Topology and scale setting for Sp($2N_c$) gauge theories

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$Sp(2N_c)$ gauge theories:

- BSM: There are possible UV-complete realizations of Composite Higgs Models (CHMs) that feature Sp($2N_c$) gauge groups. See Ferretti and Karateev 2014 and lattice studies of the Sp($2N_c$) collaboration. See also talks by J. Holligan, J.W. Lee, H. Hsiao, B. Lucini. . .)
- Large-N_c: They provide a non-trivial alternative to the series of groups SU(N_c) and SO(N_c) in which to probe universality and scaling arguments.

't Hooft 1974; Lovelace 1982

- Thermodynamics of gauge fields, Dark Matter,. . .

Holland et al. 2004; Hochberg et al. 2015

In this talk we will focus on two aspects of the Sp($2N_c$) pure gauge theory:

- Scale setting: Wilson flow scales t_0 and w_0.
- Topological susceptibility: continuum and large-N_c limits.

General setup:

- Wilson action discretization with inverse coupling β, on a hypercubical lattice of spacing a and volume $(La)^4$.
- Periodic boundary conditions in all directions.
- Update algorithm: the usual HB+OR combination implemented using the Cabibbo-Marinari technique.

In the continuum, the topological charge density $q(x)$ and the total topological charge Q and the topological susceptibility χ are defined as,

$$q(x) = \frac{1}{32\pi^2}\int \mathrm{d}^4x \tilde{F}_{\mu\nu}F^{\mu\nu}, \qquad Q = \int \mathrm{d}^4x \; q(x) \; , \qquad \chi = \frac{\langle Q^2 \rangle}{\mathcal{V}} \; .$$

$\mathcal{V}$ space-time volume.

Since the groups $\mathrm{Sp}(2N_c)$ are compact and simply connected,

$$\pi_1(\mathrm{Sp}(2N_c)) = e, \qquad \pi_3(\mathrm{Sp}(2N_c)) = \mathbb{Z} \; .$$

We thus expect an integer valued topological charge, as in the case of $\mathrm{SU}(N_c)$.

On the lattice, we adopt the clover definition of the discretized topological charge,

$$q_L(x) = \frac{1}{32\pi^2}\epsilon_{\mu\nu\rho\sigma}\mathrm{Tr}\; U_{\mu\nu}(x)U_{\rho\sigma}(x) \; , \qquad Q_L = \sum_x q_L(x) \; , \qquad \chi_L a^4 = \frac{\langle Q_L^2 \rangle}{L^4} \; .$$

Problem: UV fluctuations usually dominate $\chi_L a^4$.

Only the elements of the algebra differ w.r.t to the case of $\mathrm{SU}(N_c)$,

$$\frac{\partial V_\mu(t,\, x)}{\partial t} = -g_0^2 \left\{\partial_{x,\,\mu} S_{\mathrm{W}}\left[V_\mu\right]\right\} V_\mu(t,\, x) \ , \qquad V_\mu(t=0,\, x) = U_\mu(x)$$

Lüscher 2010

The flow equations can be integrated numerically to obtain V_μ at finite t, with a $O(\epsilon^3)$ Runge-Kutta algorithm.

Then:

- The UV fluctuations of topological charge density are smoothened out at positive flow-time:

$$q_L(t,\, x) = \frac{1}{32\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\; V_{\mu\nu}(t,\, x) V_{\rho\sigma}(t,\, x) \ , \qquad Q_L(t) = \sum_x q_L(t,\, x) \ ,$$

- The scale can be set using the two quantities

$$\mathcal{E}(t) = t^2 \langle E(t) \rangle \ , \qquad \mathcal{W}(t) = t \frac{d}{dt} t^2 \langle E(t) \rangle \ ,$$

  $\mathcal{E}(t)$ sensitive to scales up to $O(1/\sqrt{t})$, $\mathcal{W}(t)$ to scales around $O(1/\sqrt{t})$.

Lüscher 2010; Borsanyi et al. 2012

The Wilson flow for $\mathrm{Sp}(2N_c)$

In this work:

- $N_c = 1, 2, 3, 4$
- L chosen from study of glueball states so that FSE are negligible
- At least 3 values of β for each value of N_c.
- The Wilson flow was integrated up to t such that $\sqrt{8t} = L$
- $t^2 E(t)$ was computed along the flow, where two discretizations are possible for $E = -\frac{1}{2}\mathrm{Tr}\left(G_{\mu\nu}G^{\mu\nu}\right)$: plaquette and clover. Their difference informs us on the magnitude of discretization effects.

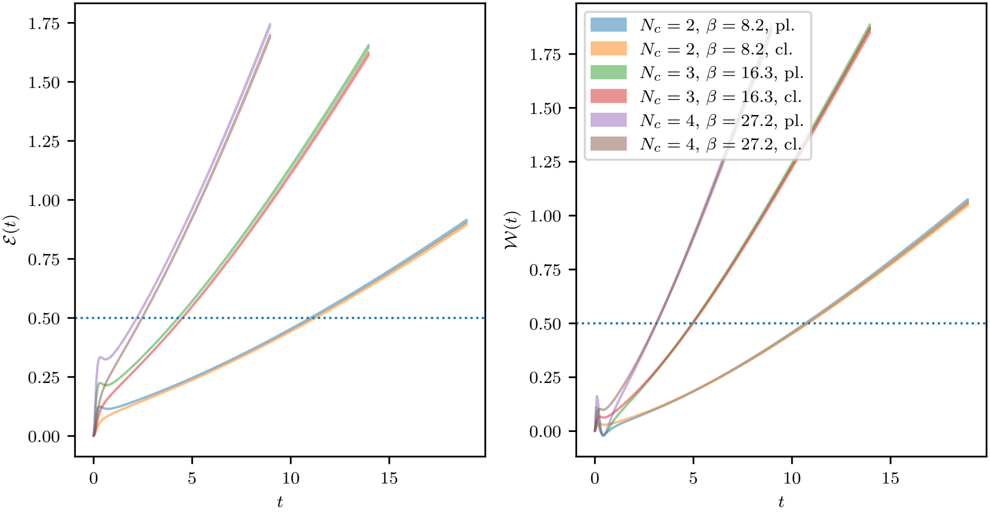
The scales t_0 and w_0 are defined from the time at which the physical quantities

$$\mathcal{E}(t)|_{t_0} = \mathcal{E}_0 \ , \qquad \mathcal{W}(t)|_{t=w_0^2} = \mathcal{W}_0$$

reach the specific values $\mathcal{E}_0$ and $\mathcal{W}_0$.

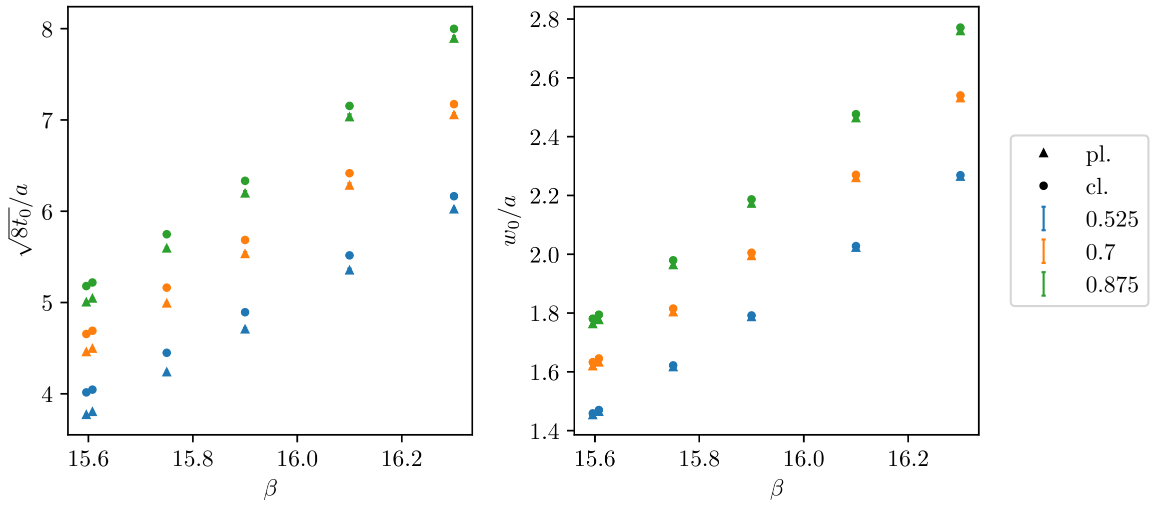
In units of t_0 or w_0,

$$\chi_L(t_0)t_0^2 = \frac{\langle Q_L^2(t_0)\rangle}{L^4}, \qquad \chi_L(w_0^2)w_0^4 = \frac{\langle Q_L^2(w_0^2)\rangle}{L^4},$$



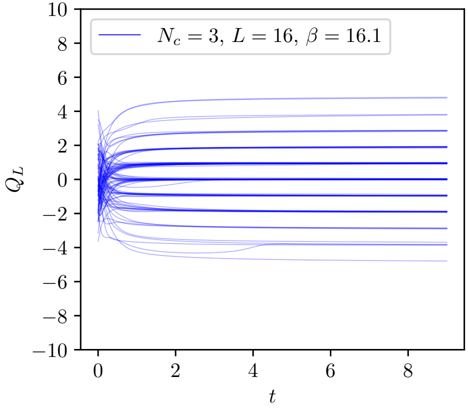
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
$\mathcal{E}(t)$
0
5
10
15
t
$N_c = 2,\ \beta = 8.2$, pl.
$N_c = 2,\ \beta = 8.2$, cl.
$N_c = 3,\ \beta = 16.3$, pl.
$N_c = 3,\ \beta = 16.3$, cl.
$N_c = 4,\ \beta = 27.2$, pl.
$N_c = 4,\ \beta = 27.2$, cl.
1.75
1.50
1.25
1.00
0.75
0.50
0.25
0.00
$\mathcal{W}(t)$
0
5
10
15
t

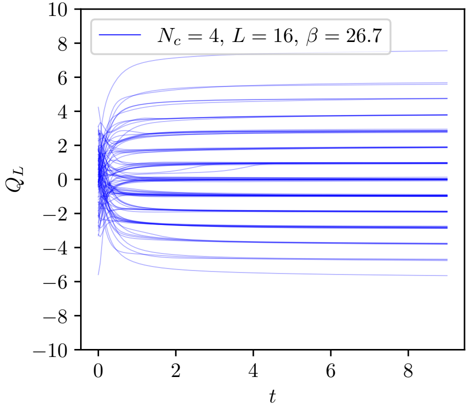
Setting the scale - $N_c = 3$



The Topological charge at positive flow time

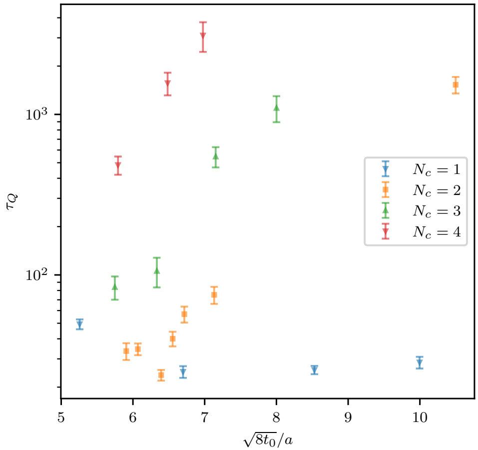
$$Q_L(t) = \frac{1}{32\pi^2} \sum_x \epsilon_{\mu\nu\rho\sigma} \mathrm{Tr}\ V_{\mu\nu}(t, x) V_{\rho\sigma}(t, x) \ ,$$





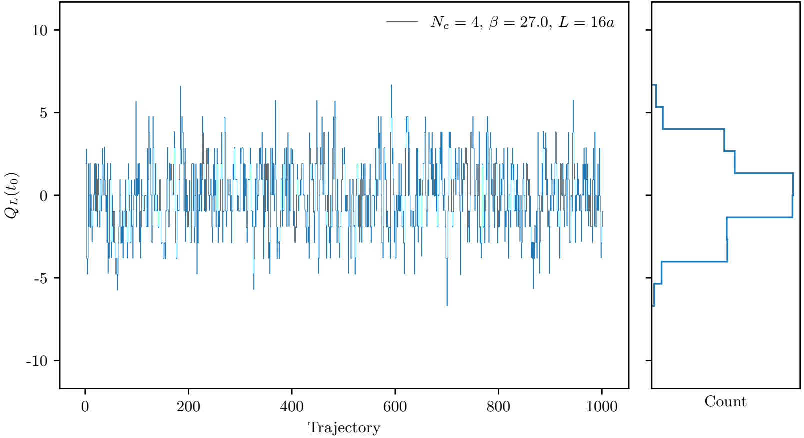
Topological critical slowing down

τ_Q is approximately exponentially dependent on the scale $\sqrt{8t_0}/a$. This poses a serious difficulty especially as N_c is increased.



The history of the Topological Charge

We expect Q_L to be approximately gaussian distributed around $Q_L = 0$. Example from data:



Note: In the figures, Q_L is computed at scale t_0.

Computation of the Topological Susceptibility

In this analysis, we considered:

$$\chi_L(t_0)t_0^2 = \frac{\langle Q_L^2(t_0)\rangle}{L^4}$$

and two working hypotheses:

- The leading discretization error is $O(a^2)$.

$$\chi(a) = \chi(a=0) + c_1 a^2$$

- The leading finite N_c correction is $O(1/N_c^2)$.

$$\chi(N_c) = \chi(N_c = \infty) + \frac{c_2}{N_c^2}$$

As for the glueball spectrum, we expect the topological susceptibility of SU(N_c) and Sp(N_c) to tend to a common limit as $N_c \to \infty$.

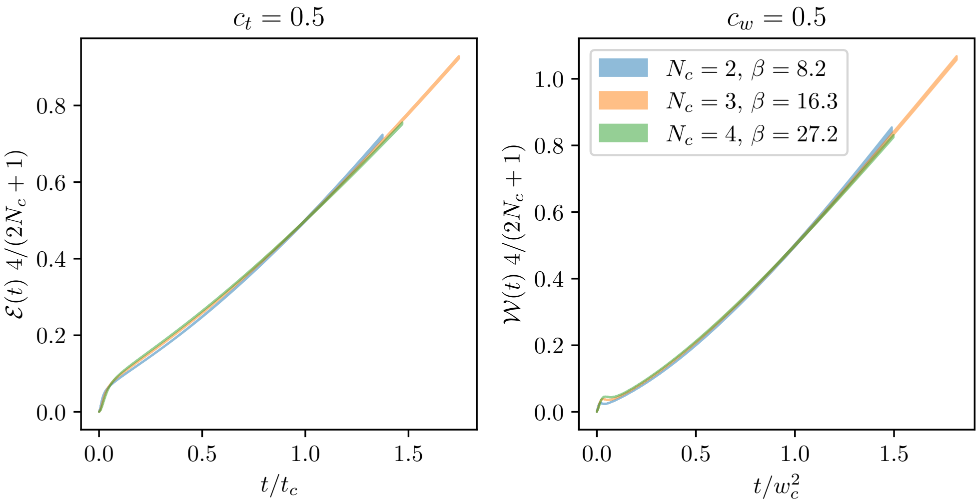
For the Wilson Flow it is found that

$$t^2\langle E\rangle = \frac{4C_2(F)}{32\pi^2}\lambda(1+O(\lambda)) \; , \qquad \text{where} \qquad C_2(F) = \frac{2N_c+1}{4}$$

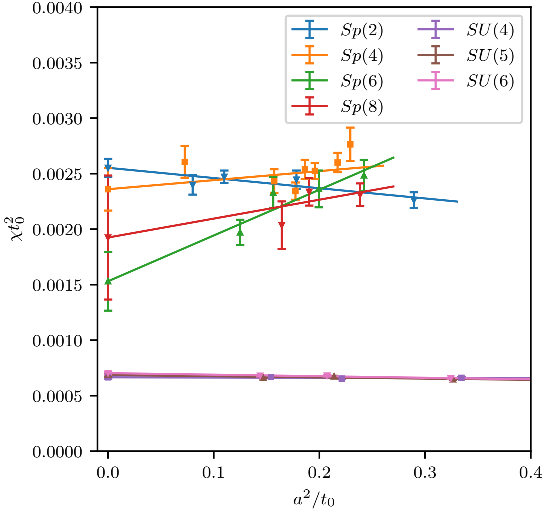
If we choose the scale reference values as follows,

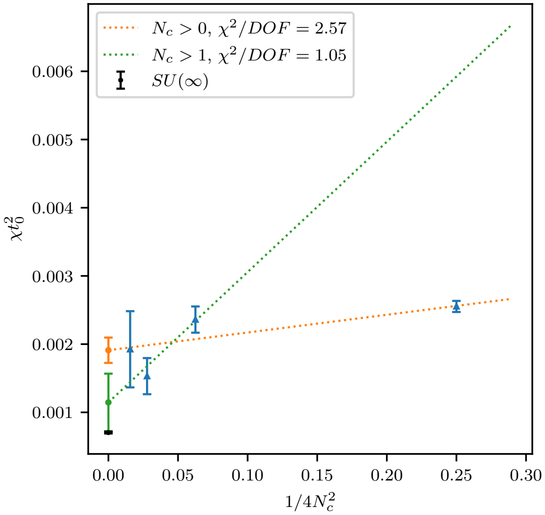
$$\mathcal{E}_0 = c_t \, \frac{2N_c+1}{4} \; , \qquad \mathcal{W}_0 = c_w \, \frac{2N_c+1}{4} \; ,$$

Scaling the flows with N_c



Continuum and Large-N_c extrapolation - PRELIMINARY





The data for SU(N_c) are taken from Cè et al. 2016.

Conclusions

These preliminary data indicate that:

- More statistics is needed on the finest lattice, at all values of N_c.
- The flows exhibit definite scaling properties with N_c
- The result for χ_L in the large-N_c limit seems compatible with the value obtained (on the lattice) for SU(N_c) (with OBC).

Thank you for your attention!

Properties of $\mathrm{Sp}(N_c)$ gauge groups

The symplectic group $\mathrm{Sp}(2N_c)$ can be defined as a subgroup of $\mathrm{SU}(2N_c)$,

$$\mathrm{Sp}(2N_c) = \left\{ U \in \mathrm{SU}(2N_c) \mid \Omega U \Omega^T = U^* \right\}$$

where Ω is the Symplectic matrix,

$$\Omega = \begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix}$$

As direct consequences of the definition:

- $\mathrm{Sp}(2) \simeq \mathrm{SU}(2)$.
- The center of the group is $\mathbb{Z}_2$ for every N_c
- All representations are pseudo-real and Charge conjugation is trivial.
- The block structure of Ω is inherited by $\mathrm{Sp}(2N_c)$ matrices,

$$U = \begin{bmatrix} A & B \\ -B^* & A^* \end{bmatrix}, \quad \left\{ \begin{array}{l} A^\dagger A + B^\dagger B = \mathbb{I}, \\ A^T B = B^T A, \end{array} \right. \quad A, B \in \mathbb{C}^{N \times N} .$$

Scaling of the Wilson Flow

We have that

$$\langle E\rangle = \frac{1}{2}\langle \partial_\mu B^a_\nu \partial_\nu B^a_\mu - \partial_\nu B^a_\nu \partial_\mu B^a_\mu\rangle + f^{abc}\langle \partial_\mu B^a_\nu B^b_\mu B^c_\nu\rangle + \frac{1}{4} f^{abc} f^{cde}\langle B^a_\mu B^b_\nu B^c_\mu B^d_\nu\rangle$$

The calculation goes as in SU(N) except for the number of generators

$$\langle E\rangle = \frac{3N_c(2N_c+1)}{32\pi^2 t^2}\alpha(q)(1+O(\alpha))$$

therefore

$$t^2\langle E\rangle = \frac{4C_2(F)}{32\pi^2}\lambda(1+O(\alpha))$$

where $\lambda = Ng^2$ the 't Hooft coupling and $C_2(F)$ the quadratic casimir of the fundamental representation of $Sp(2N_c)$. We thus expect the flow to scale with N_c according to $C_2(F) = \frac{2N_c+1}{4}$,

$$\mathcal{E}_0 = c\frac{2N_c+1}{4}$$