LATTICE 21

JULY 26-30 2021, ZOOM/GATHER@MIT

Complex Langevin Simulations for $\mathcal{PT}$-symmetric Models

Speaker: Arpith Kumar (IISER Mohali)

Collaborator: Anosh Joseph (IISER Mohali)

INDIAN INSTITUTE OF SCIENCE EDUCATION AND RESEARCH MOHALI

IN PURSUIT OF KNOWLEDGE

Overview

1. Motivation

2. Sign-problem and Complex Langevin Method

3. Scalar Field Theories on a 2d Lattice

4. Supersymmetric Models with PT Symmetry

5. Summary and Future Directions

1. Motivation

- What is supersymmetry?

  Space-time symmetry: (integer spin) bosons $\leftrightarrow$ (half-integer spin) fermions

- Why supersymmetry?

  Physics beyond the Standard Model:
  Hierarchy problem, Gauge coupling unification,
  String theory, AdS/CFT *etc.*

  Framework for unification of all forces

  Insights into strongly coupled gauge theories

- Why SUSY on lattice?

  Non-perturbative *ab-initio* definition of QFTs

  Access to dynamical SUSY breaking

1. Motivation

PHYSICAL REVIEW D VOLUME 57, NUMBER 6 15 MARCH 1998

Model of supersymmetric quantum field theory with broken parity symmetry

Carl M. Bender*
Department of Physics, Washington University, St. Louis, Missouri 63130

Kimball A. Milton†
Department of Physics and Astronomy, University of Oklahoma, Norman, Oklahoma 73019
(Received 10 October 1997; published 24 February 1998)

Non-Hermitian interactions: $W(\phi) = g(i\phi)^{2+\delta}$

$\mathscr{P}\mathscr{T}$ invariant: Physically admissible

Broken parity: $\langle \phi \rangle \neq 0$

Superpotential: $W'(\phi) = -ig(i\phi)^{1+\delta}$

Expansion parameter: δ

SUSY unbroken for $\delta > 0$

In our work:

Study the model on a 2d lattice

Explore non-perturbative regime for δ-even and δ-odd

Action is complex $\longrightarrow$ we use complex Langevin for simulations

2. MC Method and Sign Problem

- **Path integral Monte Carlo**

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}\phi \; \mathcal{O}(\phi) \; e^{-S}$$ →

Generate field configurations $\{\phi_i\}$ with a probability $e^{-S[\phi_i]}$

$e^{-S[\phi_i]}$: real positive

→ $$\langle \mathcal{O} \rangle = \frac{1}{N} \sum_{i=1}^{N} \mathcal{O}(\phi_i)$$

- **(Example) QCD**

$$Z = \int \mathcal{D}U \; \mathcal{D}\overline{\psi} \; \mathcal{D}\psi \; e^{-S[U,\psi,\overline{\psi}]}$$

$$= \int \mathcal{D}U \; e^{-S_B} \det \mathcal{M}$$

can be negative →

Numerical sign problem

(complex phase problem)

2. Sign Problem

- Few among a myriad of theories that suffer from sign problem:

Real time simulation	QCD at finite potential	SYM theory
Hubbard Model	One-Link U(1)	XY Model
Ab initio Nuclear Matter	SU(3) Spin Model	Quantum many body systems
Chern-Simons Theory	O(N) Sigma Model	……

- Tackle sign problem

Very few methods: Analytical Continuation, Re-weighting, TRG, *etc.*

Complexification of integration variables → Complex Langevin Dynamics ✓

Complexification of integration variables → Lefschetz Thimble

2. Complex Langevin Method

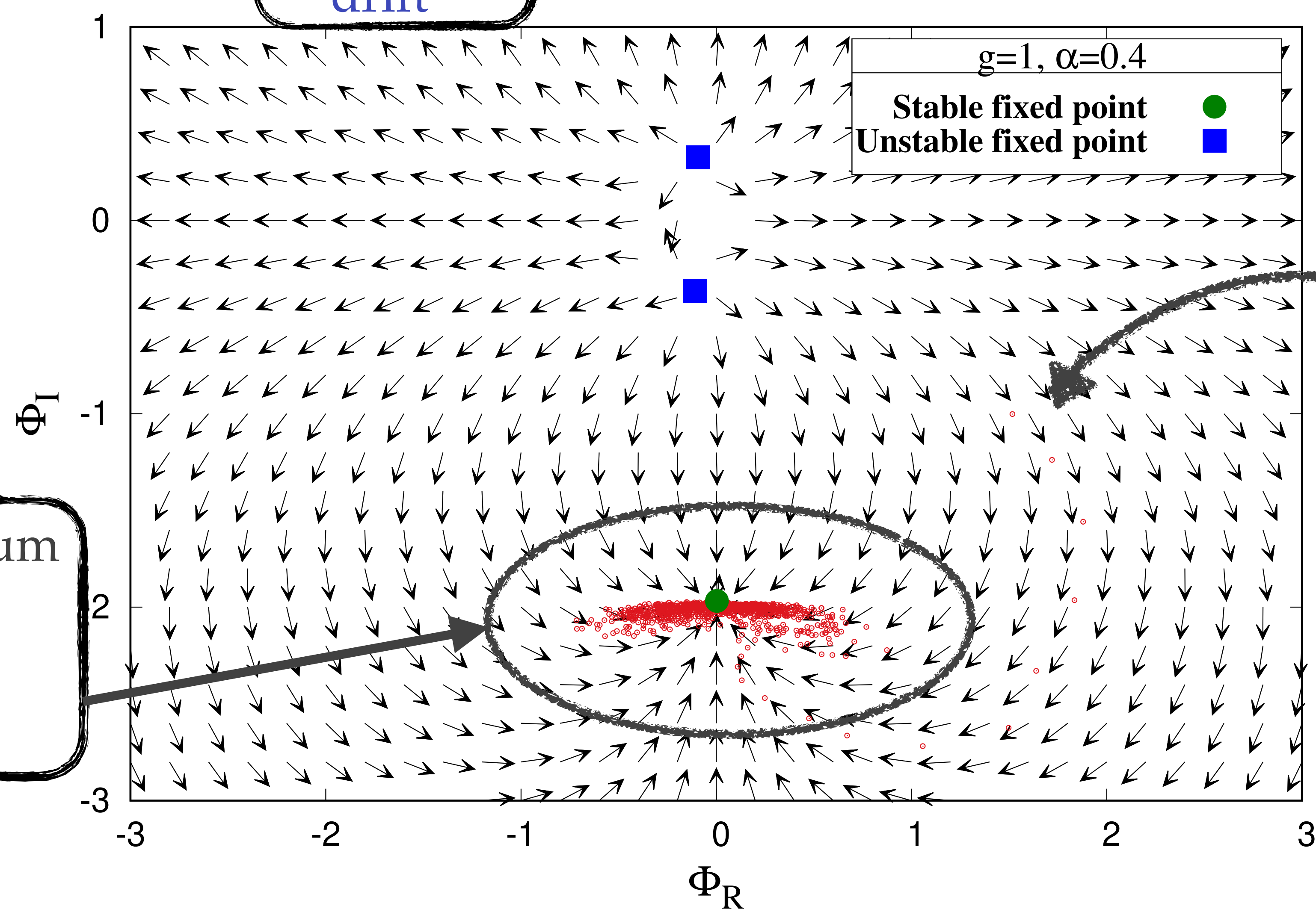
Stochastic quantization: $\frac{\partial \phi(\theta)}{\partial \theta} = -\frac{\delta S[\phi]}{\delta \phi(\theta)} + \eta(\theta)$

drift

noise $\langle \eta(\theta) \rangle = 0$

$\langle \eta(\theta)\eta(\theta') \rangle = 2\delta(\theta - \theta')$

System evolves in fictitious time (Langevin time θ)



Initial config. $\theta = 0 : \phi$

Expectation values $\equiv$ Equilibrium values

θ large : $\langle \mathcal{O}[\phi(\theta)] \rangle_\eta$

$S[\phi]$ complex: Complex Langevin

$\phi = \phi_R + i\phi_I$

$\theta \to \infty : P[\phi] \sim e^{-S[\phi]}$?

Aarts, James, Seiler, and Stamatescu (2010)

Nagata, Nishimura, and Shimasaki (2016)

3. 2d ϕ^4 Theory

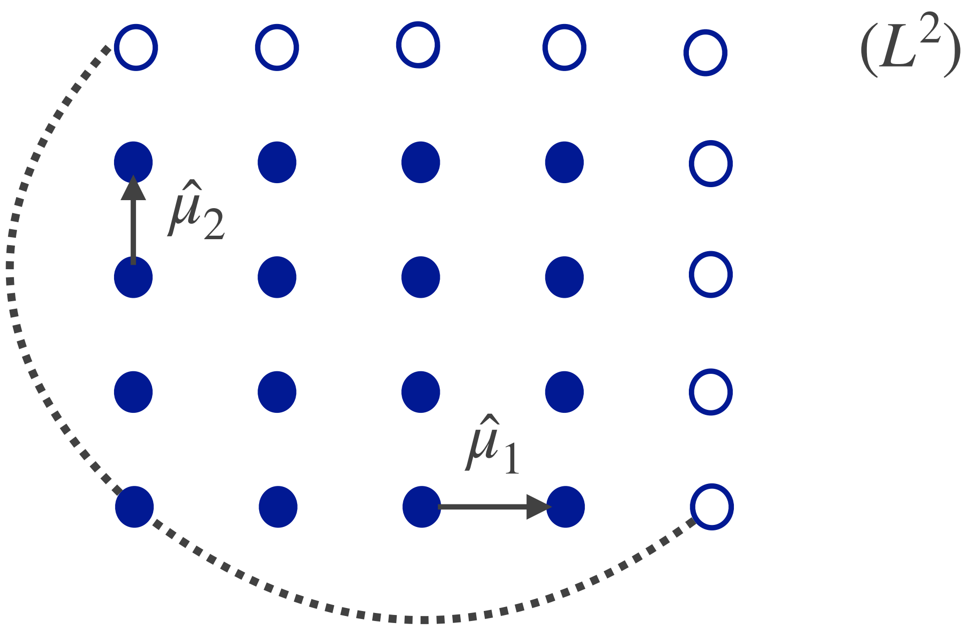
- **ϕ^4 Theory - Continuum**

$$\mathscr{L}_E = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

(ϕ and $\frac{\lambda}{m^2}$ are dimensionless)

$$S_E = \int d^2x\ \mathscr{L}_E$$

- **ϕ^4 Theory - Lattice**



$$\int d^2x = a^2 \sum_r$$

Dimensionless parameters: $m_0^2 = m^2a^2$ and $\lambda_0 = \lambda a^2$

$$S = -\sum_x \sum_\mu \phi_x\phi_{x+\mu} + (2 + \frac{m_0^2}{2}) \sum_x {\phi_x}^2 + \frac{\lambda_0}{4!}\sum_x \phi_x^4$$

3. 2d ϕ^4 Theory - Complex Langevin

- Lattice action

$$\phi \to \sqrt{2\kappa}\Phi, \quad m_0^2 \to \frac{1-2\tilde{\lambda}}{\kappa} - 4, \quad \lambda_0 \to 6\frac{\tilde{\lambda}}{\kappa^2} \qquad \text{(Lattice parameterisation)}$$

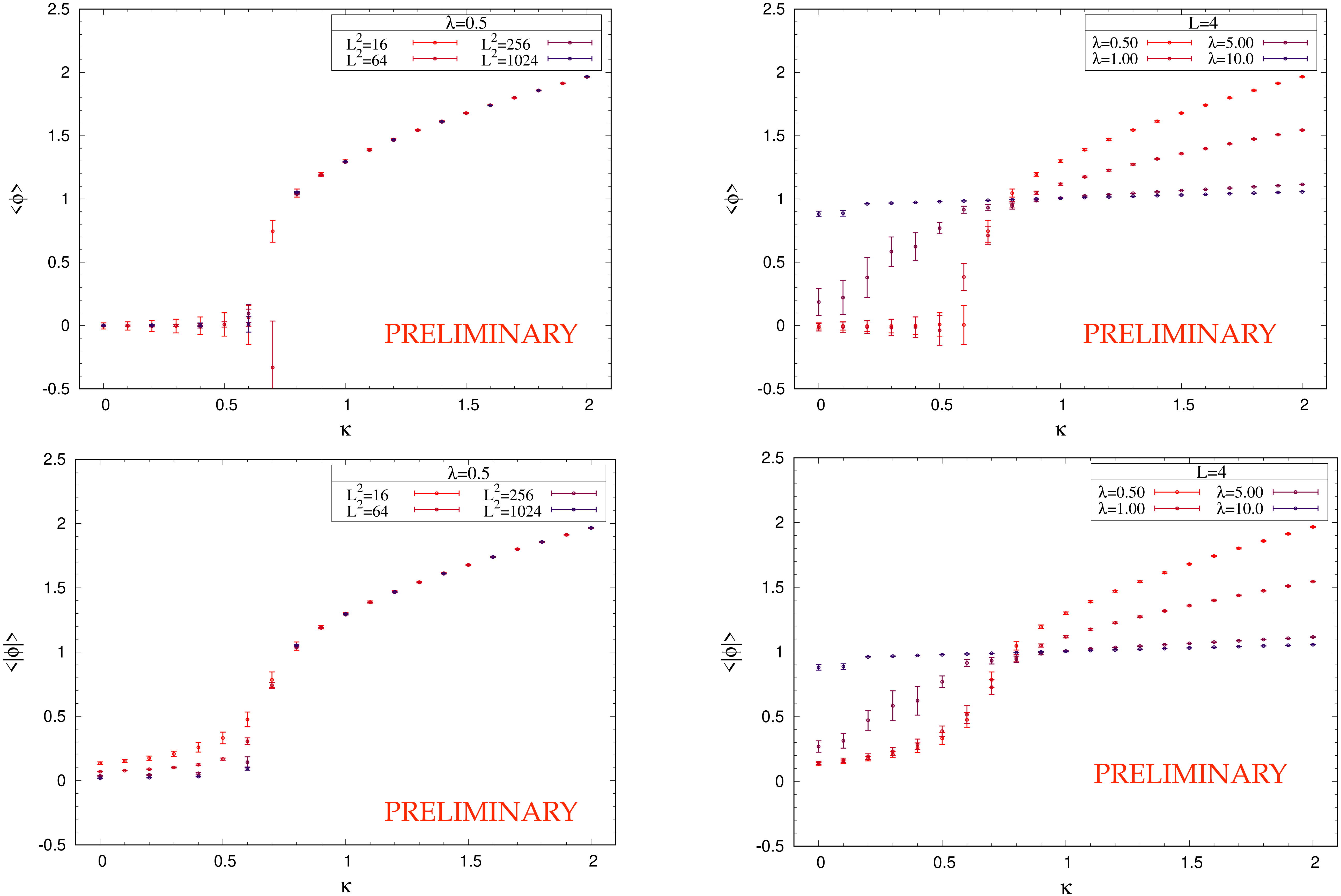
$$S = -2\kappa \sum_x \sum_\mu \Phi_x \Phi_{x+\mu} + \sum_x {\Phi_x}^2 + \tilde{\lambda} \sum_x (\Phi_x^2 - 1)^2$$

- Langevin update for field configurations:

$$\Phi_{x,\theta+\epsilon} = \Phi_{x,\theta} - \epsilon \frac{\partial S}{\partial \Phi_{x,\theta}} + \eta_{x,\theta}\sqrt{\epsilon}$$

$$\frac{\partial S}{\partial \Phi_{x,\theta}} = -\kappa \sum_\mu (\Phi_{x+\mu,\theta} + \Phi_{x-\mu,\theta}) + 2\Phi_{x,\theta} + 4\tilde{\lambda}\Phi_{x,\theta}(\Phi_{x,\theta}^2 - 1)$$

3. 2d ϕ^4 Theory - Complex Langevin



3. 2d $\mathscr{PT}$-Symmetric Theory - Complex Langevin

- **$\mathscr{PT}$-symmetric theory**

$$\mathscr{L}_E = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 - g(i\phi)^{2+\delta}$$

(Real positive spectra at $m \neq 0$)

$$S_E = \int d^2x \ \mathscr{L}_E$$

(Action is complex!)

- **Example: ϕ^3 theory**

$$\mathscr{L}_E = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + g\phi^3 \quad ;$$

Weak-coupling expansion:
Green's functions are power series in g^2.
Not Borel summable: do not alternate sign.
Non-summability reflects: theory not bounded below.

$\downarrow$ ($g \rightarrow ig$)

$$\mathscr{L}_E = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}m^2\phi^2 + ig\phi^3 \quad ;$$

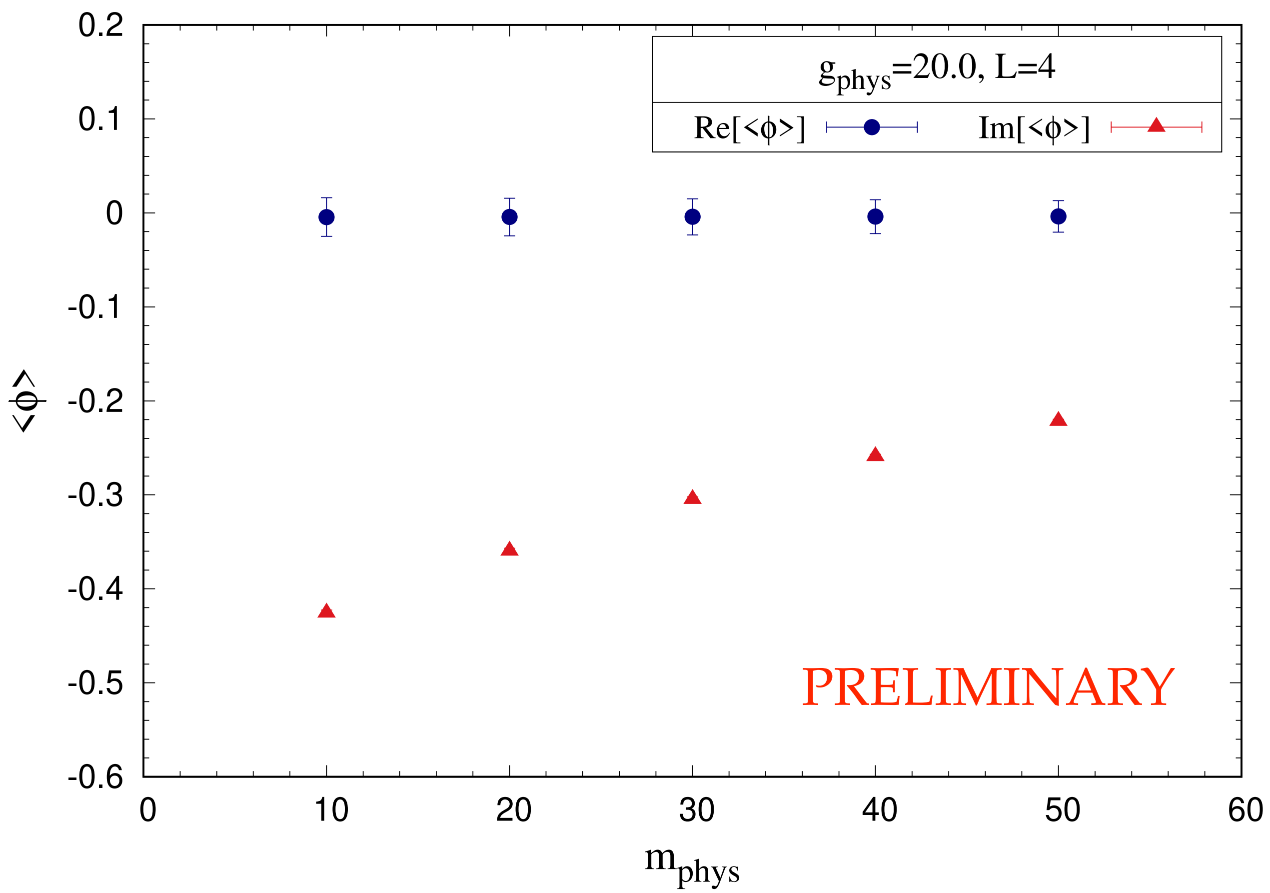
Perturbation series remains real but alternates sign.
Summable and bounded below.

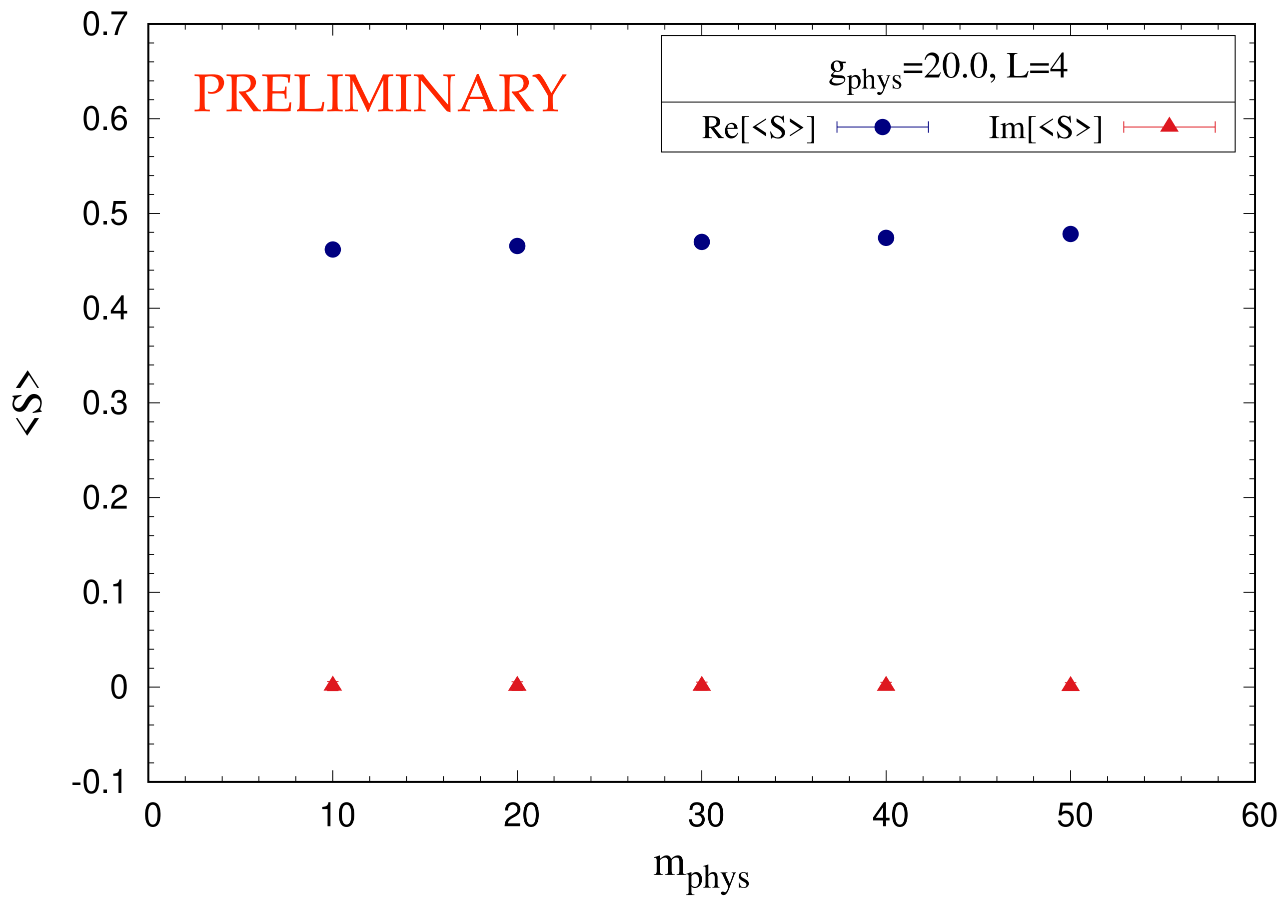
Bender and Milton (1998)

3. 2d $\mathscr{PT}$-Symmetric Theory - Complex Langevin

$$\mathscr{L}_E = \frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2} m^2 \phi^2 - g(i\phi)^{2+\delta}$$

Preliminary complex Langevin simulation results for $\delta = 1$.





4. $\mathcal{N} = 1$ Wess-Zumino Model

- $\mathcal{N} = 1$ **Wess-Zumino Model**

$$S_E = \int d^2x \; \frac{1}{2} \left[(\partial_\mu \phi)^2 + \bar{\psi} \left(\gamma^\mu \partial_\mu + P'(\phi) \right) \psi + P(\phi)^2 \right]$$

ϕ : real scalar field

ψ : Majorana spinor (2 component)

$P(\phi)$: superpotential derivative

$\overline{\psi} = \psi^T C$

C: charge conjugation operator

γ^μ: chiral representation

Action invariant under supersymmetries:

$$\delta\phi = \bar{\epsilon}\psi$$

$$\delta\psi = \left(\gamma^\mu \partial_\mu \phi - P(\phi) \right) \epsilon$$

Few crucial properties:

No $\mathcal{Q}$-exact formulation. No Nicolai map.

Witten index $\Delta = 0$ possible $\Longrightarrow$ SUSY may be broken

4. $\mathcal{N} = 1$ Wess-Zumino Model - Lattice

- Consider a symmetric lattice

  $\beta_t = La = \beta_x$

  Lattice size: L^2

Goltermann and Petcher (1988)

Catterall and Karamov (2003)

Wipf and Wozar (2011)

- Lattice action: $S = S_b + S_f$

$$S_b = \frac{1}{2}\left(-\phi_r \Box^2_{rr'} \phi_{r'} + P_r^2\right)$$

$$S_f = -\,\mathrm{tr}\left[\ln M\right] = -\ln(\det[M])$$

(After integrating out the fermions)

Here,

r, r' are 2d lattice vectors

Fermion matrix: $M \equiv M^{\alpha\beta}_{rr'} = \gamma^\mu_{\alpha\beta} D^\mu_{rr'} + \delta_{\alpha\beta} P'_{rr'}$

$$D^\mu_{rr'} = \frac{1}{2}\left[\delta_{r+e_\mu, r'} - \delta_{r-e_\mu, r'}\right] \text{ (symmetric difference)}$$

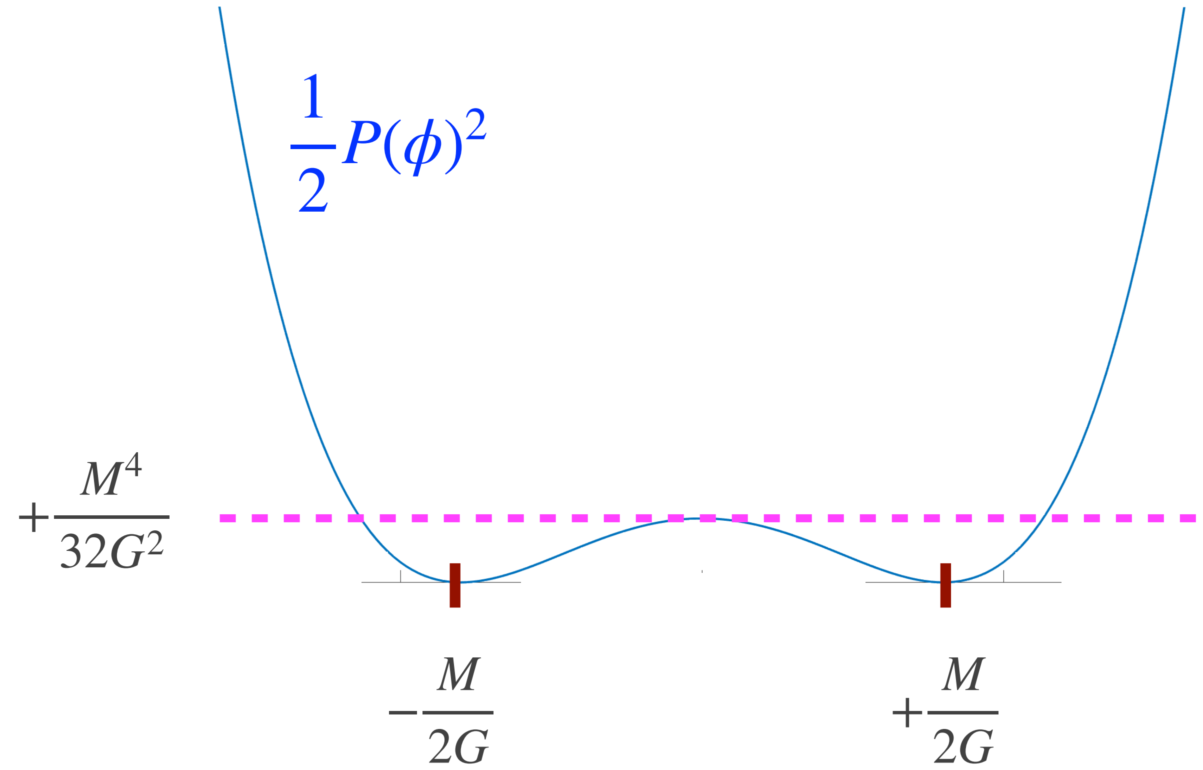
$$\Box^n_{rr'} = \frac{1}{2}\sum_\mu \left[\delta_{r+ne_\mu, r'} + \delta_{r-ne_\mu, r'} - 2\delta_{r,r'}\right]$$

4. Model with Quadratic Potential

- Consider the potential:

$$P(\phi) = G\phi^2 - \frac{M^2}{4G} \; ; \; G \neq 0$$

classical vacua: $\langle \phi \rangle = \pm \frac{M}{2G}$



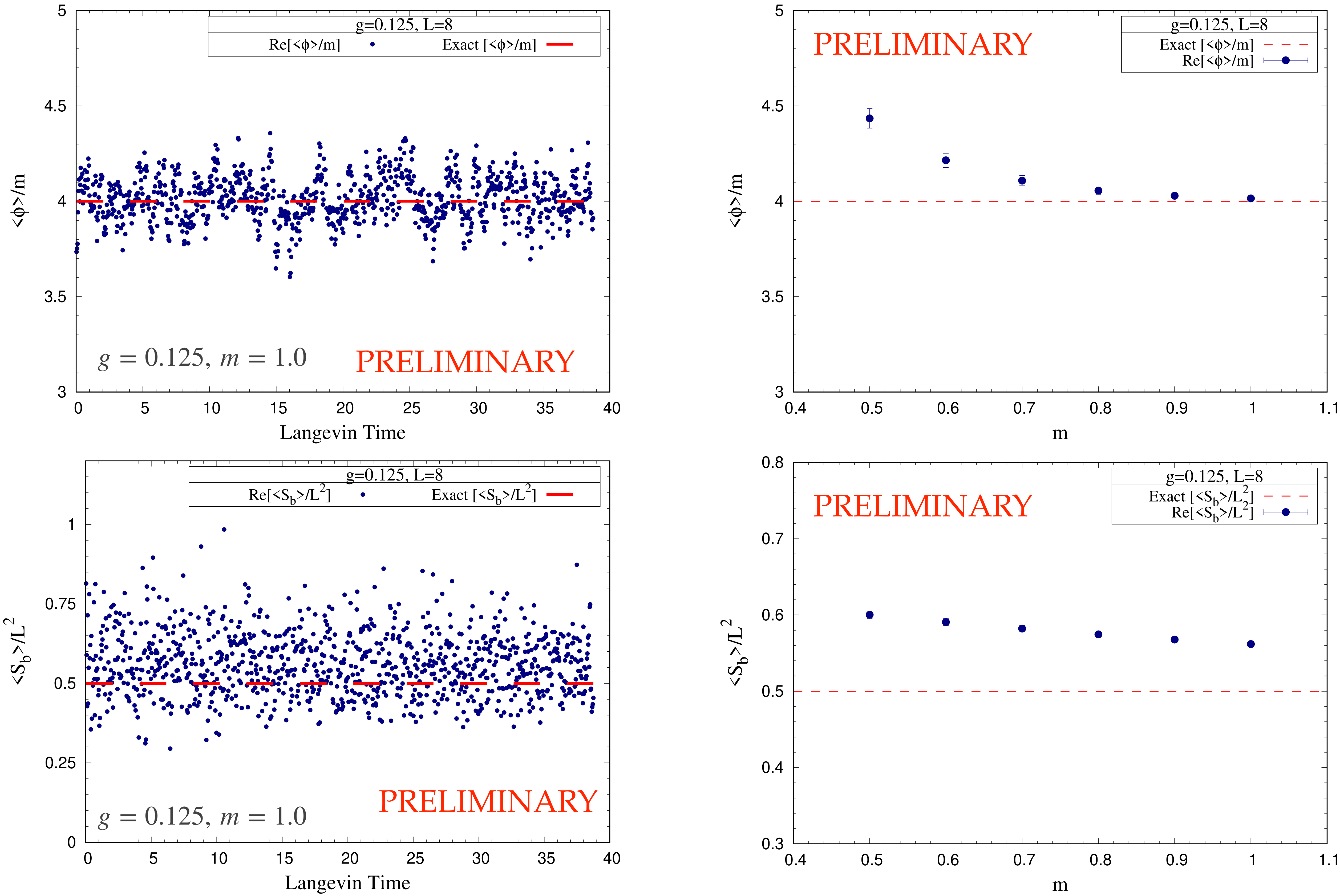
- Lattice discretised potential

Dimensionless couplings: $g = Ga$, $m = Ma$

$$P_r = g\phi_r^2 - m^2/4g - \Box^1_{rr'}\phi'_r/2 \qquad P'_{rr'} = \frac{\partial P_r}{\partial \phi'_r} = 2g\phi_r\delta_{rr'} - \Box^1_{rr'}/2$$

$$\langle \phi \rangle \sim +\frac{m}{2g} \Big/ -\frac{m}{2g}$$

4. Preliminary Simulations with Quadratic Potential

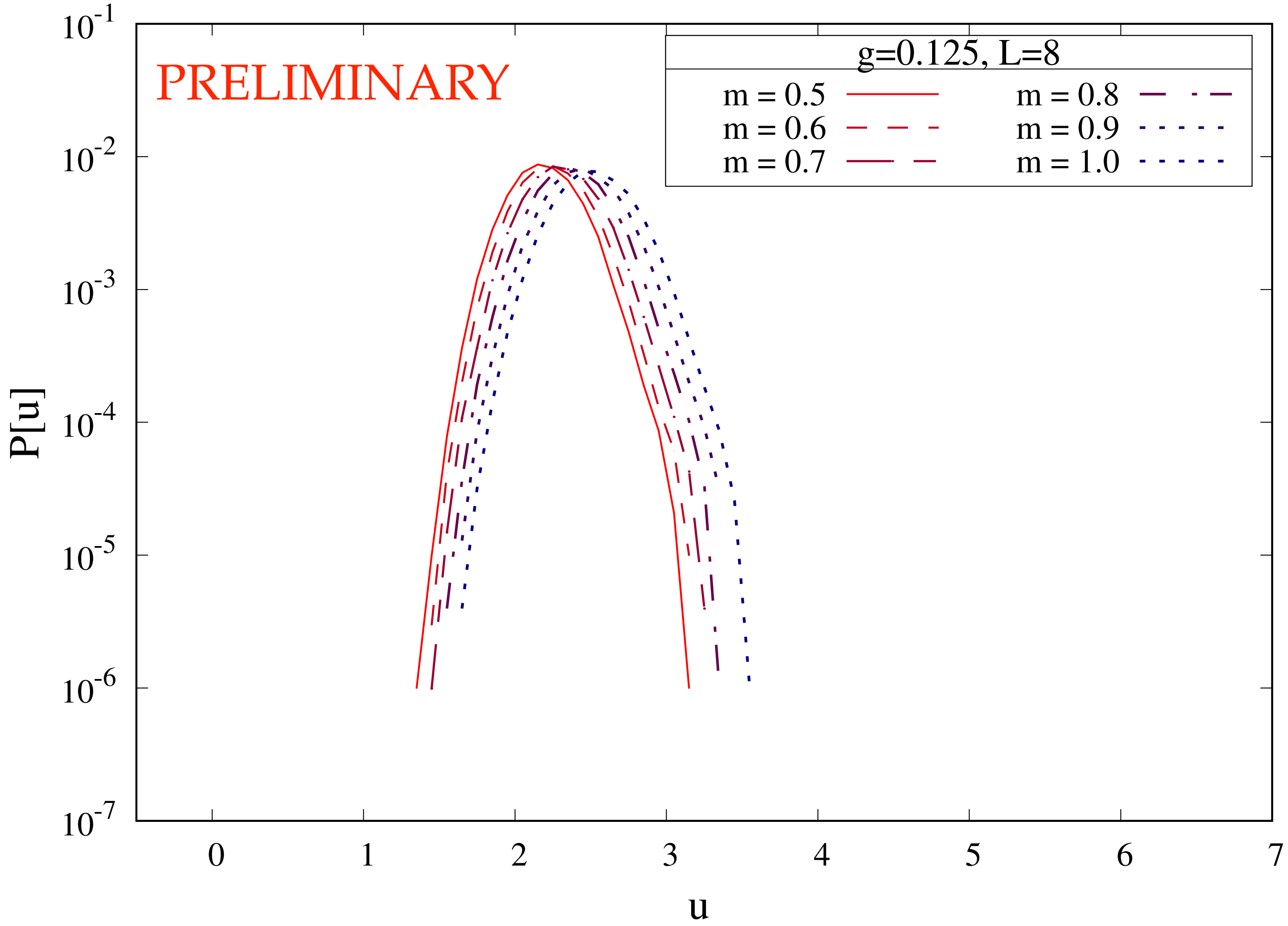


4. Model with Quadratic Potential

Reliability of simulations

Decay of the drift term: $$u = \sqrt{\frac{1}{L^2}\sum_r \left|\frac{\partial S}{\partial \phi_r}\right|^2}$$

Nagata, Nishimura, and Shimasaki (2016)



4. Model with $\mathscr{PT}$ - Symmetric Potential

Work in progress…

Want to cross-check the results of Bender and Milton (1998)

Using complex Langevin simulations

Will breaking of $\mathscr{PT}$-symmetry induce breaking of SUSY?

Summary and Future Directions

1. Can use complex Langevin to handle complex actions/sign problem.

2. Work in progress to investigate non-perturbative SUSY breaking 2d models with PT symmetry.

3. Further along the road - Implications for Higgs physics and MSSM in 4d.

Thank You!

Council of Scientific & Industrial Research
India

Indian Institute of Science Education and Research
Mohali, India

4. Complex Langevin for Lattice $\mathcal{N} = 1$ WZ Model

Update field using

$$\frac{\partial S_b}{\partial \phi_s} = -\Box^2_{rr'}\phi_{r'} + P_r \quad ; \quad \frac{\partial S_f}{\partial \phi_s} = -\operatorname{tr}\left[\frac{\partial Q}{\partial \phi_s} Q^{-1}\right]$$