THE UNIVERSITY of EDINBURGH

Renormalisation of the 3D SU(N) scalar energy-momentum tensor using the Wilson flow

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Lattice21

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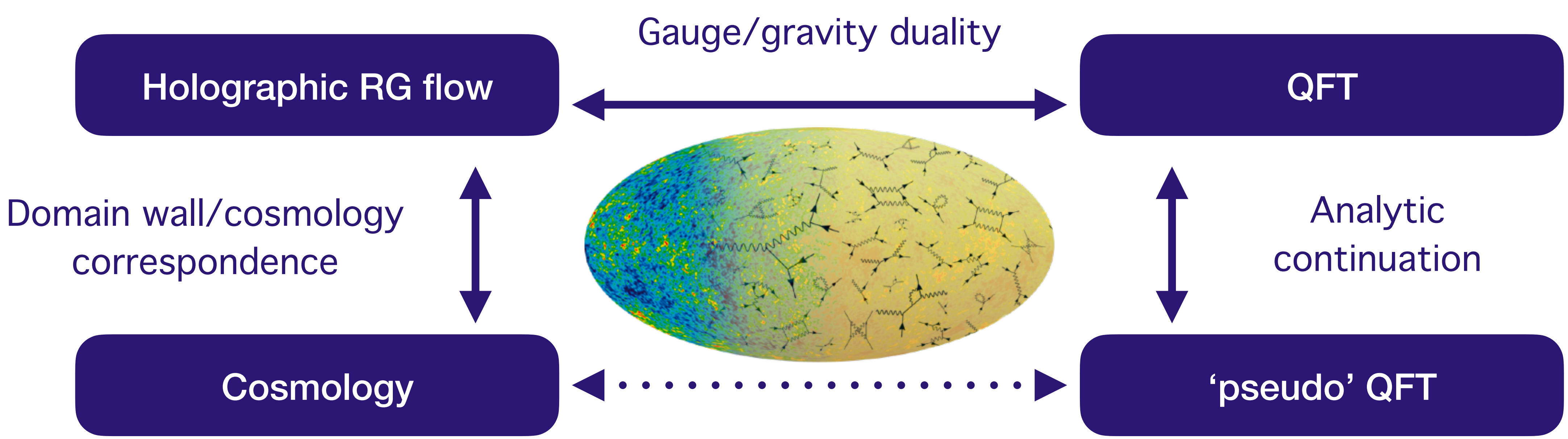
Outline

- Motivation: Holographic Cosmology
- 3D Scalar SU(N) matrix model
- Energy Momentum Tensor (EMT) on the lattice
- Contact term and the Wilson flow
- SU(2) Results
- Conclusion & outlook

Motivation: Holographic Cosmology

- Very early Universe - strong gravity: use holographic principle



P.L McFadden, K. Skenderis
[PRD 81(2) 2010]
[JPCS 222(1) 2010]
[JCAP 05 2011]

- Dual theory: 3D SU(N) gauge theory with massless scalars

- Renormalised EMT correlator related to CMB power spectrum:

$$\Delta^2_{\mathcal{R}}(q) = -\frac{q^3}{4\pi^2}\frac{1}{\langle T(q)T(-q)\rangle} \qquad (T = T_{\mu\mu})$$

Energy momentum tensor of dual theory

Scalar SU(N) matrix model

- 3D $\mathfrak{su}(N)$ -valued massless scalar matrix $\phi_i^j(x) = \phi^a(x)\,(T^a)_i^j$
- Action:

$$S = \frac{N}{g} \int d^3x \,\mathrm{Tr}\left[(\partial_\mu \phi(x))^2 + (m^2 - m_c^2)\phi(x)^2 + \phi(x)^4\right]$$

- Set critical mass m_c^2 for massless theory
  - See talk by Andreas Jüttner (28/7 Wed 14:00 EST)

ag	One loop	Two loop	Nonperturbative $(am_c)^2$
0.1	-0.03159	-0.03125	-0.0313408(38)
0.2	-0.06318	-0.06194	-0.0622974(98)
0.3	-0.09477	-0.09208	-0.092935(16)

SU(2) critical masses

LatCos - PRL 126, 221601(2021)

Simulation Set-up

- 3 lattice spacings ag
- 2 bare masses $(am)^2$
- 3 volumes N_L^3

ag	$(am)^2$
0.1	-0.0305, -0.031
0.2	-0.061, -0.062
0.3	-0.092, -0.091

N_L^3	Trajectories	Sample frequency
64^3	1,500,000	50
128^3	500,000	50
256^3	200,000	100

HMC simulation using GRID

Data analysis using Hadrons and LatAnalyze

https://github.com/paboyle/Grid

https://github.com/aportelli/hadrons

https://github.com/aportelli/LatAnalyze

Lattice Energy-Momentum Tensor

- Naive discretisation of EMT:

$$T^0_{\mu\nu} = \frac{N}{g} \,\mathrm{Tr} \left\{ 2 \left(\overline{\delta}_\mu \phi\right) \left(\overline{\delta}_\nu \phi\right) - \delta_{\mu\nu} \left[\sum_\rho \left(\overline{\delta}_\rho \phi\right)^2 + (m^2 - m_c^2)\phi^2 + \phi^4 \right] \right\}$$

- Ward identity in the continuum:

$$\langle \partial^\mu T_{\mu\nu}(x) P(y) \rangle = - \left\langle \frac{\delta P(y)}{\delta \phi(x)} \partial_\nu \phi(x) \right\rangle$$

$\rightarrow$ $\langle T_{\mu\nu} P \rangle$ Transverse and UV finite (up to contact term)

- Ward identity on the lattice

$$\langle \overline{\delta}^\mu T^0_{\mu\nu}(x) P(y) \rangle = - \left\langle \frac{\delta P(y)}{\delta \phi(x)} \overline{\delta}_\nu \phi(x) \right\rangle + \langle X_\nu(x) P(y) \rangle$$

Lattice Energy-Momentum Tensor

- Mixing with operators with same symmetry
- Divergence proportional to $\delta_{\mu\nu}\,\mathrm{Tr}\,\phi^2$

[Caracciolo et al., NPB 309(4), 1988]

$$T^R_{\mu\nu} = T^0_{\mu\nu} - c_3 \frac{N}{a} \delta_{\mu\nu}\,\mathrm{Tr}\,\phi^2$$

$$c_3^{\text{1-loop}} = \left(2 - \frac{3}{N^2}\right)\left(\frac{6Z_0 - 1}{12}\right)$$

$$Z_0 \approx 0.252731$$

- Contact term:

$$\frac{N}{g}\left\langle T^0_{\mu\nu}(\text{-}q)\,\mathrm{Tr}\,\phi^2(q)\right\rangle = \hat{C}_{\mu\nu}(q) + c_3 \frac{g}{a}\delta_{\mu\nu} C_2(q) + \frac{\kappa}{a}\delta_{\mu\nu}$$

$$\hat{C}_{\mu\nu}(q) = \frac{N}{g}\left(\left\langle T^R_{\mu\nu}\,\mathrm{Tr}\,\phi^2\right\rangle(q) - \left\langle T^R_{\mu\nu}\,\mathrm{Tr}\,\phi^2\right\rangle(0)\right)$$

← Renormalised EMT correlator

$$C_2(q) = \left(\frac{N}{g}\right)^2 \left\langle \mathrm{Tr}\,\phi^2\,\mathrm{Tr}\,\phi^2\right\rangle(q)$$

← Divergent operator mixing

↕ Need to isolate!

$$\kappa = -\frac{N^2}{2}\left(1 - \frac{1}{N^2}\right)\left(\frac{6Z_0 - 1}{12}\right)$$

← Divergent contact term

Wilson Flow

- Promote $\phi(x) \to \rho(t, x)$ Flow time
- Heat equation: $$\partial_t \rho(t,x) = \sum_\mu \partial_\mu^2 \rho(t,x) \qquad \rho(t,x)|_{t=0} = \phi(x)$$
- Dampen high-momentum modes: $\tilde{\rho}(t,k) = e^{-k^2 t}\tilde{\phi}(k)$
  - Smear field by radius $\sim \sqrt{t}$
- Dampen contact term:

$$\frac{N}{g}\left\langle T^0_{\mu\nu}(\text{-}q)\,\mathrm{Tr}\,\rho^2(t,q)\right\rangle = \hat{C}_{\mu\nu}(t,q) + c_3\frac{g}{a}\delta_{\mu\nu}C_2(t,q) + K(t)\delta_{\mu\nu}$$

$$K(t) = \frac{\omega}{\sqrt{t}} + \mathcal{O}\left(\sqrt{t}\right) \qquad \omega = -\frac{N^2}{2}\left(1-\frac{1}{N^2}\right)\left(\frac{\sqrt{2}}{24\pi^{3/2}}\right)$$

Renormalisation condition

- Impose Ward identity on renormalised EMT correlator:

$$\hat{C}_{\mu\nu}(t,q) = \left\langle T^R_{\mu\nu} \operatorname{Tr} \rho^2 \right\rangle (t,q) - \left\langle T^R_{\mu\nu} \operatorname{Tr} \rho^2 \right\rangle (t,0)$$

$$\overline{q}_\mu \hat{C}_{\mu\nu}(t,q) = 0 \quad \rightarrow \quad \hat{C}_{\mu\nu}(t,q) \propto \overline{\pi}_{\mu\nu} = \delta_{\mu\nu} - \frac{\overline{q}_\mu \overline{q}_\nu}{\overline{q}^2}$$

Ward Identity — Transverse

- Fixed longitudinal momentum: $q_l^* = (0,0,q^*)$
  - renormalised (transverse) EMT vanish
- Only contribution from c_3 and contact term $K(t)$

$$\frac{N}{g} \langle T^0_{22} \operatorname{Tr} \rho^2 \rangle (t, q_l^*) = \cancel{\hat{C}_{22}(t, q_l^*)} + c_3 \frac{N^2}{ag} \langle \operatorname{Tr} \phi^2 \operatorname{Tr} \rho^2 \rangle (t, q_l^*) + K(t)$$

Extrapolate against flow time

$$\rightarrow \quad \frac{a}{g}\frac{\langle T^0_{22}\,\mathrm{Tr}\,\rho^2\rangle(t,q_l^*)}{\langle \mathrm{Tr}\,\phi^2\,\mathrm{Tr}\,\rho^2\rangle(t,q_l^*)} = c_3 + \frac{\Omega}{g\sqrt{t}}$$

$$\Omega \approx \frac{\sqrt{2}aq_l^*}{3\pi^{3/2}}$$

Extrapolate value of c_3 in a window of finite flow time $g\sqrt{t}$:

$$ag < g\sqrt{t} < 1$$

4 choices of momenta:

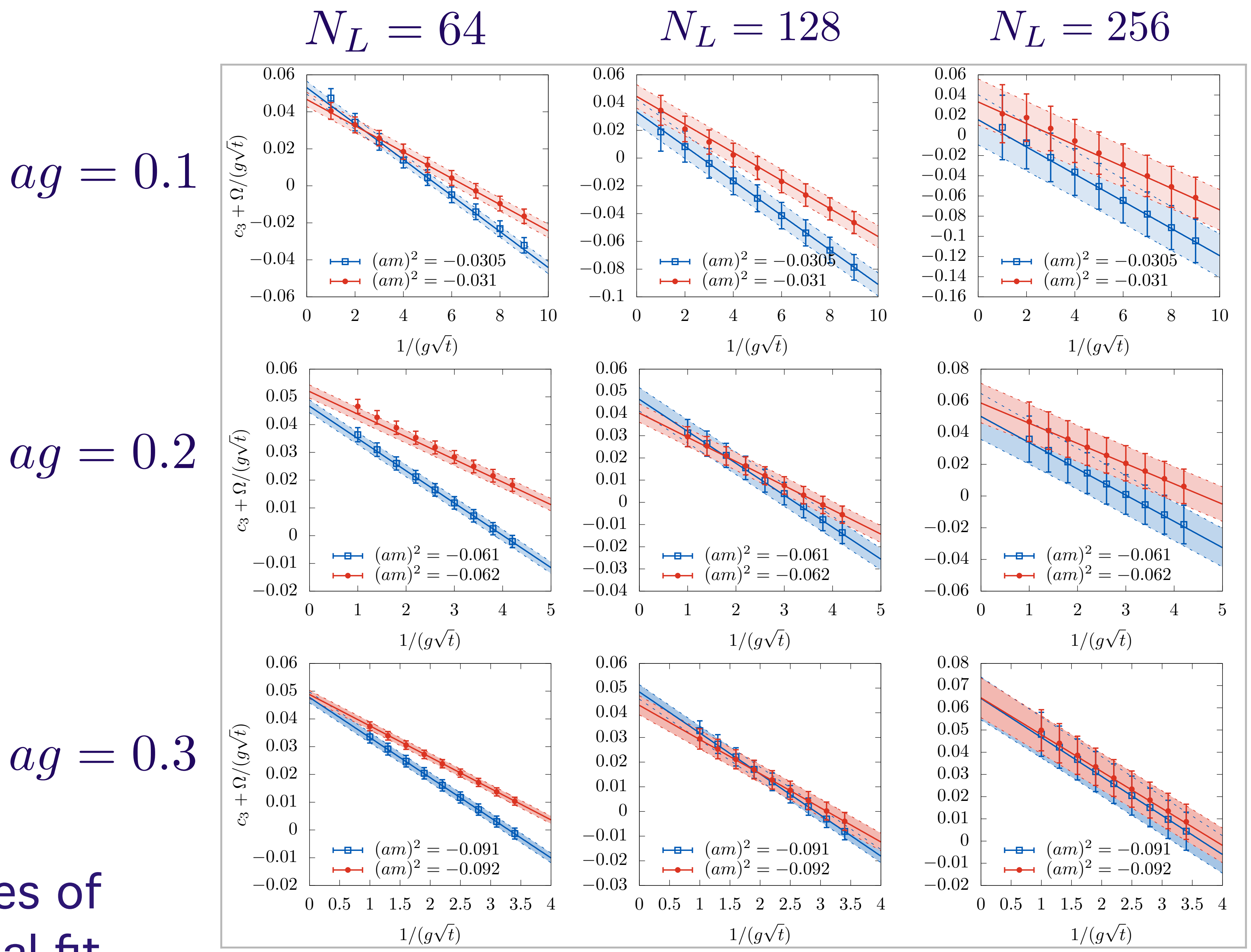
$$a|q_l^*| = 0.049, 0.098, 0.147, 0.196$$

Extrapolate against flow time

SU(2) result:

$$a|q_l^*| = 0.098$$

$$ag < g\sqrt{t} < 1$$



Collect extrapolated values of $c_3(m^2, gL, ag)$ for global fit

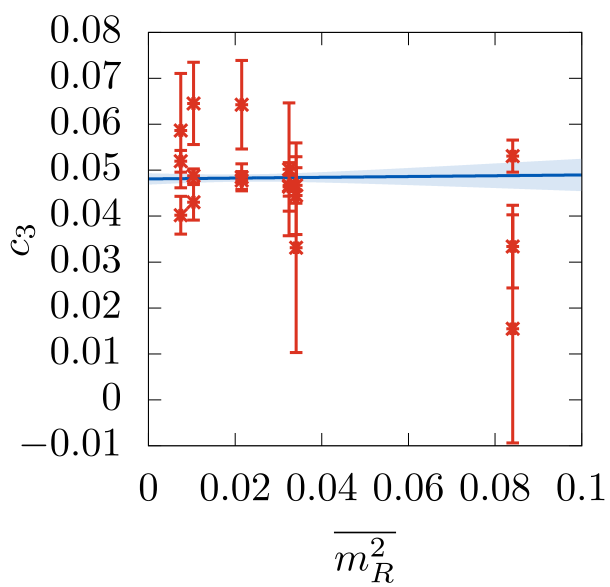
Global fit

Extrapolate to massless, infinite volume, continuum limit:

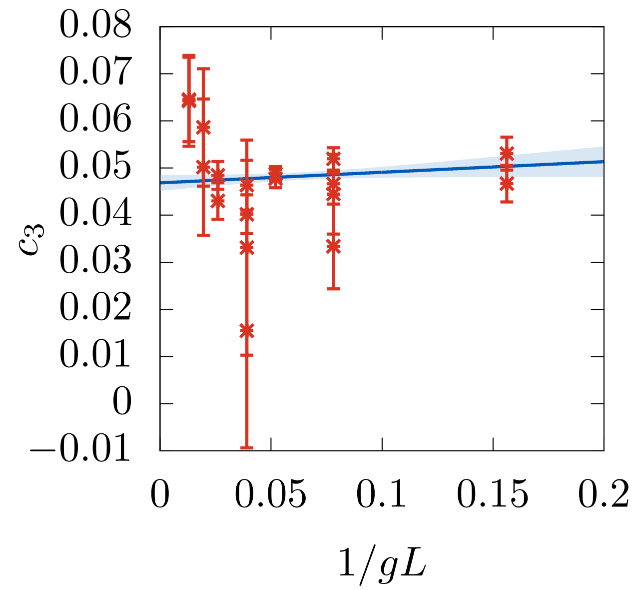
$$c_3(\overline{m_R^2}, gL, ag) = \overline{c_3} + p_0 \overline{m_R^2} + \frac{p_1}{gL} + p_2(ag)$$

$$\overline{m_R^2} = (m^2 - m_c^2)/g^2$$

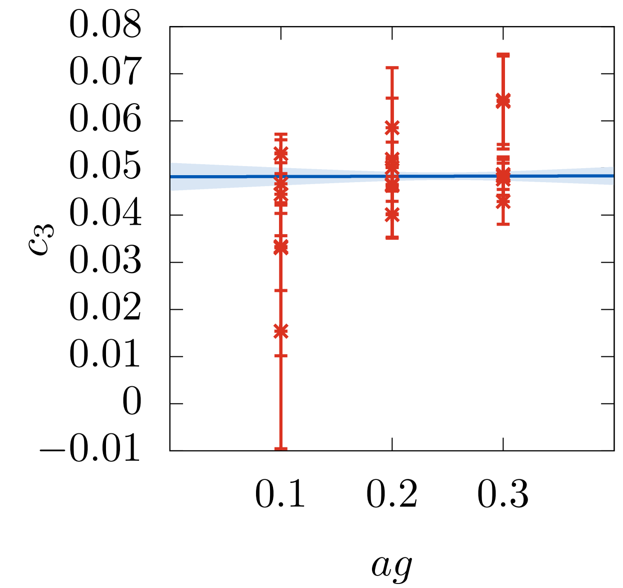
Model parameters $(\overline{c_3}, p_0)$



$(\overline{c_3}, p_1)$



$(\overline{c_3}, p_2)$



4 momenta $a|q_l^*|$, 3 on/off parameters (p_0, p_1, p_2) : 4 x 2^3 = 32 models

Global fit: Errors

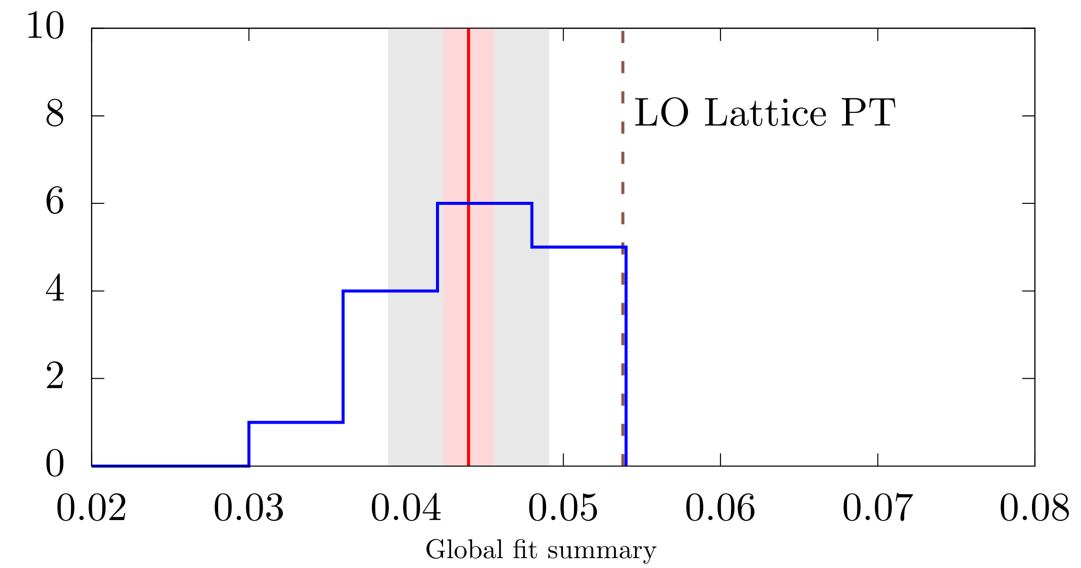
Statistical error:

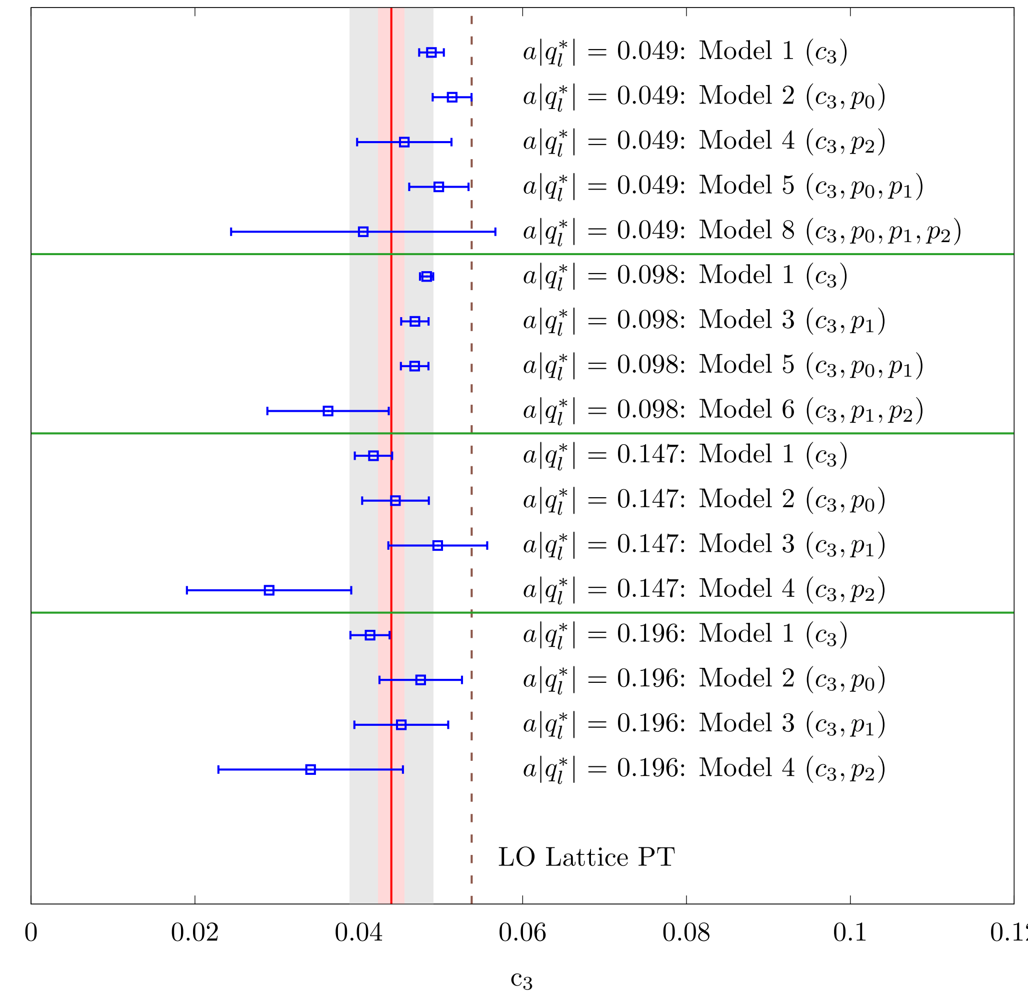
- Bootstrap resampling

Systematic error:

- Keep distribution of $\overline{c_3}$ which does not include parameter with fit value 0.5σ compatible with 0
- 17 remaining models
- Symmetrized central 68.3% interval

$$\overline{c_3} = 0.0440(16)_{\text{stat}}(51)_{\text{sys}}$$





Conclusion & Outlook

- 3D scalar SU(N) model for holographic cosmology
- Energy-momentum tensor on the lattice needs to be renormalised
- Wilson Flow for regulating contact term
- Fit results for renormalisation constant $\overline{c_3}$

- Renormalise EMT 2 point function $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$- needed for CMB spectrum
  - Exploring position space based renormalisation
- Scalar + Gauge field: Talk by Henrique Rocha (28/7 Wed 14:15 EST)

THANK YOU