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Euclidean representation of “Majorana spins”

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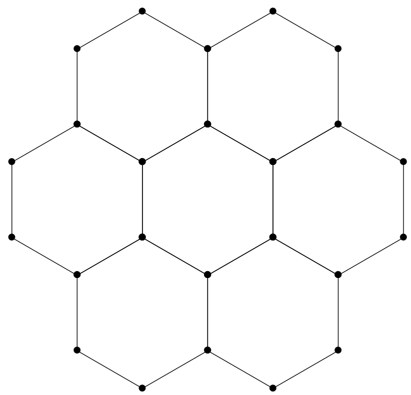
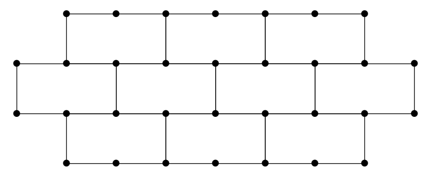
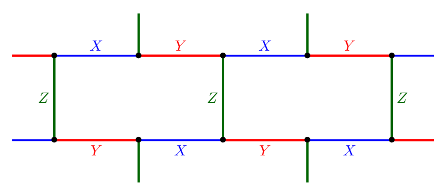
"Majorana spins" ≡ Spins resulting from bosonizing Majorana fermions

I. Lattices and Hamiltonians







Ising Hamiltonian in terms of spin variables/operators (one space-dimension only (d=1))

$$\boldsymbol{H_{Ising} = -\sum_n \sigma^1(n) + \lambda\sigma^3(n)\sigma^3(n+1) = H_{kin} + H_{pot}} \tag{1}$$

Hamiltonian of Majorana spins in two space dimensions

$$\boldsymbol{H_{Majorana} = -\sum_{l,l,l} \sigma^1(i)\sigma^1(f) + \sigma^2(i)\sigma^2(f) + \lambda\sigma^3(i)\sigma^3(f)} \tag{2}$$

Difference - elementary time step:

Ising - single flip **Majorana - double flip**

Challenge: Find euclidean action which does the same

Counting single flips on the Euclidean side

$$S_1(s', s) = \sum_n (s_n - s'_n)^2/4 \sim -\sum_n s_n s'_n, \tag{3}$$

Counting *isolated* double flips

$$S_2^{(8)} = \frac{1}{2^4} \sum_n (1 + s_{n-1}s'_{n-1})(1 - s_n s'_n)(1 - s_{n+1}s'_{n+1})(1 + s_{n+2}s'_{n+2}), \tag{4}$$

They should have the lowest weight in the continuum time limit

$$\beta_t \to \infty, \quad \epsilon = e^{-\beta_t} \to 0, \quad \beta_s = \epsilon\lambda \to 0, \quad T = 1 - \epsilon H. \tag{5}$$

Hence the two-row euclidean action

$$\beta_t L^{kin}(s', s) = \beta_t \left(p(S_1 - 2S_2) + S_2\right). \tag{6}$$

does the job, p - a penalty parameter (> 1).

Challenge II: $\sigma^2\sigma^2$ couplings - the phases

Evolution of a "row" of two spins $s = \{s_1, s_2\} \to s' = \{s'_1, s'_2\}$

$$\sigma^2\sigma^2|s_1, s_2\rangle = \eta\sigma^1\sigma^1|s_1, s_2\rangle = \exp\left(\frac{i\pi}{2}(s_1 + s_2)\right)\sigma^1\sigma^1|s_1, s_2\rangle, \qquad (7)$$

$\Longrightarrow$ **as in the $\sigma^1\sigma^1$ case but with the phase factor.**

- **Add potential terms (in the $y-$ direction).**
- **Generalize for $L_x \times L_y$ plane of spins.**
- **Convolute for L_t time slices.**

$\Longrightarrow$ **Euclidean, three dimensional system of Ising-like spins.**

The action

$$S_{3D} = \beta_t \sum_{x,y,t} O^{(6)}_{x,y,t} + \beta_s \sum_{x,y,t,\zeta_{xy}=1} O^{(2)}_{x,y,t} + \frac{i\pi}{2} \sum_{x,y,t,\zeta_{xy}=-1} O^{(7)}_{x,y,t}, \tag{8}$$

with

$$O^{(7)}_{x,y,t} = \frac{1}{2^3}(s_{x,y,t}+s_{x+1,y,t})(1+s_{x-1,y,t}s_{x-1,y,t+1})(1-s_{x,y,t}s_{x,y,t+1})(1-s_{x+1,y,t}s_{x+1,y,t+1}). \tag{9}$$

and

$$\begin{aligned} O^{(6)}_{x,y,t} &= \frac{1-2p}{8}(1+s_{x-1,y,t}s_{x-1,y,t+1})(1-s_{x,y,t}s_{x,y,t+1})(1-s_{x+1,y,t}s_{x+1,y,t+1}) \\ &+ \frac{p}{2}(1-s_{x,y,t}s_{x,y,t+1}) \\ O^{(2)}_{x,y,t} &= -s_{x,y,t}s_{x,y+1,t}. \end{aligned} \tag{10}$$

$\Longrightarrow$ **Euclidean, three dimensional system of unconstraint Ising-like spins.**

$\Longrightarrow$ **Equivalent to Majorana fermions upon implementing constraints.**

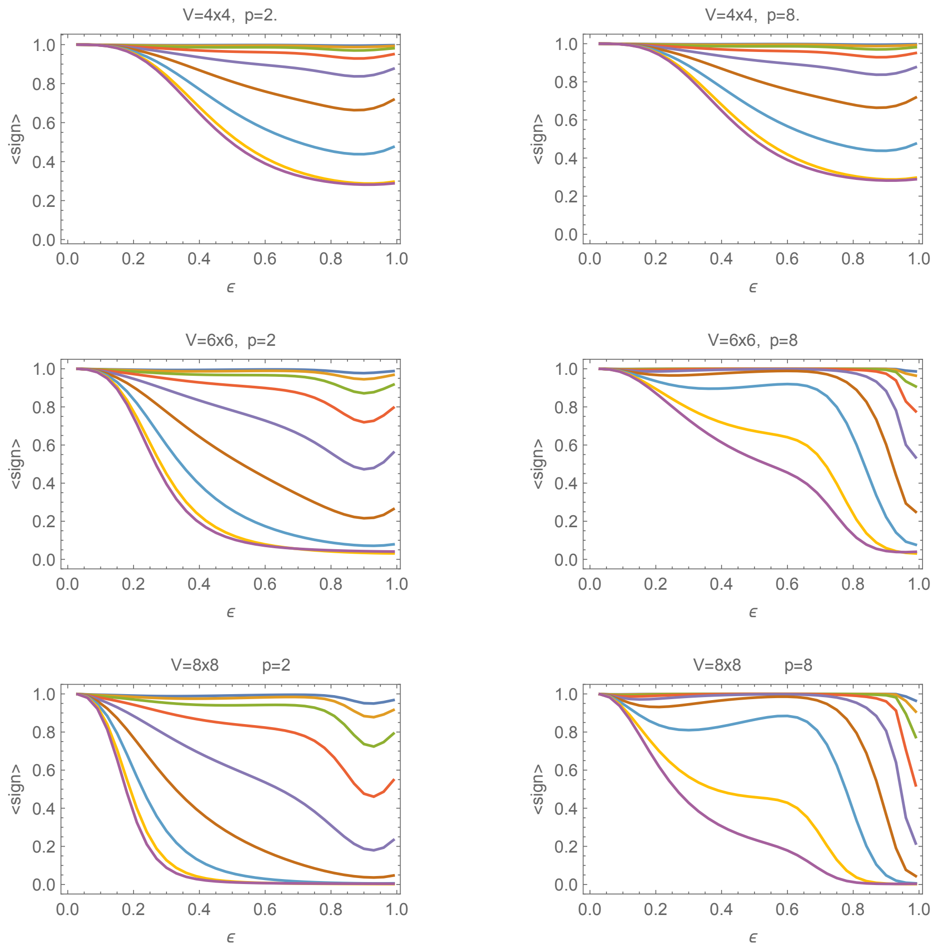
$$\exp\left(-S_{3D}(s', s)\right) \longrightarrow \langle s'|1 - \epsilon H_{Majorana}|s\rangle + O(\epsilon^2) \tag{11}$$

$\Longrightarrow$ **Can be studied with "euclidean" methods.**

$\Longrightarrow$ **Asymmetric between space and time.**

$\Longrightarrow$ **Monte Carlo – sign problem. Reweighting ?**

$$< sign > \equiv < \frac{\rho}{\rho_A} >_A = \frac{Z}{Z_A} \tag{12}$$



$$\boldsymbol{H^{ph}_{1d} = -\sum_{n\ even} \sigma^1_n \sigma^1_{n+1} - \sum_{n\ odd} \sigma^2_n \sigma^2_{n+1} - \lambda \sum_{n} \sigma^3_n \sigma^3_{n+1}, \quad \lambda \in (0.1, 2.)}$$

$D = d + 1 = 3$

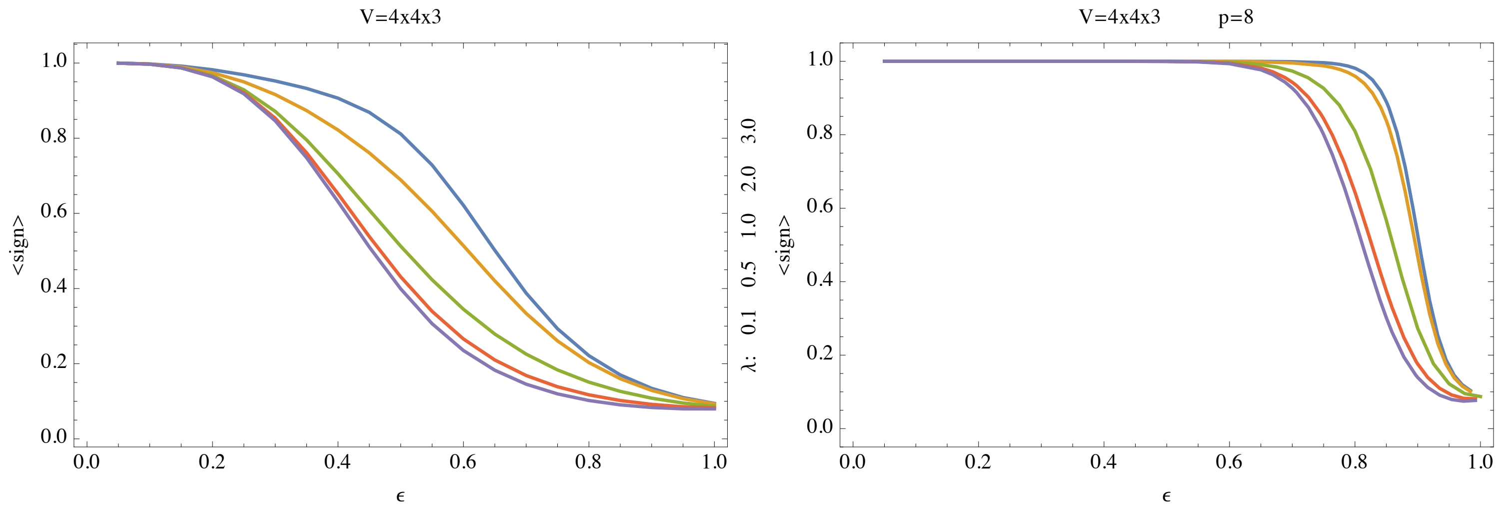


Figure 1: Average sign in the three dimensional case.

- $p = \infty$ $\leftrightarrow$ **constraints**
- **Bosonization constraints**

II. Summary

- **Majorana spins have different elementary evolution than Ising ones.**
- **Euclidean formulation has been found.**
- **The sign problem is mild for small volumes.
  MC studies at intermediate volumes seem feasible.**
- **Constraints, inherent to bosonization, might be attacked.**
- **Similar systems in higher dimensions could be explored.**