqrt{2}A_{\bar{\Xi}\pi\Xi}$

Wigner-Eckart theorem $\Longrightarrow$ 'reduced' matrix element $\times$ CG coeff

- 5 transition elements

  $A_{\bar\Sigma\pi\Lambda}$ and $A_{\bar N K\Sigma}$, $A_{\bar N K\Lambda}$, $A_{\bar\Lambda K\Xi}$, $A_{\bar\Sigma K\Xi}$

- $+5$ inverse relations:
  $A_{\bar\Lambda\pi\Sigma}$, $A_{\bar\Sigma\bar K N}$, $A_{\bar\Lambda\bar K N}$, $A_{\bar\Xi\bar K\Lambda}$, $A_{\bar\Xi\bar K\Sigma}$



I		
1	$\langle\Sigma^-\|J^{\pi^-}\|\Lambda^0\rangle$	$A_{\bar\Sigma\pi\Lambda}$
1	$\langle\Sigma^0\|J^{\pi^0}\|\Lambda^0\rangle$	$A_{\bar\Sigma\pi\Lambda}$
1	$\langle\Sigma^+\|J^{\pi^+}\|\Lambda^0\rangle$	$A_{\bar\Sigma\pi\Lambda}$
$\frac{1}{2}$	$\langle n\|J^{K^+}\|\Sigma^-\rangle$	$A_{\bar N K\Sigma}$
$\frac{1}{2}$	$\langle n\|J^{K^0}\|\Sigma^0\rangle$	$-A_{\bar N K\Sigma}/\sqrt{2}$
$\frac{1}{2}$	$\langle p\|J^{K^+}\|\Sigma^0\rangle$	$A_{\bar N K\Sigma}/\sqrt{2}$
$\frac{1}{2}$	$\langle p\|J^{K^0}\|\Sigma^+\rangle$	$A_{\bar N K\Sigma}$
$\frac{1}{2}$	$\langle n\|J^{K^0}\|\Lambda^0\rangle$	$A_{\bar N K\Lambda}$
$\frac{1}{2}$	$\langle p\|J^{K^+}\|\Lambda^0\rangle$	$A_{\bar N K\Lambda}$
$\frac{1}{2}$	$\langle\Lambda^0\|J^{K^+}\|\Xi^-\rangle$	$A_{\bar\Lambda K\Xi}$
$\frac{1}{2}$	$\langle\Lambda^0\|J^{K^0}\|\Xi^0\rangle$	$A_{\bar\Lambda K\Xi}$
$\frac{1}{2}$	$\langle\Sigma^-\|J^{K^0}\|\Xi^-\rangle$	$A_{\bar\Sigma K\Xi}$
$\frac{1}{2}$	$\langle\Sigma^0\|J^{K^+}\|\Xi^-\rangle$	$A_{\bar\Sigma K\Xi}/\sqrt{2}$
$\frac{1}{2}$	$\langle\Sigma^0\|J^{K^0}\|\Xi^0\rangle$	$-A_{\bar\Sigma K\Xi}/\sqrt{2}$
$\frac{1}{2}$	$\langle\Sigma^+\|J^{K^+}\|\Xi^0\rangle$	$A_{\bar\Sigma K\Xi}$

Matrix elements follow the schematic pattern for 2+1:

$$\langle B_i | J^{F_j} | B_k \rangle = \sum (\text{singlet mass polynomial}) \times (\text{singlet tensor})_{ijk} + \sum (\text{octet mass polynomial}) \times (\text{octet tensor})_{ijk} + \sum (\text{27-plet mass polynomial}) \times (\text{27-plet tensor})_{ijk} + \sum (\text{64-plet mass polynomial}) \times (\text{64-plet tensor})_{ijk}$$

We already know mass polynomials [QCDSF arXiv:1102.5300]

Polynomial	$SU(3)$			
1	1			
δm_l		8		
δm_l^2	1	8	27	
δm_l^3	1	8	27	64

So, for example, the 27-plet part contains $O(\delta m_l^2)$, $O(\delta m_l^3)$ terms

Determining the $T_{ijk} \sim \langle B_i | J^{F_j} | B_k \rangle$ tensors – a sketch (!)

- $SU(3)$ rotation:

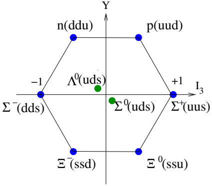
$$T'_{ijk} = U^{\dagger}_{ia} T_{abc} U_{bj} U_{ck}$$

  Consider change in T under an infinitesimal transformation by generator λ^{α}

- Isospin constraints + known Casimir eigenvalues $\Longrightarrow$
  17 independent tensors with most T_{ijk} elements zero or $\sqrt{\text{integer}}$ [Mathematica]
- Further classification:
  Define a reflection matrix $\sim C$, which inverts the outer ring of the octet
  Tensors can be divided into first or second class depending on the symmetry

first class	$T_{ijk} = +T_{kai} R_{aj}$
second class	$T_{ijk} = -T_{kai} R_{aj}$

interchanges B_i and B_k and transposes flavour matrix F^j



- Additional classification by the symmetry when R is applied to all three indices

$$d\text{−like} \qquad T_{ijk} = +R_{ia} T_{abc} R_{bj} R_{ck}$$
$$f\text{−like} \qquad T_{ijk} = -R_{ia} T_{abc} R_{bj} R_{ck}$$

Result: Matrix elements follow the schematic pattern for 2+1:

$$
\begin{aligned}
\langle B_i | J^{F_j} | B_k \rangle \quad = \quad & \sum (\text{singlet mass polynomial}) \times (\text{singlet tensor})_{ijk} \\
& + \sum (\text{octet mass polynomial}) \times (\text{octet tensor})_{ijk} \\
& + \sum (\text{27-plet mass polynomial}) \times (\text{27-plet tensor})_{ijk} \\
& + \sum (\text{64-plet mass polynomial}) \times (\text{64-plet tensor})_{ijk}
\end{aligned}
$$

Polynomial	$SU(3)$			
1	1			
δm_l		8		
δm_l^2	1	8	27	
δm_l^3	1	8	27	64

$SU(3)$	T, 1st class		T, 2nd class	
	$d-$like	$f-$like	$d-$like	$f-$like
1	d	f		
8	$r_1,\ r_2,\ r_3$	$s_1,\ s_2$	$t_1,\ t_2$	u_1
27	$q_1,\ q_2$	$w_1,\ w_2$	x_1	y_1
64	z			

17 tensors: two singlets, eight octets, six 27-plets, one 64-plet

– all contained in $8 \otimes 8 \otimes 8$ decomposition

Results I: $O(\delta m_l)$

		1, $1^{\rm st}$ class $O(1)$		8, $1^{\rm st}$ class $O(\delta m_l)$					8, $2^{\rm nd}$ class $O(\delta m_l)$		
		f	d	d	d	d	f	f	d	d	f
I	$A_{\bar{B}'FB}$	f	d	r_1	r_2	r_3	s_1	s_2	t_1	t_2	u_1
0	$\bar{N}\eta N$	$\sqrt{3}$	-1	1	0	0	0	-1	0	0	0
0	$\bar{\Sigma}\eta\Sigma$	0	2	1	0	$2\sqrt{3}$	0	0	0	0	0
0	$\bar{\Lambda}\eta\Lambda$	0	-2	1	2	0	0	0	0	0	0
0	$\bar{\Xi}\eta\Xi$	$-\sqrt{3}$	-1	1	0	0	0	1	0	0	0
1	$\bar{N}\pi N$	1	$\sqrt{3}$	0	0	-2	2	0	0	0	0
1	$\bar{\Sigma}\pi\Sigma$	2	0	0	0	0	-2	$\sqrt{3}$	0	0	0
1	$\bar{\Xi}\pi\Xi$	1	$-\sqrt{3}$	0	0	2	2	0	0	0	0
1	$\bar{\Sigma}\pi\Lambda$	0	2	0	1	$-\sqrt{3}$	0	0	1	0	0
$\frac{1}{2}$	$\bar{N}K\Sigma$	$-\sqrt{2}$	$\sqrt{6}$	0	0	$\sqrt{2}$	$\sqrt{2}$	0	0	$\sqrt{2}$	$\sqrt{6}$
$\frac{1}{2}$	$\bar{N}K\Lambda$	$-\sqrt{3}$	-1	0	1	0	$-\sqrt{3}$	1	1	$\sqrt{3}$	-1
$\frac{1}{2}$	$\bar{\Lambda}K\Xi$	$\sqrt{3}$	-1	0	1	0	$\sqrt{3}$	-1	-1	$-\sqrt{3}$	-1
$\frac{1}{2}$	$\bar{\Sigma}K\Xi$	$\sqrt{2}$	$\sqrt{6}$	0	0	$\sqrt{2}$	$-\sqrt{2}$	0	0	$-\sqrt{2}$	$\sqrt{6}$

Results II: $O(\delta m_l^2)$, $O(\delta m_l^3)$ – similar looking table

Examples

Polynomial	$SU(3)$			
1	1			
δm_l		8		
δm_l^2	1	8	27	
δm_l^3	1	8	27	64

$SU(3)$	T, 1st class d−like	T, 1st class f−like	T, 2nd class d−like	T, 2nd class f−like
1	d	f		
8	$r_1,\ r_2,\ r_3$	$s_1,\ s_2$	$t_1,\ t_2$	u_1
27	$q_1,\ q_2$	$w_1,\ w_2$	x_1	y_1
64	z			

- First-class current

[Same notation for tensor and coefficient]

$$
\begin{aligned}
\langle p|J^\eta|p\rangle &= A_{\bar{N}\eta N} \\
&= \underbrace{\sqrt{3}f - d}_{1} + (\underbrace{r_1 - s_2}_{8})\delta m_l \\
&\quad + (\underbrace{\sqrt{3}f^{\times} - d^{\times}}_{1} + \underbrace{r_1^{\times} - s_2^{\times}}_{8} + \underbrace{9q_1 + 3q_2 + 3\sqrt{3}w_2}_{27})\delta m_l^2 \\
&\quad + (\underbrace{\sqrt{3}f^{\times\times} - d^{\times\times}}_{1} + \underbrace{r_1^{\times\times} - s_2^{\times\times}}_{8} + \underbrace{9q_1^{\times} + 3q_2^{\times} + 3\sqrt{3}w_2^{\times}}_{27} + \underbrace{3\sqrt{3}z}_{64})\delta m_l^3
\end{aligned}
$$

- Second-class current

$$
\begin{aligned}
\langle n|J^{K^+}|\Sigma^-\rangle &= A_{\bar{N}K\Sigma} \\
&= (\sqrt{2}t_2 + \sqrt{6}u_1)\delta m_l + (\sqrt{2}t_2^{\times} + \sqrt{6}u_1^{\times} + \sqrt{5}x_1 + \sqrt{2}y_1)\delta m_l^2
\end{aligned}
$$

What do we gain for all this work?

Polynomial	$SU(3)$			
1	1			
δm_l		8		
δm_l^2	1	8	27	
δm_l^3	1	8	27	64

$SU(3)$	T, 1st class		T, 2nd class	
	$d-$like	$f-$like	$d-$like	$f-$like
1	d	f		
8	$r_1,\ r_2,\ r_3$	$s_1,\ s_2$	$t_1,\ t_2$	u_1
27	$q_1,\ q_2$	$w_1,\ w_2$	x_1	y_1
64	z			

- First-class current – there are $7 + 5 = 12$ As $\Rightarrow$ constraints at each order in δm_l
  - $O(1)$ has $2[1] = 2$ free parameters
  - $O(\delta m_l)$ has $2[1] + 5[8] = 7$ free parameters
  - $O(\delta m_l^2)$ has $2[1] + 5[8] + 4[27] = 11$ free parameters
  - $O(\delta m_l^3)$ has $2[1] + 5[8] + 4[27] + 1[64] = 12$ free parameters – Game Over
- Second-class current – there are 5 As $\Rightarrow$ constraints
  - $O(\delta m_l)$ has $3[8] = 3$ free parameters
  - $O(\delta m_l^2)$ has $3[8] + 2[27] = 5$ free parameters – Game Over
- So only constraints at low δm_l orders

d-Fan

- Construct ($1^{\rm st}$-class, LO)

$$
\begin{aligned}
D_1 \equiv -(A_{\bar{N}\eta N} + A_{\bar{\Xi}\eta\Xi}) &= 2d - 2r_1\delta m_l \\
D_2 \equiv A_{\bar{\Sigma}\eta\Sigma} &= 2d + (r_1 + 2\sqrt{3}r_3)\delta m_l \\
D_3 \equiv -A_{\bar{\Lambda}\eta\Lambda} &= 2d - (r_1 + 2r_2)\delta m_l \\
D_4 \equiv \frac{1}{\sqrt{3}}(A_{\bar{N}\pi N} - A_{\bar{\Xi}\pi\Xi}) &= 2d - \frac{4}{\sqrt{3}}r_3\delta m_l \\
D_5 \equiv A_{\bar{\Sigma}\pi\Lambda} &= 2d + (r_2 - \sqrt{3}r_3)\delta m_l \\
D_6 \equiv \frac{1}{\sqrt{6}}(A_{\bar{N}K\Sigma} + A_{\bar{\Sigma}K\Xi}) &= 2d + \frac{2}{\sqrt{3}}r_3\delta m_l \\
D_7 \equiv -(A_{\bar{N}K\Lambda} + A_{\bar{\Lambda}K\Xi}) &= 2d - 2r_2\delta m_l
\end{aligned}
$$

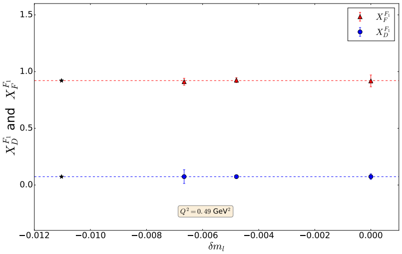
- 7 lines, but only 3 slope parameters, r_1, r_2, r_3, so splittings highly constrained
- 'Average' [Not unique, here just diagonal terms]

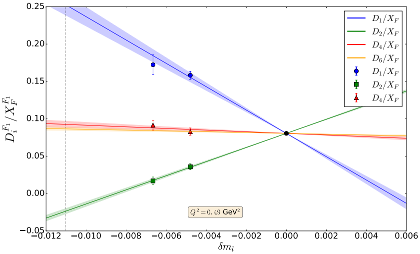
$$
X_D \equiv \frac{1}{6}(D_1 + 2D_2 + 3D_4) = 2d + O(\delta m_l^2)
$$

- Similarly for f-fan: 5 lines, but only 2 slope parameters, s_1, s_2, so splittings highly constrained

$SU(3)$ flavour symmetry breaking expansions: Baryon octet MEs

- Symanzik tree-level, $O(a)$ improved clover fermions
- $a \sim 0.074\,\text{fm}$; $M_\pi \sim 465 - 310\,\text{MeV}$
- $J \sim V$, ie vector current $\rightarrow$ form factors F_1, F_2
- Typical plots: [Presently only for diagonal MEs]



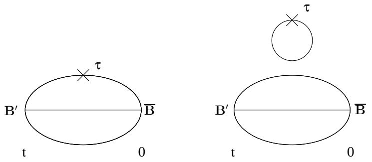


- Follows theory as expected as for masses

Further comments I

[Just given to $O(\delta m_l)$]

- Quark-line-connected and -disconnected diagrams:



$$\langle B'|J^F|B\rangle = \langle B'|J^F|B\rangle^{\rm con} + \langle B'|J^F|B\rangle^{\rm dis}$$

- Consideration of what is possible eg disconnected u, d same for $\langle p|J^{\pi^0}|p\rangle^{\rm dis}$ so vanishes $\Longrightarrow$ at $O(\delta m_l)$ only

$$r_1^{\rm dis} \neq 0$$

[ie for eg first-class currents all

$$f^{\rm dis}\,,\ d^{\rm dis}\,,\ r_2^{\rm dis}\,,\ ,\ r_3^{\rm dis}\,,\ s_1^{\rm dis}\,,\ s_2^{\rm dis} = 0$$

vanish]

Further comments II

[Just given to $O(\delta m_l)$]

- To complete the job to determine all expansions $[+\bar{d}\gamma d, \bar{s}\gamma s]$

$$\bar{u}\gamma u = \frac{1}{\sqrt{3}} J^{\eta'} + \frac{1}{\sqrt{2}} J^{\pi^0} + \frac{1}{\sqrt{6}} J^{\eta}$$

J definitions

Need singlet ie $\langle p|J^{\eta'}|p\rangle \Longrightarrow A_{\bar{N}\eta' N} \ldots$

- These are $8 \otimes 1 \otimes 8$ so already determined – masses

$$\begin{aligned} A_{\bar{N}\eta' N} &= a_0 + 3a_1\delta m_l \\ A_{\bar{\Lambda}\eta' \Lambda} &= a_0 + 3a_2\delta m_l \\ A_{\bar{\Sigma}\eta' \Sigma} &= a_0 - 3a_2\delta m_l \\ A_{\bar{\Xi}\eta' \Xi} &= a_0 - 3(a_1 - a_2)\delta m_l \end{aligned}$$

J mass expansions

- Consideration of what is possible, eg only connected pieces for $\langle p|\bar{s}s|p\rangle$ gives for example

$$\langle p|\bar{u}\gamma u|p\rangle^{\rm con} = 2\sqrt{2}f + \left(\sqrt{\frac{3}{2}} r_1^{\rm con} - \sqrt{2} r_3 + \sqrt{2} s_1 - \sqrt{\frac{3}{2}} s_2\right)\delta m_l$$

$$\langle N|\bar{u}\gamma u|N\rangle^{\rm dis} = \frac{1}{\sqrt{3}} a_0^{\rm dis} + \left(\sqrt{3} a_1^{\rm dis} + \frac{1}{\sqrt{6}} r_1^{\rm dis}\right)\delta m_l$$

Further comments III: $O(a)$-improvement

- Bhattacharya et al., [arXiv:hep-lat/0511014] determined pattern of $O(a)$ improvement coefficients
  - For example absorbing their $\bar{m}$ terms for forward $V_\mu^{\pi^0}$ into b_V, Z_V:

$$V_\mu^{\pi^0\,\mathrm{R}} = \hat{Z}_V \left[1 + \hat{b}_V \delta m_l\right] V_\mu^{\pi^0}$$

  - From (eg) diagonal matrix elements for $V_\mu^{\pi^0}$ this gives

$$\begin{aligned} s_1 &\to s_1' = s_1 + \frac{1}{2} f \hat{b}_V \,, \\ s_2 &\to s_2' = s_2 + \sqrt{3} f \hat{b}_V \,, \\ r_3 &\to r_3' = r_3 - \frac{\sqrt{3}}{2} d \hat{b}_V \end{aligned}$$

  - Additional results from $V_\mu^{\eta\,\mathrm{R}}$ and $V_\mu^{\eta'\,\mathrm{R}}$ give modified r_1, r_2
- As expected all improvement coefficients simply modify the $SU(3)$ flavour breaking expansion coefficients slightly

Conclusions

- Have developed baryon octet $SU(3)$ flavour symmetry breaking expansions for matrix elements (parallel to previous mass expansions) for $2+1$ quark flavours
- Complementary to chiral expansions (start at the quark mass symmetric point) rather than the quark mass zero point
- Again constrained expansions
- Future: eg have extended to meson octet – depending on current only f- or d-terms present

Backup

Results – Wigner-Eckart I:

I		
0	$\langle n\vert J^{\eta}\vert n\rangle$	$A_{\bar{N}\eta N}$
0	$\langle p\vert J^{\eta}\vert p\rangle$	$A_{\bar{N}\eta N}$
0	$\langle \Sigma^-\vert J^{\eta}\vert \Sigma^-\rangle$	$A_{\bar{\Sigma}\eta\Sigma}$
0	$\langle \Sigma^0\vert J^{\eta}\vert \Sigma^0\rangle$	$A_{\bar{\Sigma}\eta\Sigma}$
0	$\langle \Sigma^+\vert J^{\eta}\vert \Sigma^+\rangle$	$A_{\bar{\Sigma}\eta\Sigma}$
0	$\langle \Lambda^0\vert J^{\eta}\vert \Lambda^0\rangle$	$A_{\bar{\Lambda}\eta\Lambda}$
0	$\langle \Xi^-\vert J^{\eta}\vert \Xi^-\rangle$	$A_{\bar{\Xi}\eta\Xi}$
0	$\langle \Xi^0\vert J^{\eta}\vert \Xi^0\rangle$	$A_{\bar{\Xi}\eta\Xi}$

I		
1	$\langle n\vert J^{\pi^0}\vert n\rangle$	$-A_{\bar{N}\pi N}$
1	$\langle p\vert J^{\pi^0}\vert p\rangle$	$A_{\bar{N}\pi N}$
1	$\langle n\vert J^{\pi^-}\vert p\rangle$	$\sqrt{2}A_{\bar{N}\pi N}$
1	$\langle p\vert J^{\pi^+}\vert n\rangle$	$\sqrt{2}A_{\bar{N}\pi N}$
1	$\langle \Sigma^-\vert J^{\pi^0}\vert \Sigma^-\rangle$	$-A_{\bar{\Sigma}\pi\Sigma}$
1	$\langle \Sigma^0\vert J^{\pi^0}\vert \Sigma^0\rangle$	0
1	$\langle \Sigma^+\vert J^{\pi^0}\vert \Sigma^+\rangle$	$A_{\bar{\Sigma}\pi\Sigma}$
1	$\langle \Sigma^-\vert J^{\pi^-}\vert \Sigma^0\rangle$	$A_{\bar{\Sigma}\pi\Sigma}$
1	$\langle \Sigma^0\vert J^{\pi^-}\vert \Sigma^+\rangle$	$-A_{\bar{\Sigma}\pi\Sigma}$
1	$\langle \Sigma^0\vert J^{\pi^+}\vert \Sigma^-\rangle$	$A_{\bar{\Sigma}\pi\Sigma}$
1	$\langle \Sigma^+\vert J^{\pi^+}\vert \Sigma^0\rangle$	$-A_{\bar{\Sigma}\pi\Sigma}$
1	$\langle \Lambda^0\vert J^{\pi^0}\vert \Lambda^0\rangle$	0
1	$\langle \Xi^-\vert J^{\pi^0}\vert \Xi^-\rangle$	$-A_{\bar{\Xi}\pi\Xi}$
1	$\langle \Xi^0\vert J^{\pi^0}\vert \Xi^0\rangle$	$A_{\bar{\Xi}\pi\Xi}$
1	$\langle \Xi^-\vert J^{\pi^-}\vert \Xi^0\rangle$	$-\sqrt{2}A_{\bar{\Xi}\pi\Xi}$
1	$\langle \Xi^0\vert J^{\pi^+}\vert \Xi^-\rangle$	$-\sqrt{2}A_{\bar{\Xi}\pi\Xi}$

Results – Wigner-Eckart II:

I		
1	$\langle \Sigma^- \vert J^{\pi^-} \vert \Lambda^0 \rangle$	$A_{\bar{\Sigma}\pi\Lambda}$
1	$\langle \Sigma^0 \vert J^{\pi^0} \vert \Lambda^0 \rangle$	$A_{\bar{\Sigma}\pi\Lambda}$
1	$\langle \Sigma^+ \vert J^{\pi^+} \vert \Lambda^0 \rangle$	$A_{\bar{\Sigma}\pi\Lambda}$
$\frac{1}{2}$	$\langle n \vert J^{K^+} \vert \Sigma^- \rangle$	$A_{\bar{N}K\Sigma}$
$\frac{1}{2}$	$\langle n \vert J^{K^0} \vert \Sigma^0 \rangle$	$-A_{\bar{N}K\Sigma}/\sqrt{2}$
$\frac{1}{2}$	$\langle p \vert J^{K^+} \vert \Sigma^0 \rangle$	$A_{\bar{N}K\Sigma}/\sqrt{2}$
$\frac{1}{2}$	$\langle p \vert J^{K^0} \vert \Sigma^+ \rangle$	$A_{\bar{N}K\Sigma}$
$\frac{1}{2}$	$\langle n \vert J^{K^0} \vert \Lambda^0 \rangle$	$A_{\bar{N}K\Lambda}$
$\frac{1}{2}$	$\langle p \vert J^{K^+} \vert \Lambda^0 \rangle$	$A_{\bar{N}K\Lambda}$
$\frac{1}{2}$	$\langle \Lambda^0 \vert J^{K^+} \vert \Xi^- \rangle$	$A_{\bar{\Lambda}K\Xi}$
$\frac{1}{2}$	$\langle \Lambda^0 \vert J^{K^0} \vert \Xi^0 \rangle$	$A_{\bar{\Lambda}K\Xi}$
$\frac{1}{2}$	$\langle \Sigma^- \vert J^{K^0} \vert \Xi^- \rangle$	$A_{\bar{\Sigma}K\Xi}$
$\frac{1}{2}$	$\langle \Sigma^0 \vert J^{K^+} \vert \Xi^- \rangle$	$A_{\bar{\Sigma}K\Xi}/\sqrt{2}$
$\frac{1}{2}$	$\langle \Sigma^0 \vert J^{K^0} \vert \Xi^0 \rangle$	$-A_{\bar{\Sigma}K\Xi}/\sqrt{2}$
$\frac{1}{2}$	$\langle \Sigma^+ \vert J^{K^+} \vert \Xi^0 \rangle$	$A_{\bar{\Sigma}K\Xi}$

I		
1	$\langle \Lambda^0 \vert J^{\pi^+} \vert \Sigma^- \rangle$	$A_{\bar{\Lambda}\pi\Sigma}$
1	$\langle \Lambda^0 \vert J^{\pi^0} \vert \Sigma^0 \rangle$	$A_{\bar{\Lambda}\pi\Sigma}$
1	$\langle \Lambda^0 \vert J^{\pi^-} \vert \Sigma^+ \rangle$	$A_{\bar{\Lambda}\pi\Sigma}$
$\frac{1}{2}$	$\langle \Sigma^- \vert J^{K^-} \vert n \rangle$	$A_{\bar{\Sigma}\bar{K}N}$
$\frac{1}{2}$	$\langle \Sigma^0 \vert J^{\bar{K}^0} \vert n \rangle$	$-A_{\bar{\Sigma}\bar{K}N}/\sqrt{2}$
$\frac{1}{2}$	$\langle \Sigma^0 \vert J^{K^-} \vert p \rangle$	$A_{\bar{\Sigma}\bar{K}N}/\sqrt{2}$
$\frac{1}{2}$	$\langle \Sigma^+ \vert J^{\bar{K}^0} \vert p \rangle$	$A_{\bar{\Sigma}\bar{K}N}$
$\frac{1}{2}$	$\langle \Lambda^0 \vert J^{\bar{K}^0} \vert n \rangle$	$A_{\bar{\Lambda}\bar{K}N}$
$\frac{1}{2}$	$\langle \Lambda^0 \vert J^{K^-} \vert p \rangle$	$A_{\bar{\Lambda}\bar{K}N}$
$\frac{1}{2}$	$\langle \Xi^- \vert J^{K^-} \vert \Lambda^0 \rangle$	$A_{\bar{\Xi}\bar{K}\Lambda}$
$\frac{1}{2}$	$\langle \Xi^0 \vert J^{\bar{K}^0} \vert \Lambda^0 \rangle$	$A_{\bar{\Xi}\bar{K}\Lambda}$
$\frac{1}{2}$	$\langle \Xi^- \vert J^{\bar{K}^0} \vert \Sigma^- \rangle$	$A_{\bar{\Xi}\bar{K}\Sigma}$
$\frac{1}{2}$	$\langle \Xi^- \vert J^{K^-} \vert \Sigma^0 \rangle$	$A_{\bar{\Xi}\bar{K}\Sigma}/\sqrt{2}$
$\frac{1}{2}$	$\langle \Xi^0 \vert J^{\bar{K}^0} \vert \Sigma^0 \rangle$	$-A_{\bar{\Xi}\bar{K}\Sigma}/\sqrt{2}$
$\frac{1}{2}$	$\langle \Xi^0 \vert J^{K^-} \vert \Sigma^+ \rangle$	$A_{\bar{\Xi}\bar{K}\Sigma}$

Table: Left: The 'forward' $I = 1$ and $\frac{1}{2}$ relations; Right: the inverse relations.

Results II: $O(\delta m_l^2)$, $O(\delta m_l^3)$

		27, $1^{\rm st}$ class $O(\delta m_l^2)$				64, $1^{\rm st}$ $O(\delta m_l^3)$	27, $2^{\rm nd}$ class $O(\delta m_l^2)$	
		d	d	f	f	d	d	f
I	$A_{\bar{B}'FB}$	q_1	q_2	w_1	w_2	z	x_1	y_1
0	$\bar{N}\eta N$	9	3	0	$3\sqrt{3}$	$3\sqrt{3}$	0	0
0	$\bar{\Sigma}\eta\Sigma$	-6	-10	0	0	$-\sqrt{3}$	0	0
0	$\bar{\Lambda}\eta\Lambda$	-18	18	0	0	$-9\sqrt{3}$	0	0
0	$\bar{\Xi}\eta\Xi$	9	3	0	$-3\sqrt{3}$	$3\sqrt{3}$	0	0
1	$\bar{N}\pi N$	$-5\sqrt{3}$	$\sqrt{3}$	4	-1	1	0	0
1	$\bar{\Sigma}\pi\Sigma$	0	0	-4	2	0	0	0
1	$\bar{\Xi}\pi\Xi$	$5\sqrt{3}$	$-\sqrt{3}$	4	-1	-1	0	0
1	$\bar{\Sigma}\pi\Lambda$	14	-6	0	0	$-\sqrt{3}$	4	0
$\frac{1}{2}$	$\bar{N}K\Sigma$	0	$2\sqrt{6}$	$-3\sqrt{2}$	$2\sqrt{2}$	$\sqrt{2}$	$\sqrt{6}$	$\sqrt{2}$
$\frac{1}{2}$	$\bar{N}K\Lambda$	-6	0	$3\sqrt{3}$	0	$3\sqrt{3}$	-3	$3\sqrt{3}$
$\frac{1}{2}$	$\bar{\Lambda}K\Xi$	-6	0	$-3\sqrt{3}$	0	$3\sqrt{3}$	3	$3\sqrt{3}$
$\frac{1}{2}$	$\bar{\Sigma}K\Xi$	0	$2\sqrt{6}$	$3\sqrt{2}$	$-2\sqrt{2}$	$\sqrt{2}$	$-\sqrt{6}$	$\sqrt{2}$

Full Results I: $O(\delta m_l)$

		1, $1^{\rm st}$ class $O(1)$		8, $1^{\rm st}$ class $O(\delta m_l)$					8, $2^{\rm nd}$ class $O(\delta m_l)$		
		f	d	d	d	d	f	f	d	d	f
I	$A_{\bar{B}'FB}$	f	d	r_1	r_2	r_3	s_1	s_2	t_1	t_2	u_1
0	$\bar{N}\eta N$	$\sqrt{3}$	-1	1	0	0	0	-1	0	0	0
0	$\bar{\Sigma}\eta\Sigma$	0	2	1	0	$2\sqrt{3}$	0	0	0	0	0
0	$\bar{\Lambda}\eta\Lambda$	0	-2	1	2	0	0	0	0	0	0
0	$\bar{\Xi}\eta\Xi$	$-\sqrt{3}$	-1	1	0	0	0	1	0	0	0
1	$\bar{N}\pi N$	1	$\sqrt{3}$	0	0	-2	2	0	0	0	0
1	$\bar{\Sigma}\pi\Sigma$	2	0	0	0	0	-2	$\sqrt{3}$	0	0	0
1	$\bar{\Xi}\pi\Xi$	1	$-\sqrt{3}$	0	0	2	2	0	0	0	0
1	$\bar{\Sigma}\pi\Lambda$	0	2	0	1	$-\sqrt{3}$	0	0	1	0	0
1	$\bar{\Lambda}\pi\Sigma$	0	2	0	1	$-\sqrt{3}$	0	0	-1	0	0
$\frac{1}{2}$	$\bar{N}K\Sigma$	$-\sqrt{2}$	$\sqrt{6}$	0	0	$\sqrt{2}$	$\sqrt{2}$	0	0	$\sqrt{2}$	$\sqrt{6}$
$\frac{1}{2}$	$\bar{N}K\Lambda$	$-\sqrt{3}$	-1	0	1	0	$-\sqrt{3}$	1	1	$\sqrt{3}$	-1
$\frac{1}{2}$	$\bar{\Lambda}K\Xi$	$\sqrt{3}$	-1	0	1	0	$\sqrt{3}$	-1	-1	$-\sqrt{3}$	-1
$\frac{1}{2}$	$\bar{\Sigma}K\Xi$	$\sqrt{2}$	$\sqrt{6}$	0	0	$\sqrt{2}$	$-\sqrt{2}$	0	0	$-\sqrt{2}$	$\sqrt{6}$
$\frac{1}{2}$	$\bar{\Sigma}\bar{K}N$	$-\sqrt{2}$	$\sqrt{6}$	0	0	$\sqrt{2}$	$\sqrt{2}$	0	0	$-\sqrt{2}$	$-\sqrt{6}$
$\frac{1}{2}$	$\bar{\Lambda}\bar{K}N$	$-\sqrt{3}$	-1	0	1	0	$-\sqrt{3}$	1	-1	$-\sqrt{3}$	1
$\frac{1}{2}$	$\bar{\Xi}\bar{K}\Lambda$	$\sqrt{3}$	-1	0	1	0	$\sqrt{3}$	-1	1	$\sqrt{3}$	1
$\frac{1}{2}$	$\bar{\Xi}\bar{K}\Sigma$	$\sqrt{2}$	$\sqrt{6}$	0	0	$\sqrt{2}$	$-\sqrt{2}$	0	0	$\sqrt{2}$	$-\sqrt{6}$

Full Results II: $O(\delta m_l^2)$, $O(\delta m_l^3)$

		27, $1^{\rm st}$ class $O(\delta m_l^2)$				64, $1^{\rm st}$ $O(\delta m_l^3)$	27, $2^{\rm nd}$ class $O(\delta m_l^2)$	
		d	d	f	f	d	d	f
I	$A_{\bar{B}'FB}$	q_1	q_2	w_1	w_2	z	x_1	y_1
0	$\bar{N}\eta N$	9	3	0	$3\sqrt{3}$	$3\sqrt{3}$	0	0
0	$\bar{\Sigma}\eta\Sigma$	-6	-10	0	0	$-\sqrt{3}$	0	0
0	$\bar{\Lambda}\eta\Lambda$	-18	18	0	0	$-9\sqrt{3}$	0	0
0	$\bar{\Xi}\eta\Xi$	9	3	0	$-3\sqrt{3}$	$3\sqrt{3}$	0	0
1	$\bar{N}\pi N$	$-5\sqrt{3}$	$\sqrt{3}$	4	-1	1	0	0
1	$\bar{\Sigma}\pi\Sigma$	0	0	-4	2	0	0	0
1	$\bar{\Xi}\pi\Xi$	$5\sqrt{3}$	$-\sqrt{3}$	4	-1	-1	0	0
1	$\bar{\Sigma}\pi\Lambda$	14	-6	0	0	$-\sqrt{3}$	4	0
1	$\bar{\Lambda}\pi\Sigma$	14	-6	0	0	$-\sqrt{3}$	-4	0
$\frac{1}{2}$	$\bar{N}K\Sigma$	0	$2\sqrt{6}$	$-3\sqrt{2}$	$2\sqrt{2}$	$\sqrt{2}$	$\sqrt{6}$	$\sqrt{2}$
$\frac{1}{2}$	$\bar{N}K\Lambda$	-6	0	$3\sqrt{3}$	0	$3\sqrt{3}$	-3	$3\sqrt{3}$
$\frac{1}{2}$	$\bar{\Lambda}K\Xi$	-6	0	$-3\sqrt{3}$	0	$3\sqrt{3}$	3	$3\sqrt{3}$
$\frac{1}{2}$	$\bar{\Sigma}K\Xi$	0	$2\sqrt{6}$	$3\sqrt{2}$	$-2\sqrt{2}$	$\sqrt{2}$	$-\sqrt{6}$	$\sqrt{2}$
$\frac{1}{2}$	$\bar{\Sigma}\bar{K}N$	0	$2\sqrt{6}$	$-3\sqrt{2}$	$2\sqrt{2}$	$\sqrt{2}$	$-\sqrt{6}$	$-\sqrt{2}$
$\frac{1}{2}$	$\bar{\Lambda}\bar{K}N$	-6	0	$3\sqrt{3}$	0	$3\sqrt{3}$	3	$-3\sqrt{3}$
$\frac{1}{2}$	$\bar{\Xi}\bar{K}\Lambda$	-6	0	$-3\sqrt{3}$	0	$3\sqrt{3}$	-3	$-3\sqrt{3}$
$\frac{1}{2}$	$\bar{\Xi}\bar{K}\Sigma$	0	$2\sqrt{6}$	$3\sqrt{2}$	$-2\sqrt{2}$	$\sqrt{2}$	$\sqrt{6}$	$-\sqrt{2}$

f-Fan

- Construct (1^{st}-class, LO)

$$
\begin{aligned}
F_1 \equiv \frac{1}{\sqrt{3}}(A_{\bar{N}\eta N} - A_{\bar{\Xi}\eta\Xi}) &= 2f - \frac{2}{\sqrt{3}} s_2 \delta m_l \\
F_2 \equiv (A_{\bar{N}\pi N} + A_{\bar{\Xi}\pi\Xi}) &= 2f + 4s_1 \delta m_l \\
F_3 \equiv A_{\bar{\Sigma}\pi\Sigma} &= 2f + (-2s_1 + \sqrt{3}s_2)\delta m_l \\
F_4 \equiv \frac{1}{\sqrt{2}}(A_{\bar{\Sigma}K\Xi} - A_{\bar{N}K\Sigma}) &= 2f - 2s_1 \delta m_l \\
F_5 \equiv \frac{1}{\sqrt{3}}(A_{\bar{\Lambda}K\Xi} - A_{\bar{N}K\Lambda}) &= 2f + \frac{2}{\sqrt{3}}(\sqrt{3}s_1 - s_2)\delta m_l
\end{aligned}
$$

- 5 lines, but only 2 slope parameters, s_1, s_2, so splittings highly constrained
- 'Average' [Not unique, here just diagonal terms]

$$
X_F \equiv \frac{1}{6}(3F_1 + F_2 + 2F_3) = 2f + O(\delta m_l^2)
$$