Bosonization of Majorana modes in arbitrary geometries

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References

- A. Bochniak and B. Ruba, *Bosonization based on Clifford algebras and its gauge theoretic interpretation*, J. High Energ. Phys. **2020**, 118 (2020).
- A. Bochniak, B. Ruba and J. Wosiek, *Bosonization of Majorana modes and edge states*, arXiv:2107.06335.

Fermionic system

- Majorana fermions $\psi_\alpha(x)$ on lattice with arbitrary geometry.
- Number of fermions may be site-dependent ($\alpha = 0, \ldots, n(x)$).
- $\{\psi_\alpha(x), \psi_\beta(y)\} = 2\delta_{x,y}\delta_{\alpha,\beta}$.
- $n = \sum_x (n(x) + 1)$ is an even number.
- We are free to choose any Hamiltonian which conserves $(-1)^F$.

Algebra of even operators

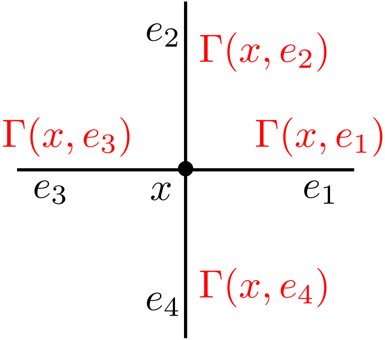
- Even operator $\equiv$ operator commuting with $(-1)^F \equiv$ linear combination of products of bilinears $S(e)$, $T_\alpha(x)$, where
  - $S(e) = \psi_0(x)\psi_0(y)$ for an edge e from x to y,
  - $T_\alpha(x) = \psi_0(x)\psi_\alpha(x)$ for $\alpha \neq 0$.
- $S(e)^2 = T_\alpha(x)^2 = -1$ and $S(e)$, $T_\alpha(x)$ are skew-hermitian.
- $S(e)S(e') = \pm S(e')S(e)$, with the minus sign only if e shares exactly one endpoint with e',
- $S(e)T_\alpha(x) = \pm T_\alpha(x)S(e)$, with the minus sign only if x is incident to e,

Algebra of even operators

- $T_\alpha(x)T_\beta(y) = \pm T_\beta(y)T_\alpha(x)$, with the minus sign only if $x = y$ and $\alpha \neq \beta$,
- Loop relations: $S(e_1)\ldots S(e_m) = 1$ for any loop $e_1 \ldots e_m e_1$.
- The commutation relations together with the loop ones generate the algebra of even operators.

Bosonization

- To any site x associate Clifford algebra with generators $\{\Gamma(x,e) |\, e - \text{edge incident to } x\} \cup \{\Gamma'_\alpha(x)\}_{\alpha=0}^{n(x)}$.



Bosonization

- Relations:
  - $\Gamma(x,e)\Gamma(x,e') + \Gamma(x,e')\Gamma(x,e) = 2\delta_{e,e'}$,
  - $\Gamma'_\alpha(x)\Gamma'_\beta(x) + \Gamma'_\beta(x)\Gamma'_\alpha(x) = 2\delta_{\alpha,\beta}$,
  - $\Gamma(x,e)\Gamma'_\alpha(x) + \Gamma'_\alpha(x)\Gamma(x,e) = 0$,
  - $\Gamma(x,\cdot)\Gamma(x',\cdot) = \Gamma(x',\cdot)\Gamma(x,\cdot)$ for $x' \neq x$.
- Hilbert space: $\mathcal{H} = \bigotimes_x \mathcal{H}_x$,
- In the above sense the system is bosonic.

Mapping between bilinears

- $\widehat{S}(e) = i\,\Gamma(x,e)\Gamma(y,e)$ for an edge e from x to y,
- $\widehat{T}_\alpha(x) = i\,\Gamma'_\alpha(x)$ for $\alpha \neq 0$.
- $\widehat{S}$ and $\widehat{T}$ operators satisfy all relations obeyed by S and T, except for the loop relations.
- We impose them by introducing contraints

$$W(\ell)|\mathrm{phys}\rangle = |\mathrm{phys}\rangle \text{ for every loop } \ell$$

for the operator $W(\ell) = \widehat{S}(e_1)\ldots\widehat{S}(e_m)$.

Relation to gauge theory

- Modification of the constraint:

$$W(\ell)|\mathrm{phys}\rangle = \omega(\ell)|\mathrm{phys}\rangle$$

  with $\omega(\ell) = \pm 1 \Leftrightarrow$ coupling fermions to a background $\mathbb{Z}_2$ gauge field for the $(-1)^F$ symmetry.

- In this case $\omega(\ell)$ is the holonomy along ℓ.

Representation - geometry relation

- Let $N(x) = \deg(x) + n(x)$,
- Define $\gamma(x) \propto \prod_e \Gamma(x,e) \prod_{\alpha \neq 0} \Gamma'_\alpha(x)$ (phase: $\gamma(x)^2 = 1$).
- For $N(x)$ odd $\gamma(x)$ commutes with all gamma matrices $\Rightarrow$ one can impose $\gamma(x) = 1$ or $\gamma(x) = -1$.
- For $N(x)$ even $\gamma(x)$ anticommutes with all gamma matrices $\Rightarrow$ it defines an additional $\Gamma'_{n(x)+1}(x)$ matrix $\Rightarrow$ There is one additional Majorana fermion here.

Equivalence of systems

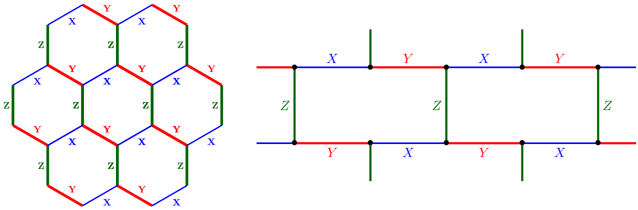
$$\{ \text{ Bosonic system } + \text{constraints}\}$$

$$\Updownarrow$$

$$\left\{ \text{ Sector of fermionic system given by one of the two values of } (-1)^F \right\}$$

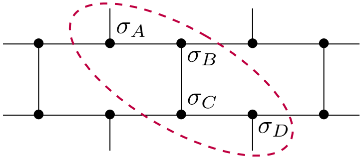
- On the fermionic side we may have those additional fermions.

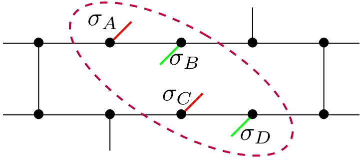
Example 1: Kitaev model



- $H = i \sum_{I \in \{X,Y,Z\}} \sum_{\substack{\text{type } I \\ \text{edges}}} J_I \, \psi(x)\psi(y)$,
- Here $\Gamma(x, e) = \sigma_I(x)$ for $I \in \{X, Y, Z\}$.
- The constraint takes the form $W_P|\mathrm{phys}\rangle = -|\mathrm{phys}\rangle$.
- $\widehat{H} = -\sum_{I \in \{X,Y,Z\}} \sum_{\substack{\text{type } I \\ \text{edges}}} J_I \, \sigma_I(x)\sigma_I(y)$.

Example 2: decagonal lattice





Example 3: Hubbard model on square lattice

- $H = H_0 + V$,
- $H_0 = -t \sum_{\langle xy \rangle} \sum_{\sigma=\uparrow,\downarrow} \Big(c_\sigma^\dagger(x) c_\sigma(y) + c_\sigma^\dagger(y) c_\sigma(x) \Big)$,
- $V = U \sum_x n_\uparrow(x) n_\downarrow(x)$, where $n_\sigma(x) = c_\sigma^\dagger(x) c_\sigma(x)$.
- $t, U \in \mathbb{R}$.

Example 3: Hubbard model on square lattice

- We introduce Majorana fermions ψ by

$$c_\uparrow(x) = \frac{1}{2}(\psi_0(x) + i\psi_1(x)), \quad c_\uparrow^\dagger(x) = \frac{1}{2}(\psi_0(x) - i\psi_1(x)),$$
$$c_\downarrow(x) = \frac{1}{2}(\psi_2(x) + i\psi_3(x)), \quad c_\downarrow^\dagger(x) = \frac{1}{2}(\psi_2(x) - i\psi_3(x)).$$

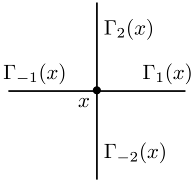
- We need 7 gamma matrices per site to bosonize the system, 6 of them being independent.
- We can immediately express the Hamiltonian in terms of Γ matrices.
- W_P does not involve Γ' matrices (this feature is characteristic for square geometry).

Example 3: Hubbard model on square lattice - Symmetries

- Total number of particles: $\sum_\sigma c_\sigma^\dagger c_\sigma \leftrightarrow 1 - \frac{1}{2}\Gamma'_1 + \frac{i}{2}\Gamma'_2\Gamma'_3$,
- Spin ($SU(2)$ generators):
  - $c_\uparrow^\dagger c_\uparrow - c_\downarrow^\dagger c_\downarrow \longleftrightarrow -\frac{1}{2}\Gamma'_1 - \frac{i}{2}\Gamma'_2\Gamma'_3$,
  - $c_\uparrow^\dagger c_\downarrow \longleftrightarrow \frac{i}{4}(1 - \Gamma'_1)(\Gamma'_2 + i\Gamma'_3)$,
  - $c_\downarrow^\dagger c_\uparrow \longleftrightarrow -\frac{i}{4}(1 + \Gamma'_1)(\Gamma'_2 - i\Gamma'_3)$.
- The corresponding charges are expressible in terms of Γ' matrices.

Boundary effects

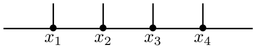
- $L_x \times L_y$ rectangular lattice.
- Two Majorana fermions ψ_0, ψ_1 per lattice site.
- Bulk:



- $\Gamma'_1(x) = \Gamma_{-1}(x)\Gamma_1(x)\Gamma_{-2}(x)\Gamma_2(x)$.
- Constraints identical as for the Hubbard model.

Boundary effects

- $i\Gamma_1(x)\Gamma_{-1}(y) \leftrightarrow \psi_0(x)\psi_0(y)$ if y is the eastern neighbor of x,
- $i\Gamma_2(x)\Gamma_{-2}(y) \leftrightarrow \psi_0(x)\psi_0(y)$ if y is the northern neighbor of x,
- $i\Gamma_{-1}(x)\Gamma_1(x)\Gamma_{-2}(x)\Gamma_2(x) \leftrightarrow \psi_0(x)\psi_1(x)$.
- On the boundary we have:



  - No neighbors in the direction -2
  - $\Gamma_{-2} \leftrightarrow$ an additional Γ', corresponding to a spurious Majorana fermion $\chi_{\mathrm{S}}(x_i)$ at x_i: $\Gamma_{-2}(x_i) \longleftrightarrow \psi_0(x_i)\chi_{\mathrm{S}}(x_i)$.
  - On each of the four corners there are two χ fermions.

Boundary effects

- For every $1 \leq b \leq L_y$ and $1 \leq a \leq L_x$, respectively:

$$\psi_0(1,b)\psi_0(L_x,b) \longleftrightarrow i^{L_x-1}\Gamma_{-1}(1,b)\left(\prod_{a=1}^{L_x}\Gamma_{-1,1}(a,b)\right)\Gamma_1(L_x,b),$$

$$\psi_0(a,1)\psi_0(a,L_y) \longleftrightarrow i^{L_y-1}\Gamma_{-2}(a,1)\left(\prod_{b=1}^{L_y}\Gamma_{-2,2}(a,b)\right)\Gamma_1(a,L_y).$$

- Constraint: $(-1)^{F_\psi} = \kappa\chi_{\partial_S}\chi_{\partial_N}\chi_{\partial_W}\chi_{\partial_E}$ with $\kappa = i^{-(L_x-L_y)^2+2(L_x+L_y)}$.
- Starting from ψ fermionic system we ended up with bosonic theory equivalent to ψ fermions with additional χ fermions on the boundary.

Conclusions and outlook

- We presented the bosonization method which generalizes the old Γ model scheme and solves the problem of the existence of a boundary.
- For two Majoranas per lattice site the duality to the higher gauge theory is known, but the general case remains a open problem.