Coupling Yang–Mills with Causal Dynamical Triangulations

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Coupling gravity with Yang–Mills gauge fields

Main motivation:
move the first steps in the direction of simulating more realistic systems: gravity coupled to bosonic (and ultimately also fermionic) fields.

We start by minimally coupling Yang–Mills gauge bosons with compact Lie group G.
Continuous action:

$$S_{\mathrm{EH+YM}} = \int_M d^d x \sqrt{|g_{\mu\nu}|} \left[\frac{1}{16\pi G}(R - 2\Lambda) + \frac{1}{2} g^{\mu\alpha} g^{\nu\beta} Tr[F_{\mu\nu} F_{\alpha\beta}] \right],$$

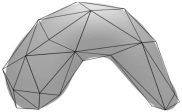
where $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu + ig\,[A_\mu, A_\nu]$ is in Lie(G) and g is the gauge coupling parameter.

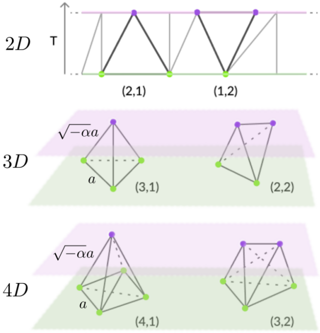
In the lattice regularization, the field $A_\mu(x) \in \mathrm{Lie}(G)$ is replaced by finite parallel transporters called **link variables** $U_{\vec{n},\mu} \in G$.

Model for gravity: Causal Dynamical Triangulation

Causal Dynamical Triangulations (CDT): non-perturbative Monte-Carlo approach to Quantum Gravity.

- **Simplicial manifolds** approximate smooth ones;





- A **causality condition** is enforced by considering only foliated manifolds (fixed slice topology, S^3 here);
- The action is the **Regge discretization** of the Einstein-Hilbert one;
- Wick rotation of timelike links to the Euclidean. Luckily no sign problem!
- **Continuum limit**: search for a second order critical point in the phase diagram.

Reviews: J. Ambjorn et al., Phys.Rept. 519 (2012)
R. Loll, 1905.08669 [hep-th] (2019)

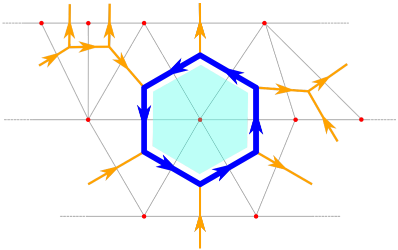
Minimal coupling: the discretized action

$\exists$ analytical studies in 2D on the primal lattice [J. Ambjorn and A. Ipsen (2013)], but the choice of the dual graph accounts for the correct counting for the density of gauge degrees of freedom.
[J. Ambjorn, K.N. Anagnostopoulos and J. Jurkiewicz (1999)]

For a discretized configuration of triangulation and fields $(\mathcal{T}, \Phi_{\mathcal{T}})$:

$$S[\mathcal{T}, \Phi_{\mathcal{T}}] = S_{CDT}[\mathcal{T}] - \beta \sum_{b \in \mathcal{T}^{(d-2)}} \frac{1}{n_b} \Big[\frac{1}{N} Re Tr \Pi_b[\Phi_{\mathcal{T}}] - 1 \Big],$$

where the **plaquette** Π_b is the oriented product of the n_b link variables in loop around a $(d-2)$-simplex b (called **bone**).

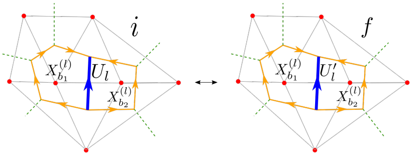


A plaquette in the dual graph of a 2D triangulation.

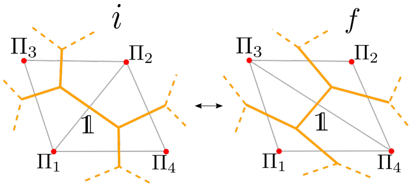
Monte Carlo moves for CDT coupled to Yang–Mills in 2D

The space of all composite configurations is made of pairs $(\mathcal{T}, \Phi_{\mathcal{T}})$, where $\Phi_{\mathcal{T}}$ is a gauge configuration in the fixed triangulation $\mathcal{T}$.

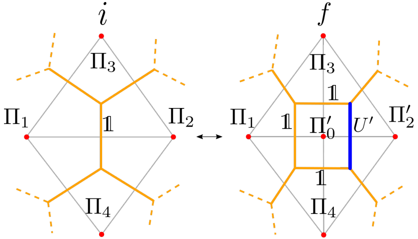
1. The action changes locally at the level of plaquettes involved;
2. Each move is accepted according to the detailed balance conditions;
3. Gauge invariance allows convenient gauge transformations (GT).



Yang–Mills move (fixed $\mathcal{T}$)



Flip of timelike link



Vertex creation/destruction

Complexity for higher dimensions and gauge groups

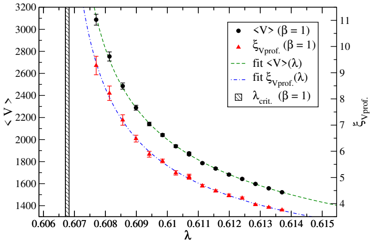
Gauge invariance helps in reducing the number of degrees of freedom involved in the detailed balance, and for $G = U(1)$ and $G = SU(2)$ the group integrals appearing have convenient closed forms [R. Brower, P. Rossi and C. I. Tan (1981)]

but for higher dimensions and for generic Lie groups the complexity of the algorithm explodes!

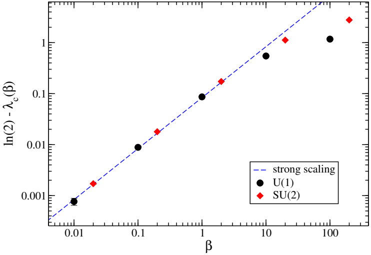
- **In higher dimensions** the number of links which cannot be fixed by gauge transformations increases with the dimension;
- **For higher gauge groups** $U(N)$ and $SU(N)$ For $U(N)$ and $SU(N)$ groups with $N > 2$ a numerical evaluation of Haar integrals has to performed;
- **In both cases** Metropolis-Hasting has very poor acceptance, while heat-bath requires integrations on many group variables.

Gravity-related observables

We focused on the numerical study of the gauge groups $U(1)$ and $SU(2)$ coupled to CDT in 2D: $S_{CDT,2D} = \lambda N_2[\mathcal{T}]$,
$\lambda \leq \lambda_c$: volumes diverge, $\lambda \gtrsim \lambda_c$: shifted power law $A(\lambda - \lambda_c)^{-\nu}$.



$\langle V \rangle$ and 2-point correlation length of volume profiles $\xi_{\text{Vprof.}}$ as function of λ ($\beta = 1$ fixed, $U(1)$).

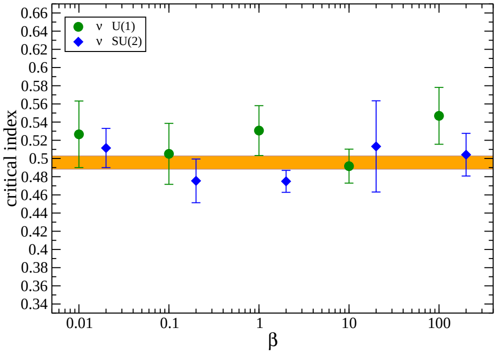


Critical points $\lambda_c(\beta)$ for both $U(1)$ and $SU(2)$ groups. Dashed slope: expected strong coupling behavior.

In the **strong coupling limit** ($\beta \to 0$) $\lambda_c \to \log 2$ (no fields).

Gravity-related observables are independent from YM fields

Besides an unobservable shift in the critical value λ_c, we observed no β-dependence on gravity-related observables.

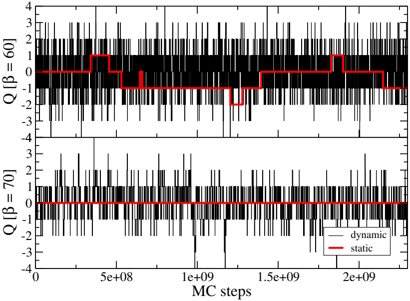


Critical indices for the total volume $\langle V \rangle$ at different λ from the shifted power law $A(\lambda - \lambda_c)^{-\nu}$.

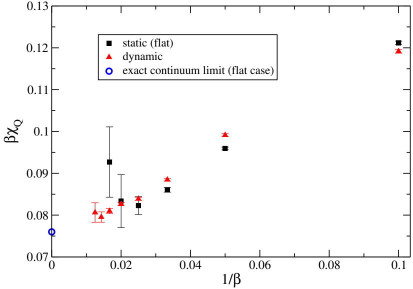
Numerical Results: topology for the $U(1)$ gauge group

$U(1)$ gauge configurations in a toroidal space has non-trivial homotopy, with interesting **topological charge** and **susceptibility**:

$$Q \equiv \frac{1}{2\pi} \sum_{b \in \mathcal{T}^{(d-2)}} \arg[Tr(\Pi_b)] \qquad \chi_Q \equiv \langle Q^2 \rangle / V.$$



Comparison of topological charge histories for a static (flat) and dynamic simulations at same $\langle V \rangle$.



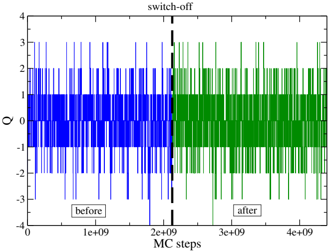
The extrapolated topological susceptibility $\beta\chi_Q = 0.0758(14)$ ($\chi^2/dof = 1.6/3$) agrees with the exact (flat) limit $3/(4\pi^2)$.

Mitigation of topological freezing

Observation: susceptibility is comparable for both the flat static and curved dynamical cases, but the first suffers freezing while the second don't.

As a test, we started a dynamical simulation where we switched off all geometry updates at a certain Monte Carlo step after which the (locally non-flat) geometry is fixed and only the YM part changes.

Unexpectedly, mitigation seems induced by metric inhomogeneities in the geometry, and its fluctuations in Monte Carlo time don't seem to have a role!



Since topological freezing is just an algorithmic problem, this observation has potentially fruitful applications to improve algorithms also in other contexts (e.g. QCD).

Conclusions

- Coupling CDT with compact gauge (and eventually fermionic) fields brings us closer to more realistic systems;
- Algorithm increasingly complex for higher dimensions and for higher order groups (large N);
- The YM topology behaves similarly to flat case, but with an algorithmically mitigated freezing.

Current and future developments:

- Implement the coupling of CDT with higher dimensions and gauge group orders.
- Couple the system with fermion fields (challenging).

Conclusions

- Coupling CDT with compact gauge (and eventually fermionic) fields brings us closer to more realistic systems;
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Thank you for the attention!

additional slides

From dual graph to the continuum action

As in hypercubic lattices, the plaquette term in the YM part of the action contributes with an area term of the type:

$$\Pi_b \simeq \exp\left[igAn_b F_{\mu\nu}\right],$$

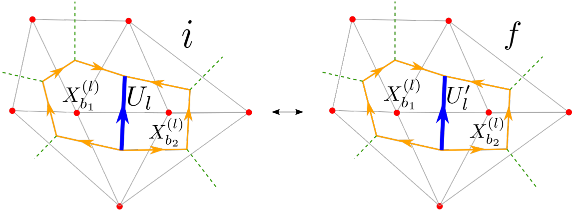
where An_b is the area of the plaquette (b dictates the pair μ, ν). By expanding for small g (continuum limit) we have

$$\Pi_b \to 1 + igAn_b F^a_{\mu,\nu} T_a - \frac{1}{2} g^2 A^2 n_b^2 \sum_{a,b} F^a_{\mu,\nu} F^b_{\mu,\nu} T_a T_b + O(g^3),$$

with $\{T_a\}_{a=1}^{N_G}$ generators of Lie(G) ($N_{U(1)} = 1$, $N_{SU(N)} = N^2 - 1$); by inserting this expression in the discretized action only the third term survives:

$$\implies \frac{1}{n_b}\left[\frac{1}{N} ReTr \Pi_b[\Phi_{\mathcal{T}}] - 1\right] \to -\frac{1}{4}\left(\frac{g^2 A}{N}\right)(An_b) \sum_a F^a_{\mu,\nu} F^a_{\mu,\nu}$$

Detailed balance for the YM moves



$$\Delta S^{(\text{gauge})} = -\frac{\beta}{N} ReTr\left[\left(U'_l - U_l\right) F_l^\dagger\right], \text{ with } F_l \equiv \sum_{b \ni l} \frac{X_b^{(l)}}{n_b} \text{ (force)},$$

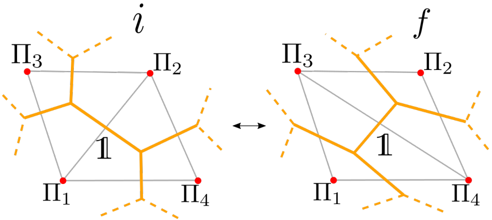
with $X_b^{(l)}$ being the so called *staple* (the two oriented yellow paths)
Extracting U' can be done either by **Metropolis-Hastings** or by a **heat-bath** procedure:

$$\rho_{\text{sel.}}^{\widetilde{U}}(\Phi) = \frac{1}{Z_G(F_l)} \exp\left\{\frac{\beta}{N} ReTr\left[\widetilde{U} F_l^\dagger\right]\right\}, \text{ with } \widetilde{U} \text{ either } U_l \text{ or } U'_l,$$

Normalization $Z_G(F)$ is computed as a Haar integral on G:

$$Z_G(F) \equiv \int_G dU'' \exp\left\{\frac{\beta}{N} ReTr\left[U'' F^\dagger\right]\right\}.$$

Detailed balance for the $(2,2)$ move



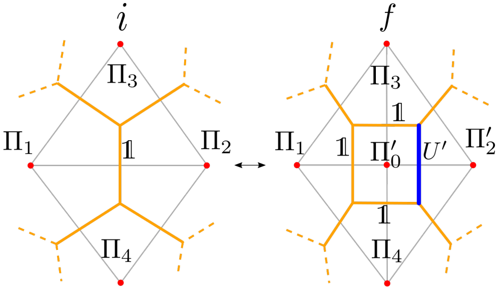
We can use gauge transformations to fix the central link variable to $\mathbb{1}$.

$$\Delta S^{(2,2)} \equiv S^{(2,2)}[f] - S^{(2,2)}[i] = \beta\left[\frac{\widetilde{\Pi}_1}{n_1(n_1+1)} + \frac{\widetilde{\Pi}_2}{n_2(n_2+1)} - \frac{\widetilde{\Pi}_3}{n_3(n_3-1)} - \frac{\widetilde{\Pi}_4}{n_4(n_4-1)}\right],$$

where we defined $\widetilde{\Pi}_b \equiv \left[\frac{1}{N} ReTr\Pi_b - 1\right]$.
Selection probability is trivial: $P_{\text{sel.,YM}}[i] = P_{\text{sel.,YM}}[f] = 1$, but acceptance is not (because of the change in the n_bs):

$$A^{(2,2)} = \min\left(1, e^{-\Delta S^{(2,2)}} \cdot \frac{P_{\text{sel.,CDT}}[\mathcal{T}]}{P_{\text{sel.,CDT}}[\mathcal{T}']}\right).$$

Detailed balance for the (2, 4)–(4, 2) moves



On i we can use gauge transformations to fix the central link to $\mathbb{1}$, but only 3 out of 4 links in f (plaquettes are gauge invariants!!).

$$\Delta S_{\text{YM}}^{(2,4)} = \beta \Big\{ \frac{\widetilde{\Pi}_3}{n_3(n_3+1)} + \frac{\widetilde{\Pi}_4}{n_4(n_4+1)} + \frac{\widetilde{\Pi}_2}{n_2} + \frac{1}{4} - \frac{1}{n_2} - \frac{1}{N} ReTr \Big[U' \Big(\frac{\mathbb{1}}{4} + \frac{X_2^\dagger}{n_2} \Big) \Big] \Big\},$$

where we can isolate the U'-dependent part by defining:

$$\widetilde{\Delta S}_{\text{YM}}^{(2,4)}(U', \{\Pi_b\}) = -\frac{\beta}{N} ReTr \Big[U' \mathcal{F}^\dagger \Big],$$

$$\widehat{\Delta S}_{\text{YM}}^{(2,4)}(\{\Pi_b\}) = +\beta \Big[\frac{\widetilde{\Pi}_3}{n_3(n_3+1)} + \frac{\widetilde{\Pi}_4}{n_4(n_4+1)} + \frac{\widetilde{\Pi}_2}{n_2} + \frac{1}{4} - \frac{1}{n_2} \Big],$$

Detailed balance for the $(2,4)$–$(4,2)$ moves (cont'd)

We choose to select a random value for the new link variable $U' \in G$ with a heat-bath approach:

$$P_{\text{sel.,YM}}[f] = \frac{\exp\Big[-\Delta S^{(2,4)}_{\text{YM}}(U')\Big]}{\int_G dU'' \exp\Big[-\Delta S^{(2,4)}_{\text{YM}}(U'')\Big]} = \frac{1}{Z_G(\mathcal{F})} \exp\left(\frac{\beta}{N} ReTr\Big[U'\mathcal{F}^\dagger\Big]\right),$$

where $\mathcal{F} \equiv \mathbb{1}/4 + X_2/n_2$. Using heat-bath selection, the final Metropolis acceptance probability simplifies:

$$A^{(2,4)} = \min\left\{1, \frac{P_{\text{sel.,CDT}}(\mathcal{T}_i)}{P_{\text{sel.,CDT}}(\mathcal{T}_f)} e^{-2\lambda - \widehat{\Delta S}^{(2,4)}_{\text{YM}}} Z_G(\mathcal{F})\right\},$$

$$A^{(4,2)} = \min\left\{1, \frac{P_{\text{sel.,CDT}}(\mathcal{T}_f)}{P_{\text{sel.,CDT}}(\mathcal{T}_i)} e^{+2\lambda + \widehat{\Delta S}^{(2,4)}_{\text{YM}}} / Z_G(\mathcal{F})\right\}.$$

No dependence on the selected U', but the integral $Z_G(\mathcal{F})$ must be computed each time!

Complexity for higher dimensions

The same ideas sketched on 2D can be applied for higher dimensions and for generic Lie groups, but the complexity explodes! About higher dimensions, gauge transformations can fix only part of the involved link variables.

E.g. the final configuration of move $(2, 6)$ used 3D CDT has dual links structured into a triangular prism: only 5 out of 9 links can be fixed to $\mathbb{1}$.

All unfixed links are degrees of freedom involved in the detail balance selection: Metropolis-Hasting has very poor acceptance, while heat-bath requires integrations on many group variables.

Complexity for higher gauge groups

For $G = U(1)$ and $G = SU(2)$ we have closed forms

[R. Brower, P. Rossi and C. I. Tan (1981)]

$$Z_{U(1)}(\mathcal{F}) = I_0\Big(2\,det(J)\Big); \tag{1}$$

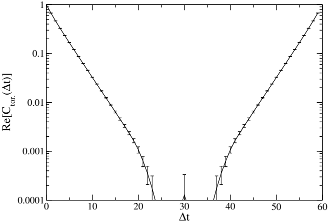
$$Z_{SU(2)}(\mathcal{F}) = \frac{1}{\sqrt{K(J)}} I_1\Big(2\sqrt{K(J)}\Big), \tag{2}$$

where $J \equiv \frac{\beta}{2N}\mathcal{F}$, I_k are the modified Bessel functions of the first kind, while $K(J) \equiv Tr\big(J^\dagger J\big) + det(J) + det(J^\dagger)$.

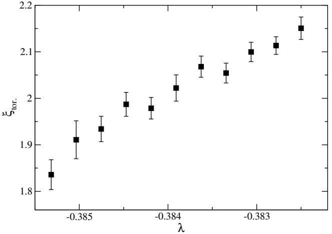
For $U(N)$ and $SU(N)$ groups with $N > 2$ a numerical evaluation of the Haar integral has to performed.

Torelons

Another YM-related observable is the **torelonic profile** $\Pi(t)$: for a 2D torus, it is the oriented product of link variables on the loop of dual-links that wraps around spatially at (slab) time t.



Typical torelon correlation function ($\lambda = -0.44244$ and $\beta = 90$).



Dependence on λ, at fixed $\beta = 70$, for the torelon correlation length.

$$C_{\text{tor}}(\Delta t) \equiv \frac{\langle Tr\Pi(t)\, Tr\Pi^{\dagger}(t+\Delta t)\rangle - |\langle Tr\Pi(t)\rangle|^2}{\langle |Tr\Pi(t)|^2\rangle - |\langle Tr\Pi(t)\rangle|^2} \, .$$

From this we can extract a toreonic 2-pt correlation length $\xi_{tor.}$ as we did for the volume profiles.

A recipe for the continuum limit

Our two length scales for our system, $\xi_{\mathsf{Vprof.}}$ and $\xi_{tor.}$. are related to the physical ones $\xi_{X,r}$ by means of the unobservable lattice spacing a, i.e., $\xi_{X,r} = a\,\xi_X$.
The correct procedure for building a sequence of simulations corresponding to a continuum limit is to keep their ratio $\rho \equiv \xi_{\mathsf{tor.},r}/\xi_{\mathsf{Vprof.},r} = \xi_{\mathsf{tor.}}/\xi_{\mathsf{Vprof.}}$ invariant, while the critical line is approached for $\beta \to \infty$ ($a \to 0$).

Physically, keeping ρ invariant means that the relation between Yang–Mills and gravitational ranges is kept the same, while $a \to 0$ since $\xi_X \to \infty$ in lattice spacing units.

Then, we can forget about the original bare couplings λ and β, since each value of ρ identifies a distinct continuum limit physics, while a correlation length (let's say $\xi_{\mathsf{tor.}}$), used as parameter dictating the common scale for every phenomena of the system.